A modified stochastic volatility model based on Gamma Ornstein–Uhlenbeck process and option pricing

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Abstract

Stochastic volatility model of the Gamma Ornstein–Uhlenbeck possess authentic capability of both capturing some stylized features of financial time series and pricing European options. In this work we modify the Gamma OU model from the viewpoint of Monte Carlo simulation, which is crucial in both model inference and exotic option pricing. We discuss topics related to the measure transformation between objective and risk-neutral measures, arbitrage-free and market incompleteness of the new model. Furthermore, we investigate the performance of this model in European options pricing and an empirical application is presented.

Keywords: Gamma Ornstein–Uhlenbeck processes; gamma-beta relationship; objective and risk-neutral measures; incomplete markets; option pricing; Monte Carlo.

1. Introduction

The standard models for portfolio allocation (Merton, 1973) and for option pricing (Black and Scholes, 1973) both assume that continuously compounded returns are normally distributed. The central limit theorem is often invoked as a primary motivation for this assumption. If returns are realized as the sum of a large number of independent influences, then one can anticipate that returns will in fact be normally distributed.

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Unfortunately, it is well documented that the assumed normality of the return distribution is violated in both the time-series data and in option prices. It is widely accepted that financial time series of different assets share a common set of well-established stylized features. For example, log-returns are known to display heavy tailed and high kurtosis distributions. In financial derivative market, it has been clear that the standard option pricing model of Black and Scholes has been inconsistent with options data for at least a decade. The model in fact implies that the informational content of the option surface is one-dimensional. If the model were to be empirically relevant, then one could infer the prices of options at all strikes and maturities from the price of any single option. This property calls into question the associated activity of price under the Black and Scholes model.

On the theoretical side, arguments have been proposed by Geman *et al.* (2001) which suggest that price processes for financial assets must have a jump component but they need not have a diffusion component. Their argument rests on recognizing that all price processes of interest may be regarded as Brownian motion subordinated to a random clock. An example of infinite activity pure jump Levy processes is variance gamma (VG) model studied by Madan and Seneta (1990) and its asymmetric extension studied by Madan *et al.* (1998). The main feature missing from the Levy model described above is the fact that volatility is changing stochastically over time and negatively correlated with the stock price. It has been observed that the estimated volatilities change stochastically over time and are clustered. In derivative market, the pure jump Levy processes can not perform well when one considers the variation of option prices across maturity (Konikov and Madan, 2002).

These empirical facts in both the time-series data and in option prices are successfully captured by a new class of models in which the stochastic variance of log-returns is constructed via a mean-reverting stationary process of the Ornstein–Uhlenbeck (OU) type driven by a subordinator. These stochastic volatility models have been introduced and thoroughly studied by Barndorff-Nielsen and Shephard (2001) and will be, hereafter, termed BNS models.

Traditional inference approaches for BNS model to empirical financial time-series data are Bayesian inference based on Markov Chain Monte Carlo (MCMC) algorithm which involve the posterior distributions of all parameters given some priors. Due to the evolution of dynamics of volatility in BNS model is complicate, the simulation for the posterior distribution of each parameter is computationally burdensome (Barndorff-Nielsen and Shephard, 2001).

Again, the path-dependent exotic options have become popular in the OTC market in the last two decades. Examples of these exotic options are lookback options, barrier options and asian options. The prices of barrier and lookback options under Black–Scholes model have closed formular. But the asian option

does not have closed formular even in Black–Scholes model. If we use Levy model or more advanced stochastic volatility models to price exotic options, the closed formulars do not exist. Besides, if the payoff of option is dependent of several underlying variables, the closed pricing formular can not be derived generally. In both case, we need to make use of numerical procedure such as Monte Carlo techniques. The Monte Carlo method is based on the simulation of the stock-price process. [Barndorff-Nielsen and Shephard \(2001\)](#) showed that the simulation of BNS model needs series representation. The procedure in this simulation involves approximation to the stochastic intergral and may introduce large variance and reduce efficiency in pricing path-dependent exotic options.

In this paper we first focus on the measure transformation and option pricing of BNS model with Gamma OU process. And then we proposed a modified stochastic volatility model SVBGAR based on BNS model. The new model has same properties as BNS model such that the financial returns display fatter tailed distribution and volatility clustering effect. Moreover, we investigate the option pricing framework of SVBGAR model. Typically, financial markets where investors can only trade in a riskless asset and in an underlying stock with stochastic volatility are incomplete, and in SVBGAR model is no exception. In this work we derived measure transformation between the objective measure and the equivalent martingale measure (or risk-neutral measure). After that, the performance of SVBGAR model in pricing European option was investigated.

The paper proceeds as follows. Section 2 reviews the variance gamma model which gives the foundation of SVBGAR model in terms of the returns distribution. Section 3 goes through the measure transformation and option pricing techniques of BNS model. Section 4 presents the SVBGAR model and its performance in option pricing. Section 5 concludes with a few remarks.

