Superluminal Tunneling of a Relativistic Half-Integer Spin Particle Through a Potential Barrier

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This paper investigates the problem of a relativistic Dirac half-integer spin free particle tunneling through a rectangular quantum-mechanical barrier. If the energy difference between the barrier and the particle is positive, and the barrier width is large enough, there is proof that the tunneling may be superluminal. For first spinor components of particle and antiparticle states, the tunneling is always superluminal regardless the barrier width. Conversely, the second spinor components of particle and antiparticle states may be either subluminal or superluminal depending on the barrier width. These results derive from studying the tunneling time in terms of phase time. For the first spinor components of particle and antiparticle states, it is always negative while for the second spinor components of particle and antiparticle states, it is always positive, whatever the height and width of the barrier. In total, the tunneling time always remains positive for particle states while it becomes negative for antiparticle ones. Furthermore, the phase time tends to zero, increasing the potential barrier both for particle and antiparticle states. This agrees with the interpretation of quantum tunneling that the Heisenberg uncertainty principle provides. This study's results are innovative with respect to those available in the literature. Moreover, they show that the superluminal behaviour of particles occurs in those processes with high-energy confinement.

Keywords: Quantum tunneling; evanescent waves; Hartman effect; relativistic particles.

1. Introduction

Several theoretical and experimental studies in the past decades have examined phenomena involving superluminal waves and objects because of their implication in quantum and cosmological physics [1-9]. Among them, the study of the tunneling time problem is one of the topics that has most attracted the interest of quantum physicists [10-15]. Researchers have approached this issue both from the perspective of non-relativistic [16] and relativistic [17] quantum theory. In both cases the tunneling time does not depend on the barrier width (at least for large enough barriers), thus proving superluminal behaviour of the quantum object (wave or particle). However, the tunneling time problem remains a controversial one in quantum physics. A comprehensive and clear theory to explain how long does it take a particle to tunnel through a barrier still does not exist [18]. As is well known, classical quantum mechanics does not treat time as an Hermitian operator but rather as a parameter [19]. Time does not appear in the commutation relationships typical of the Hermitian operators, even if it appears in one of the forms of

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the Heisenberg uncertainty principle, being a physical variable conjugated to the energy. For this reason we have to give up directly knowing the tunneling time. We may bypass the obstacle by assuming that the wave packet inside the barrier is stationary, with an imaginary wave vector. We can then interpret the tunneling time as the phase variation of the evanescent stationary wave that crossing the potential barrier induces. This is the definition of tunneling phase time [20-21] and is the most widely used quantity in the studies of tunneling phenomena.

In this paper we investigate, by the study of the phase time, the one-dimensional scattering process of a relativistic half-integer spin free particle through a rectangular barrier. Because the motion takes place in one dimension, particle and antiparticle states are bi-spinor (there is no spin-flip in one-dimensional motion). Thus, we investigate the tunneling time for each of the spinor components. When the barrier height is greater than the particle energy, for the first component of particle and antiparticle states, we prove that the tunneling process always occurs at superluminal velocity (negative tunneling times). This transpires regardless of the width of the barrier and the energy gap between the barrier and the relativistic particle. Second spinor components of particle and antiparticle states, on the contrary, behave in a different way. In these, the scattering through the barrier may be subluminal or superluminal depending on the barrier width (positive tunneling time). In total, the tunneling time is almost always positive for particle states, while it becomes negative for antiparticle ones. Therefore, the Hartman effect always occurs for both particles and antiparticles. Furthermore, the study shows that when the energy gap increases, the tunneling time tends to zero for both particle and antiparticle states, and the crossing velocity of the barrier diverges to infinite values. Finally, we prove that superluminality is more accentuated for antiparticle states than particle states, even if this peculiarity disappears as the particle velocity increases.