2. Variance Gamma Model in Finance

The variance gamma (VG) Levy processes were proposed in Carr *et al.* (1998) to model the log return process $X_t := \log(S_t/S_0)$ of a financial asset. The VG model can be seen as a Brownian motion with drift time-deformed by a random clock. That is, X_t enjoys the representation

$$X_t = bt + \theta\tau(t) + \sigma W(\tau(t)), \quad (1)$$

where $\sigma > 0$, $\theta \in R$ are given constants, W is Wiener process, and $\tau(t)$ is a gamma Levy subordinator (nondecreasing Levy process), which has gamma distributed with scale parameter $\beta := \kappa$ and shape parameter $\alpha := t/\kappa$. A well known fact is that Levy process has independent and stationary increments. By the definition

of variance gamma process, the increments $X_{t+\delta} - X_t$ has Variance gamma distribution, which is gamma variance–mean mixture of normal distribution as: if $(X_\delta|\tau_\delta) \sim N(b\delta + \theta\tau_\delta, \sigma^2\tau_\delta)$ and $\tau_\delta \sim \text{Gamma}(\delta/\kappa, \kappa)$, then X_δ has same distribution as any increments in $X_{t+\delta} - X_t$ of variance gamma process.

Variance gamma model has been researched in different financial literatures. The empirical property of variance gamma model such that it has fatter tail and higher kurtosis than Brownian motion was investigated in Madan *et al.* (1990). The option pricing framework was given in Carr *et al.* (1998). But VG model does not characterize the dependence of the financial returns and volatility clustering effect which are observed in historical time series data.

3. Non-Gaussian Ornstein–Uhlenbeck-Based Stochastic Volatility Model

One way to incorporate both variance gamma distribution and stochastic volatility effect is using non-Gaussian Ornstein–Uhlenbeck-based stochastic volatility model (Barndorff-Nielsen and Shephard, 2001). The key points of this approach are the presence of jumps in the volatility, flexibility in the choice of marginal distributions for volatility and the availability of closed form expression for characteristic functions of integrated volatility and returns. In this model, the stock process S_t under objective measure is given by

$$S_t = S_0 e^{X_t}, \quad (2)$$

$$dX_t = (\mu + \beta\sigma_t^2)dt + \sigma_t dW_t + \rho dz_t, \quad (3)$$

$$d\sigma_t^2 = -\lambda\sigma_t^2 dt + dz_t, \quad \sigma_0^2 > 0, \quad (4)$$

where μ, β, ρ and λ are real constants with $\lambda > 0$ and $\rho < 0$, W_t is a standard Brownian motion and z_t is the background driving Levy process, a subordinator without drift. The term $\beta\sigma^2$ in Eq. (3) corresponds to the volatility risk premium and the last term ρdz_t accounts for the leverage effect. In the sequel we will call such model BNS model.

In the BNS model (3), the volatility at increases by jumps and decays exponentially between two jumps. When $\rho \neq 0$, each jump in the volatility is associated to a jump in the price process, whose size is proportional to the size of volatility jump. The jumps can be interpreted as the arrival new information on the market, triggering an increase in volatility and a simultaneous fall of asset price. When $\rho = 0$, the volatility still moves by jumps but the price process has continuous sample paths.

Non-Gaussian Ornstein–Uhlenbeck processes in (4) are closely linked to a class of infinitely divisible distributions called self-decomposable distributions.

In essence, the process σ_t^2 is strictly stationary on the positive half-line, i.e. there exists a law D , called the stationary law or the marginal law, such that σ_t^2 will follow the law D for every t if the initial σ_0^2 is chosen according to D . The process σ_t^2 moves up entirely by jumps and then tails off exponentially. The gamma distribution is self-decomposable. In this respect it is interesting to consider an OU process which admits a gamma distribution as its stationary measure and compare the corresponding stochastic volatility models based on the two processes. In particular, this will allow us to see, whether, in addition to marginal properties of at, jumps in the volatility are important. Suppose that the σ_t^2 has stationary gamma distribution with shape c and rate α , which has density $f_{\text{Gamma}}(x) = \frac{\alpha^c}{\Gamma(c)} x^{c-1} e^{-\alpha x}$ for $x > 0$. Barndorff-Nielsen and Shephard (2001) showed that the z_t is a compound poisson variable $\sum_{j=1}^{N_t} G_j$, where N_t is a Poisson process with intensity parameter λc , i.e. $E[N_t] = \lambda c$ and $\{G_j, j = 1, 2, \dots\}$ is an independent and identically distributed sequence; each G_j follows a $\text{Gamma}(1, \alpha)$ law. Since the z_t is compound Poisson, it only jumps a finite number of times in every compact interval. Hence, the process σ_t^2 also jumps a finite number of times in every compact (time) interval.

3.1. BNS model under risk-neutral measure

Recall that a probability measure Q is called a risk-neutral measure if it is equivalent to the objective measure and if under Q the discounted price process $e^{-rt}S_t$ is a martingale. Under one of such measure the models (2), (3) and (4) has the form

$$S_t = S_0 e^{X_t}, \quad (5)$$

$$dX_t = \left(r - l(\rho) - \frac{1}{2} \sigma_t^2 \right) dt + \sigma_t dW_t^Q + \rho dz_t^Q, \quad (6)$$

$$d\sigma_t^2 = -\lambda \sigma_t^2 dt + dz_t^Q, \quad \sigma_0^2 > 0, \quad (7)$$

where r is the interest rate and the superscript Q refers to the fact that W_t and Z_t are, respectively, standard Brownian motion and subordinator with Laplace exponent $l(u) = \log(E[e^{uz_1}])$. In the case of that $\sigma_t^2 \sim \text{Gamma}(\tilde{c}, \tilde{\alpha})$ under the risk-neutral probability, $z_t^Q = \sum_{j=1}^{N_t} G_j$, where N_t is a Poisson process with intensity parameter $\tilde{\lambda} \tilde{c}$, and $\{G_j, j = 1, 2, \dots\}$ is an independent and identically distributed $\text{Gamma}(1, \tilde{\alpha})$ variables. The characteristic function of z_t is by simple calculation

$$\begin{aligned} E[e^{iuz_t}] &= E[e^{iu \sum_{j=1}^{N_t} G_j}] = \exp \left\{ \tilde{\lambda} \tilde{c} t \int_{z>0} (e^{iuz} - 1) f(z) dz \right\} \\ &= \exp \left\{ t \left(\tilde{\lambda} \tilde{c} \int_{z>0} (e^{iuz} - 1) f(z) dz \right) \right\} = \exp \left\{ t \left(\int_{z>0} (e^{iuz} - 1) \tilde{v}(dz) \right) \right\}, \end{aligned} \quad (8)$$

since the Levy measure of z_t has a density $\tilde{\nu}(dz) = \tilde{\nu}(z)dz = \tilde{\lambda} \tilde{c}f(z)dz$. Similarly the moment generating function of z_t is

$$E[e^{uz_t}] = \exp \left\{ t \left(\int_{z>0} (e^{uz} - 1) \tilde{\nu}(dz) \right) \right\}. \quad (9)$$

Then

$$\tilde{l}(u) = \log E[e^{uz_t}] = \int_{z>0} (e^{uz} - 1) \tilde{\nu}(dz) = \frac{u\tilde{\lambda} \tilde{c}}{\tilde{\alpha} - u}. \quad (10)$$

To see why (5)–(7) hold under the risk-neutral measure. Let $x_t = -rt + X_t$, consider $Y_t = e^{-rt+X_t} = e^{x_t}$, where $x_t = \int_0^t (-l(\rho) - \frac{1}{2}\sigma_s^2)ds + \int_0^t \sigma_s dW_s^Q + \rho z_t^Q = \int_0^t (-l(\rho) - \frac{1}{2}\sigma_s^2)ds + \int_0^t \sigma_s dW_s^Q + \rho \sum_{j=1}^{N_t} G_j$, and the x_t 's continue part $\hat{X}_t^c = \int_0^t (-l(\rho) - \frac{1}{2}\sigma_s^2)ds + \int_0^t \sigma_s dW_s^Q$. Apply Ito's formula for Levy process (Cont 2004, Theorem 8.14) with respect function of $f(x_t) = e^{x_t}$,