2. The scattering model

Let us consider a half-integer spin relativistic free particle of rest mass m_0 , energy $\pm E$ (depending on its particle or antiparticle state) and velocity u , scattering through a potential barrier of width a and potential energy U . The potential energy is always greater than the particle energy ($U > \pm E$). In regions where the potential energy is zero, the particle moves freely and behaves like a relativistic monochromatic de Broglie wave plane [24], which spreads in space-time. To simplify the study we suppose a two-dimensional space-time, so that the plane wavefront impinging on the barrier has the two-component spinor form (there is no spin-flip):

$$\begin{cases} |\Psi\rangle_+ = N \left(\frac{1}{\frac{cp_x}{E+m_0c^2}} \right) \exp\{i(k_0ct - k_x x)\} = N \begin{pmatrix} 1 \\ \beta \end{pmatrix} \exp\{i(k_0ct - k_x x)\} \\ |\Psi\rangle_- = N \left(-\frac{\frac{cp_x}{E+m_0c^2}}{1} \right) \exp\{-i(k_0ct - k_x x)\} = N \begin{pmatrix} -\beta \\ 1 \end{pmatrix} \exp\{-i(k_0ct - k_x x)\} \end{cases} \quad (1)$$

In Eq. (1), β is the relativistic factor (u/c), while $|\Psi\rangle_+$ and $|\Psi\rangle_-$ are respectively the particle and antiparticle spinors. N is the normalization constant that, for a given finite

volume, $\sqrt{(\gamma + 1)/2\gamma V}$ provides, where γ is the Lorentz factor. Wave vectors are $k_0 = p_0/\hbar$ and $k_x = p_x/\hbar$.

At the beginning of the barrier ($x = x_0$), the de Broglie wave may be reflected or transmitted. In the latter case, the wave behaves like an evanescent stationary wave [16]. Since the wave is confined within the barrier (evanescent waves are not propagating waves), its wave vector has only a spatial component $\chi_+ = \sqrt{2m_0(U - E)}/\hbar$ for particle and $\chi_- = \sqrt{2m_0(U + E)}/\hbar$ for antiparticle.

At the end of the barrier, the evanescent wave is transmitted and returns to be a free relativistic particle with half-integer spin. Since the wave function (1) is a two-component vector, we must study the scattering process for both of them. At the barrier edges, each component of the Dirac spinor has to match the continuity conditions that quantum theory imposes.

We will investigate the tunneling time by the phase time (which is asymptotic and non-local) defined as [14]:

$$\tau = \hbar \frac{\partial \arg\{C_T \exp[i(k_0 c t_a - k_x x_a)]\}}{\partial E} \quad (2)$$

Therefore, the study object of this research preliminarily requires the calculation of the transmission amplitude C_T for the spinor component of the particle and antiparticle states.

3. Transmission and reflection amplitudes

Let us calculate the transmission and reflection amplitudes of the scattering process that we will use for tunneling time investigation. Considering initially the first component of the particle spinor (1), the continuity conditions are:

$$\begin{cases} \Psi(I)|_{x_0} = \varphi(II)|_{x_0} \\ \Psi(III)|_{x_0+a} = \varphi(II)|_{x_0+a} \\ \Psi'(I)|_{x_0} = \varphi'(II)|_{x_0} \\ \Psi'(III)|_{x_0+a} = \varphi'(II)|_{x_0+a} \end{cases} \quad (3)$$

where:

$$\begin{cases} \Psi(I) = \exp\{i(k_0 c t - k_x x)\} + c_R \exp\{-i(k_0 c t - k_x x)\} \\ \varphi(II) = \alpha e^{\chi x} + \delta e^{-\chi x} \\ \Psi(III) = c_T \exp\{i(k_0 c t - k_x x)\} \end{cases} \quad (4)$$

c_R and c_T are respectively the reflection and the transmission coefficients of the particle wave, while α and δ are the coefficients of the evanescent wave. Substituting functions (4) in (3) and solving the system by the Cramer algorithm, we get:

$$c_T = \frac{4i\chi\Delta k}{\exp\{i(k_0 c t_a - k_x x_a)\}[e^{\chi x_a}(\chi - i\Delta k)^2 + e^{-\chi x_a}(\Delta k - i\chi)^2]} \quad (5)$$

$$C_R = \frac{\exp\{2i(k_0 c t_0 - k_x x_0)\} \exp\{i(k_0 c t_a - k_x x_a)\}}{4(\chi^2 + \Delta k^2) \sinh^2(\chi x_a)} [-2 \sinh(\chi x_a)(\chi^2 - \Delta k^2) - 4i \exp\{i(k_0 c t_0 - k_x x_0)\} \exp\{i(k_0 c t_a - k_x x_a)\} \cosh(\chi x_a)] \quad (6)$$

where $\Delta k = (k_0 - k_1) = \gamma m_0(c - u)/\hbar$. For antiparticle states, Eq. (5) provides the transmission amplitude, replacing Δk with $-\Delta k$ and using the evanescent wave vector χ_- . Through some algebraic manipulations, we use the explicit forms of k_0 , k_x , χ_+ , χ_- wave vectors and set $m_0 \equiv c \equiv \hbar = 1$. Eq. (5) thus provides the probability transmission of the particle:

$$C_T^2 = \frac{\gamma^2(1-u)^2(U-E)}{[\gamma^2(1-u)^2 + 2(U-E)] \sinh^2(x_a \sqrt{2(U-E)})} \quad (7)$$

When increasing the barrier height, i.e., increasing the potential energy U , the probability that the particle is transmitted through the barrier rapidly goes to zero, while if $U = E$ this probability tends to one. These limits also hold for the antiparticle state, but for $U \rightarrow \infty$ the probability tends to zero more rapidly. The reflection probability C_R^2 has an opposite trend to the transmitted one, which the quantum relation $(C_T^2 + C_R^2) = 1$ has to satisfy.

4. Tunneling time for particle states

As mentioned earlier, the phase time (2) helps perform the investigation of the tunneling time. The first step for its calculation is to determine the argument of the complex number $C_T \exp[i(k_0 c t_a - k_x x_a)]$. To simplify the formalism, from now on the Greek letter θ will represent this. Through some algebraic calculations, we get:

$$\theta = \text{arctg} \left[-2\chi \Delta k \frac{1}{\Delta k^2 + \chi^2 \frac{e^{2\chi x_a} - 1}{e^{2\chi x_a} + 1}} \right] \quad (8)$$

Assuming that the barrier width is large enough, we may set the term $(e^{2\chi x_a} - 1)/e^{2\chi x_a} + 1$ equal to one, so that the argument function becomes:

$$\theta = \text{arctg} \left[-\frac{2\chi \Delta k}{\Delta k^2 + \chi^2} \right] \quad (9)$$

Performing its derivative respect for the particle energy and substituting the result in Eq. (2), we get the phase time:

$$\tau^{(1^o)} = \frac{2m_0 \Delta k E (1 - 2\chi^2) + 2\hbar^2 \chi^2 \Delta k (2\Delta k^2 - 1)}{\hbar \chi E [(\Delta k^2 + \chi^2)^2 + 4\Delta k^2 \chi^2]} \quad (10)$$

where the superscript (1^o) means that the phase time refers the first component of the Dirac spinor. Eq. (10) is independent from the barrier width a , and as expected, the Hartman effect [11] occurs. Moreover, the value of the phase time is always negative whatever the particle energy and the barrier height. According to references [25-26], this means the tunneling always occurs at superluminal velocities. When increasing the barrier height, the phase time goes rapidly to zero. This result is in agreement with the superluminality that the Heisenberg uncertainty principle predicts. The ratio between the

barrier width and the tunneling time provides the barrier-crossing speed, at least in principle. From the obtained results, we thus conclude that the higher the barrier is, the faster the scattering through it is. This phenomenon occurs as if the potential barrier boosts the particle towards higher velocities, even if its occurrence probability (that the squared transmission amplitude provides) goes to zero as the barrier height decreases.