$$\begin{aligned} Y_t - Y_0 &= \int_0^t e^{x_s} dx_s^c + \frac{1}{2} e^{x_s} dx_s^c dx_s^c + \sum_{s \leq t} (e^{x_s} - e^{x_{s-}}) \\ &= \int_0^t e^{x_s} (-l(\rho) - \frac{1}{2}\sigma_s^2) ds + \int_0^t e^{x_s} \sigma_s dW_s^Q \\ &\quad + \frac{1}{2} \int_0^t e^{x_s} \sigma_s^2 ds + \sum_{s \leq t} (e^{x_s} - e^{x_{s-}}) \\ &= \int_0^t e^{x_s} (-l(\rho)) ds + \int_0^t e^{x_s} \sigma_s dW_s^Q + \sum_{s \leq t} e^{x_{s-}} (e^{\Delta x_s} - 1) \\ &= \int_0^t e^{x_s} (-l(\rho)) ds + \int_0^t e^{x_s} \sigma_s dW_s^Q + \sum_{s \leq t} e^{x_{s-}} (e^{\rho \Delta z_s} - 1) \\ &= \int_0^t e^{x_s} (-l(\rho)) ds + \int_0^t e^{x_s} \sigma_s dW_s^Q + \int_0^t \int_{z>0} e^{x_{s-}} (e^{\rho z} - 1) J(dz, ds) \\ &= \int_0^t e^{x_s} (-l(\rho)) ds + \int_0^t e^{x_s} \sigma_s dW_s^Q \\ &\quad + \int_0^t \int_{z>0} e^{x_{s-}} (e^{\rho z} - 1) (J(dz, ds) - \tilde{\nu}(dz) ds) \\ &\quad + \int_0^t \int_{z>0} e^{x_s} (e^{\rho z} - 1) \tilde{\nu}(dz) ds \\ &= \int_0^t e^{x_s} \left(\int_{z>0} (e^{\rho z} - 1) \tilde{\nu}(dz) - l(\rho) \right) ds + \int_0^t e^{x_s} \sigma_s dW_s^Q \\ &\quad + \int_0^t \int_{z>0} e^{x_{s-}} (e^{\rho z} - 1) (J(dz, ds) - \tilde{\nu}(dz) ds) \\ &= \int_0^t e^{x_s} \sigma_s dW_s^Q + \int_0^t \int_{z>0} e^{x_{s-}} (e^{\rho z} - 1) (J(dz, ds) - \tilde{\nu}(dz) ds), \end{aligned} \quad (11)$$

where $J(dz, dx)$ is the Jump random measure of z_t and the two terms in the last equation are martingales (Cont 2004, Theorem 8.7). Then Y_t is a martingale and $e^{-rt}S_t = S_0 Y_t$ is also a martingale.

Next we can check if the risk-neutral measure exists for this model. Under object measure, $Y_t = e^{-rt}S_t/S_0 = \exp\{\int_0^t (\mu - r + \beta\sigma_s^2)ds + \int_0^t \sigma_s dW_s + \rho z_t\}$. Apply Ito's lemma to the Y_t , we have

$$Y_t - Y_0 = \int_0^t Y_s \left(\mu - r + \left(\frac{1}{2} + \beta \right) \sigma_s^2 \right) ds + \int_0^t Y_{s-} (e^{\rho z} - 1) J(ds, dz) + \int_0^t Y_s \sigma_s dW_s. \quad (12)$$

Define $\theta_s = \frac{\mu - r + (\frac{1}{2} + \beta)\sigma_s^2 + \tilde{l}(\rho)}{\sigma_s} = \frac{\mu - r + (\frac{1}{2} + \beta)\sigma_s^2 + \frac{\rho \tilde{\lambda} \tilde{c}}{\alpha - \rho}}{\sigma_s}$, we have

$$\mu - r = \theta_s \sigma_s - \tilde{l}(\rho) = \theta_s \sigma_s - \int_{z>0} (e^{uz} - 1) \tilde{v}(dz). \quad (13)$$

We also define

$$Z_1(t) = \exp \left\{ - \int_0^t \theta_s dW_s - \frac{1}{2} \int_0^t \theta_s^2 ds \right\},$$

$$Z_2(t) = \exp \left\{ (\lambda c - \tilde{\lambda} \tilde{c})t + \sum_{s \leq t} \log \left(\frac{\tilde{v}(dz)}{v(dz)} \right) \right\} = e^{(\lambda c - \tilde{\lambda} \tilde{c})t} \prod_{i=1}^{N_t} \frac{\tilde{\lambda} \tilde{c} G_i(1, \tilde{\alpha})}{\lambda c G_i(1, \alpha)},$$

$$Z_3(t) = Z_1(t)Z_2(t).$$

Fix a positive T and define $dQ = Z_3(T)dP$, by Girsanov's Theorem (Shreve, 2004, Theorem 11.6.9),

$$W_t^Q = W_t + \int_0^t \theta_s ds \quad (14)$$

is a Brownian motion under Q , and z_t is a compound poisson process with intensity $\tilde{\lambda} \tilde{c}$ and jump distribution of $G_i(1, \tilde{\alpha})$. We plug (13) and 14 into (12) and then Y_t has the dynamics in (11). So the risk-neutral measure exists for this model. But the solution of θ_s is not unique, then there exist many different risk-neutral measure and the market under this model is not complete.

3.2. Option pricing under BNS model

In this section we assume the model is under risk-neutral measure. One way to price the European options given the Characteristic function of the log S_t is using Fourier transform (Carr *et al.*, 1999).

As stated in Barndorff-Nielsen *et al.* (2002) and Carr *et al.* (2003), the Characteristic function of the log S_t under models (5)–(7) is given by

$$\begin{aligned}\phi(u) = E[e^{iu \log S_t}] = \exp \left\{ iu \log S_0 + iut \left(r - \frac{\lambda c \rho}{\alpha - \rho} \right) - \sigma_0^2 \frac{(1 - e^{-\lambda t})}{\lambda} \frac{(u^2 + iu)}{2} \right. \\ \left. + \frac{\alpha \lambda c}{\lambda(\alpha - iu\rho) + \frac{(u^2 + iu)}{2}} \log \left(1 + \frac{(u^2 + iu)}{2} \frac{1}{\alpha - iu\rho} \frac{(1 - e^{-\lambda t})}{\lambda} \right) \right. \\ \left. + \lambda c t \frac{2i\lambda u \rho - u^2 + iu}{2\lambda(\alpha - iu\rho) + u^2 + iu} \right\}. \quad (15)\end{aligned}$$

Let α_0 be a positive constant such that the α th moment of the stock exists. Carr *et al.* (1998) showed that the price a European call option with strike K at the time of maturity T is

$$C(K, T) = \frac{e^{-\alpha_0 K}}{\pi} \int_0^\infty e^{-ivK} \Psi(v) dv, \quad (16)$$

where

$$\Psi(v) = \frac{e^{-rT} \phi(v - (\alpha_0 + 1)i)}{\alpha_0 + \alpha_0 - v^2 + i(2\alpha + 1)v}. \quad (17)$$

The fast Fourier transform can be used to invert the generalized Fourier transform of the call price. Put options can be priced using the put-call parity. Barndorff-Nielsen *et al.* (2002) and Carr *et al.* (2003) calibrated this model and then price European options. The results showed that the option prices of BNS model are strongly consistent with market prices and gave the evidence of the validity of BNS model.

The Fourier transformation method is applied to European options well and efficiently, but it can not be used in American and Exotic options. Another method for pricing options is to use a Monte Carlo simulation. The main advantage of this method is that complex and exotic derivatives can be treated easily which is very important in applications, since little is known about functionals of stochastic volatility models. Moreover, options on several assets can also be handled easily using Monte Carlo simulations. The main drawback of Monte Carlo methods is the slow computational speed. The European call option price is provided by the discounted expected payoff under a risk-neutral measure, i.e.

$$C(K, T) = e^{-rT} E[(S_T - K)^+].$$

The crux of pricing European options with Monte Carlo methods is to simulate the terminal value of asset price S_T . Let S_{T_k} for $k = 1, \dots, N$ denote the simulated values; then, the option price $C(K, T)$ is estimated by the average of the prices for

the simulated asset values, that is

$$\hat{C}(K, T) = e^{-rT} \sum_{k=1}^N (S_{T_k} - K)^+,$$

and by the Law of Large Numbers we have that

$$\hat{C}(K, T) \rightarrow C(K, T) \text{ as } N \rightarrow \infty.$$

If we want to simulate the stock process, we assume that the log return process $X_t = \log(S_t)$ is sampled during a fixed time interval $[0; T]$ at evenly spaced times $t_i = i\delta$, $i = 1, \dots, n$, where $\delta = T/n$.

Since we know $z_t = \sum_{j=1}^{N(t)} G_j(1, \alpha) = \sum_{j=1}^{N(t)} \text{Exp}_j(\alpha)$, the σ_t^2 of (7) can be expressed as by Ito's lemma

$$\begin{aligned} \sigma_t^2 &= \sigma_0^2 e^{-\lambda t} + e^{-\lambda t} \int_0^t e^{-\lambda s} dz_s = \sigma_0^2 e^{-\lambda t} + e^{-\lambda t} \sum_{s \leq t} e^{\lambda s} \Delta z_s \\ &= \sigma_0^2 e^{-\lambda t} + e^{-\lambda t} \sum_{j=1}^{N(t)} e^{\lambda s_j} G_j(1, \alpha) \\ &= \sigma_0^2 e^{-\lambda t} + e^{-\lambda t} \sum_{j=1}^{N(t)} e^{\lambda s_j} \left(-\frac{1}{\alpha} \log(u_j) \right), \end{aligned}$$

where u_j are i.i.d. uniform variables in $[0, 1]$. Then we have

$$\begin{aligned} \sigma_{t_i}^2 &= \sigma_0^2 e^{-\lambda t_i} + e^{-\lambda t_i} \int_0^{t_i} e^{-\lambda s} dz_s, \\ \sigma_{t_{i-1}}^2 &= \sigma_0^2 e^{-\lambda t_{i-1}} + e^{-\lambda t_{i-1}} \int_0^{t_{i-1}} e^{-\lambda s} dz_s, \end{aligned}$$

and

$$\begin{aligned} \sigma_{t_i}^2 &= e^{-\lambda \delta} \sigma_{t_{i-1}}^2 + e^{-\lambda t_i} \int_{t_{i-1}}^{t_i} e^{\lambda s} dz_s \\ &= e^{-\lambda \delta} \sigma_{t_{i-1}}^2 + e^{-\lambda t_i} \sum_{j=N(t_{i-1})+1}^{N(t_i)} e^{\lambda s_j} \left(-\frac{1}{\alpha} \log(u_j) \right). \end{aligned} \quad (18)$$

Given the $\sigma_{t_i}^2$, we have an approximate discrete version of (5):

$$\begin{aligned} \log(S_{t_i}) - \log(S_{t_{i-1}}) &= \left(r - l(\rho) - \frac{1}{2} \sigma_{t_i}^2 \right) \delta + \sigma_{t_i} \sqrt{\delta} \epsilon_i + \rho \sum_{t_{i-1} \leq s \leq t_i} \Delta z_s \\ &= \left(r - l(\rho) - \frac{1}{2} \sigma_{t_i}^2 \right) \delta + \sigma_{t_i} \sqrt{\delta} \epsilon_i + \rho \sum_{j=N(t_{i-1})+1}^{N(t_i)} G_j(1, \alpha), \end{aligned} \quad (19)$$

where ϵ_i are i.i.d $N(0, 1)$. From (19) and (18), the log-returns have variance gamma distribution when $\rho = 0$. Also, this model has volatility clustering effect due to the stochastic volatility. We can use this model to simulate final price S_T to price European options and the path of $(S_t)_{0 \leq t \leq T}$ to price path-dependent exotic options.

4. A Modified Non-Gaussian OU Stochastic Volatility

Model: SVBGAR

The simulation of random term $e^{-\lambda t_i} \sum_{j=N(t_{i-1})+1}^{N(t_i)} e^{\lambda s_j} (-\frac{1}{\alpha} \log(u_j))$ in (18) is complex due to stochastic integral with respect to z_t which is a compound poisson process. We use gamma subordinator and gamma-beta algebra to build a simple model such that the increments of log-stocks under the new model still has variance gamma distribution and volatility clustering effect.

We state two lemmas which give the relation between gamma and beta distribution, which is the fundamental principle to propose new model. The proof can be found in Lewis *et al.* (1989).

Lemma 1. Suppose that $X \sim \text{Gamma}(\alpha, r)$, $Y \sim \text{Gamma}(\beta, r)$, and X and Y are independent. Then

- (a) $X + Y \sim \text{Gamma}(\alpha + \beta, r)$, and
- (b) $\frac{X}{X+Y}$ follows $\text{Beta}(\alpha, \beta)$ and is independent of $X + Y$.

Lemma 2. If random variables X and Y are independent with $X \sim \text{Beta}(\alpha, \beta)$ and $Y \sim \text{Gamma}(\alpha + \beta, r)$, then $XY \sim \text{Gamma}(\alpha, r)$.

Let e_t is a Gamma process with Levy density $v(dx) = \frac{ae^{-\alpha x}}{x} 1_{x>0} dx$, then the probability density of e_t is

$$f_{e_t}(x) = \frac{\alpha^{at}}{\Gamma(at)} x^{at-1} e^{-\alpha x}.$$

Then the increments of e_t at $[t_{i-1}, t_i]$ is $e_i = e_{t_i} - e_{t_{i-1}}$, which has following distribution as $e_\delta \sim \text{Gamma}(a\delta, \alpha)$. We know $\sigma_{t_{i-1}}^2$ in (18) follows $\text{Gamma}(c, \alpha)$ distribution and let $b_i \sim \text{Beta}(c - a\delta, \delta)$ which is independent with $\sigma_{t_{i-1}}^2$ and e_i , then $b_i \sigma_{t_{i-1}}^2 \sim \text{Gamma}(c - a\delta, \alpha)$ by Lemma 2. Then we define

$$\sigma_{t_i}^2 = b_i \sigma_{t_{i-1}}^2 + e_i. \quad (20)$$

By Lemma 1, $\sigma_{t_i}^2 \sim \text{Gamma}(c, \alpha)$. The $\sigma_{t_i}^2$ in (18) and (20) have same marginal distribution and auto-regression structure but the simulation in (20) is much simpler.

In order to consider the leverage effect of stock price, we consider the discrete model for S_t as

$$\begin{aligned} & \log(S_{t_i}) - \log(S_{t_{i-1}}) \\ &= \left(r - l(\rho) - \frac{1}{2} \sigma_{t_i}^2 \right) \delta + \sigma_{t_i} \sqrt{\delta} \epsilon_i + \rho(e_{t_i} - e_{t_{i-1}}), \end{aligned} \quad (21)$$

where ϵ_i are i.i.d $N(0, 1)$. From (21) and (20), the log-returns still have variance gamma distribution when $\rho = 0$. Also, this model has leverage and volatility clustering effect. We call this model as stochastic volatility model with beta-gamma autoregressive (SVBGAR).

The continuous model corresponding to (21) is

$$S_t = S_0 e^{X_t} \quad (22)$$

$$dX_t = \left(r - l(\rho) - \frac{1}{2} \sigma_t^2 \right) dt + \sigma_t dW_t + \rho de_t. \quad (23)$$

We can use similar skills as (11) and show that if

$$l(\rho) = \log(E[e^{\rho e_1}]) = -a \log \left(1 - \frac{\rho}{\alpha} \right),$$

then $e^{-rt} S_t$ in (22) and (23) is a martingale and the model is risk-neutral. This model is more efficient than BNS model when we price European and exotic options using Monte Carlo simulation.

4.1. SVBGAR model under objective and risk-neutral measure

In this section, we use different notation as previous section. Assume the stock process S_t under objective measure is given by

$$S_t = S_0 e^{X_t}, \quad (24)$$

$$dX_t = (\mu + \beta \sigma_t^2) dt + \sigma_t dW_t + \rho de_t, \quad (25)$$

$$\sigma_{t_i}^2 = b_i \sigma_{t_{i-1}}^2 + e_i, \sigma_0^2 > 0, \quad (26)$$

where W_t is Brownian motion, e_t is a gamma process with Levy density $v(dx) = \frac{ae^{-\alpha x}}{x} 1_{x>0} dx$ and $b_i \sim \text{Beta}(c - a\delta, \delta)$. We have

$$e^{-rt} S_t = S_0 e^{Y_t},$$

where

$$Y_t = \int_0^t (\mu - r + \beta \sigma_s^2) ds + \int_0^t \sigma_s dW_s + \rho e_t. \quad (27)$$

Under risk-neutral measure, the stock process S_t is given from (22), (23) and (20):

$$S_t = S_0 e^{X_t}, \quad (28)$$

$$dX_t = \left(r - l(\rho) - \frac{1}{2} \sigma_t^2 \right) dt + \sigma_t dW_t^Q + \rho d e_t^Q, \quad (29)$$

$$\sigma_{t_i}^2 = b_i^Q \sigma_{t_{i-1}}^2 + e_i^Q, \quad \sigma_0^2 > 0, \quad (30)$$

where W_t^Q is Brownian motion under risk neutral measure, e_t^Q is a Gamma process with Levy density $\tilde{v}(dx) = \frac{\tilde{a}e^{-\tilde{a}x}}{x} 1_{x>0} dx$ and $b_i \sim \text{Beta}(\tilde{c} - \tilde{a}\delta, \delta)$. We have

$$e^{-rt} S_t = S_0 e_t^Y,$$

where

$$Y_t = \int_0^t \left(-l(\rho) - \frac{1}{2} \sigma_s^2 \right) ds + \int_0^t \sigma_s dW_s^Q + \rho e_t^Q. \quad (31)$$

The main theorem we used to measure transformation is stated as below. The proof can be found in Sato (1999, Theorems 33.1 and 33.2).

Theorem 1. *Let (X_t, P) and (X_t, P') be two Levy processes on R with characteristic triplets $(\sigma_t^2, \nu, \gamma_t)$ and $(\sigma_t'^2, \nu', \gamma_t')$. Then P and P' are equivalent for all $t > 0$ if and only if all following conditions are satisfied:*

- (i) $\sigma_t = \sigma_t'$.
- (ii) *The Levy measure are equivalent with*

$$\int_{-\infty}^{\infty} (e^{\phi(x)/2} - 1)^2 \nu(dx) < \infty, \quad (32)$$

where $\phi(x) = \log(\frac{d\nu'}{d\nu})$. When P and P' are equivalent, the Radon–Nikodum derivative is

$$\frac{dP'}{dP} = e^{U_T}, \quad (33)$$

with

$$\begin{aligned} U_t = & \int_0^t \eta_s dX_s^c - \frac{\int_0^t \eta_s^2 \sigma_s^2 ds}{2} - \int_0^t \eta_s \gamma_s ds \\ & + \lim_{\epsilon \rightarrow 0} \left(\sum_{s \leq t, |\Delta X_s| > \epsilon} \phi(\Delta X_s) - t \int_{|x| > \epsilon} (e^{\phi(x)} - 1) \nu(dx) \right). \end{aligned} \quad (34)$$

Here (X_t^c) is the continuous part of (X_t) and η_t is such that

$$\gamma'_t - \gamma_t - \int_{-1}^1 x(v' - v)(dx) = \sigma_t^2 \eta_t. \quad (35)$$

Then we have following theorem of equivalent measures for SVBGAR model.

Theorem 2. Assume that $a = \tilde{a}$. Let

$$\eta_s = \frac{1}{\sigma_s^2} \left(\tilde{a} \log \left(1 - \frac{\rho}{\tilde{\alpha}} \right) - \mu + r - \left(\beta + \frac{1}{2} \right) \sigma_s^2 - \left(\frac{\tilde{a}}{\tilde{\alpha}} (1 - e^{-\tilde{\alpha}}) \right) - \frac{a}{\alpha} (1 - e^{-\alpha}) \right) \quad (36)$$

and

$$U_t = \int_0^t \eta_s \sigma_s dW_s - \frac{1}{2} \int_0^t \eta_s^2 \sigma_s^2 ds + \lim_{\epsilon \rightarrow 0} \left(\int_0^t \int_{\epsilon}^{\infty} (\alpha - \tilde{\alpha}) x J(dx, ds) - t \int_{\epsilon}^{\infty} \left(\frac{\tilde{a} e^{-\tilde{\alpha}x}}{x} - \frac{a e^{-\alpha x}}{x} \right) dx \right), \quad (37)$$

where $J(dx, ds)$ is the jump random measure of e_t . If we define $\frac{dQ}{dP} = e^{U_T}$, then Q is a risk-neutral measure which is equivalent with P , so the model is arbitrage free. But the solution of U_T is not unique, the model is not complete as other stochastic volatility models.

Proof. If $a = \tilde{a}$, then $\phi(x) = \log\left(\frac{dv'}{dv}\right) = (\alpha - \tilde{\alpha})x$. Since $v(dx) = 0$ and $\tilde{v}(dx) = 0$ for $x < 0$, the condition (32) in Theorem 1 is satisfied as

$$\int_{-\infty}^{\infty} (e^{\phi(x)/2} - 1)^2 v(dx) = a \int_0^{\infty} \frac{e^{-\tilde{\alpha}x/2} - e^{-\alpha x/2}}{x} dx < \infty,$$

since $\lim_{x \rightarrow 0} \frac{e^{-\tilde{\alpha}x/2} - e^{-\alpha x/2}}{x} = \frac{\alpha - \tilde{\alpha}}{2} < \infty$.

We use the drift $\gamma'_t = -l(\rho) - \frac{1}{2} \sigma_s^2$ and $\gamma_t = \mu - r + \beta \sigma_s^2$ and from (35) the η_s in (36) holds. Since $\sum_{s \leq t, |\Delta X_s| > \epsilon} \phi(\Delta X_s) = \int_0^t \int_{\epsilon}^{\infty} \phi(x) J(dx, ds) = \int_0^t \int_{\epsilon}^{\infty} (\alpha - \tilde{\alpha}) J(dx, ds)$, (37) is calculated directly from (34).

The difference of measure transformation between BNS and SVBGAR is that the Levy measure of z_t in BNS model is finite since it is a compound process but the Levy measure of e_t in SVBGAR model is infinite since it is a Gamma subordinator process. $\square$

4.2. European option pricing under SVBGAR model and numerical implementation

In this section, we assume the model is in the risk-neutral measure. To illustrate the empirical performance of SVBGAR models for derivatives pricing, we

Table 1. The parameter estimates of SBGAR model from call options.

a	c	α	σ_0^2	ρ
0.0985	2.362	8.347	0.014	-1.145

Fig. 1. Actural and predicted prices of the SVBGAR model.

consider a data set of 48 European call options for three different maturities on the IBM stock.

We followed a simple Monto Carlo procedure for constructing the option prices. We calibrated the SVBGAR model by minimizing the mean squared error (MSE) defined as the average squared difference between model-produced option prices and observed prices. We also compute the average absolute error as a percentage of the mean price:

$$\text{APE} = \frac{1}{\text{mean option price}} \sum_{\text{options}} \frac{|\text{market price} - \text{model price}|}{\text{number of options}}.$$

The fitted parameters are given in Table 1 and the observed and fitted option prices are illustrated in Fig. 1. The mean pricing error is 4.485%. The model seems to perform well.

5. Conclusion

We started a special Levy process as variance gamma model and BNS stochastic volatility model based on Gamma Ornstein–Uhlenbeck (Γ -OU) process. The log

returns in BNS model with Γ -OU process have approximately variance gamma distribution as well as volatility clustering effect. We provide a detailed analysis regarding the measure transformations and option pricing of BNS model.

A new SVBGAR model was formulated based on the relationship between gamma and beta variables. The log returns in SVBGAR model still have approximately variance gamma distribution as well as volatility clustering effect. Moreover, the simulation in SVBGAR is much simpler than BNS model. We expect this model can give more efficient way in pricing exotic options by Monte Carlo simulation. As is typical for the more advanced stochastic volatility models, the model is arbitrage free but incomplete, which means that there exists more than one equivalent martingale measure. The measure transformation between objective and risk-neutral measures were investigated in this paper.

At last, we researched the performance of SVBGAR model in option pricing. The parameters of SVBGAR model were calibrated from European option prices in market. The model prices were calculated based on the calibration. The comparison between market and model prices delivered promising performance in option pricing and we hope to provide additional empirical support in future work.

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