Let us calculate the phase time for the second component of the spinor (1) for the particle state following the same physico-mathematical approach used for the first component:

$$\tau^{(2^{\circ})} = \hbar \frac{2(1-\beta^2)(\Delta k^2 + \chi^2)}{E[(\Delta k^2 + \chi^2)^2 + 4\beta^4 \Delta k^2 \chi^2]} + \tau^{(1^{\circ})} \quad (11)$$

This shows that the two components of the spinor emerge from the far side of the barrier with a phase time difference given by:

$$\Delta\tau = \hbar \frac{2(1-\beta^2)(\Delta k^2 + \chi^2)}{E[(\Delta k^2 + \chi^2)^2 + 4\beta^4 \Delta k^2 \chi^2]} \quad (12)$$

This quantity, which from now we will call phase time delay, is always positive. This delay is considerable at non-relativistic regimes ($u \ll c$), at which the magnitude of the secondary component of spinor (1) is very low, and tends to zero when the particle velocity approaches the speed of light. For this reason, whatever the barrier width, the secondary spinor component always emerges later than the principal one. This phenomenon leads to an anomalous distortion of the Dirac wave function that disappears when the relativistic behaviour of the particle becomes relevant. Overall, the phase time (11) is always positive, except when the particle velocity approaches the speed of light.

5. Tunneling time for antiparticle states

Let us calculate the tunneling time for the antiparticle states following the same approach used in the previous section. The negative values of energy E and the relativistic wave number Δk represent the only difference. Eq. (10) can then provide the phase time by replacing E with $-E$, Δk with $-\Delta k$ and using the second variant of Eq. (3):

$$\tau_{-}^{(1^{\circ})} = - \frac{2m_0 \Delta k E (2\chi^2 - 1) + 2\hbar^2 \chi^2 \Delta k (1 - 2\Delta k^2)}{\hbar \chi E [(\Delta k^2 + \chi^2)^2 + 4\Delta k^2 \chi^2]} \quad (13)$$

The negative sign of Δk is due to the fact that the sign of the argument of the exponential function for the antiparticle state is opposite to that of the particle state. Eq. (13) does not depend on the barrier width, and the Hartman effect occurs.

Unlike the first spinor component of the particle state, the phase time is always positive, and the tunneling through the barrier could be subluminal or superluminal depending on the barrier width. This is expected behaviour considering that the first spinor component of the antiparticle is the opposite of the second spinor component of the particle. After setting the barrier height, the phase time tends to zero as the velocity approaches the speed of light, just like what occurs for the particle state.

Similarly, Eq. (12) provides the phase time difference between the two components of the antiparticle spinor, replacing E with $-E$ and using the second part of Eq. (3). Once again unexpectedly, the secondary component of the antiparticle spinor emerges from the

far side of the barrier before the principal one. This always leads to an anomalous distortion of the Dirac antiparticle spinor that, however, is opposite to the one that occurs for the particle state. This distortion disappears as the antiparticle velocity approaches the speed of light but is slower than what occurs for the particle spinor.

In total, the antiparticle phase time given by the contribution of the two spinor components is always negative and turns out to be positive only when the particle velocity approaches the speed of light.

6. Conclusion

Summarizing, we have considered tunneling of a relativistic free particle with half-integer spin through a one-dimensional potential barrier. The analysis suggests that the total transmission is almost always superluminal for antiparticle states and may be subluminal or superluminal for particle states. Within the potential barrier, the two components of the spinor behave as evanescent modes, and the high energy confinement accelerates the particle/antiparticle further towards the speed of light. Considering the interference between forward and backward evanescent waves may explain the phenomenon, but an easier explanation holds if we account for the Heisenberg uncertainty principle. As the particle/antiparticle velocity approaches the speed of light, the tunneling time tends to zero, making the acceleration divergent towards infinite values. The study also proves that the two spinor-transmitted components emerge from the potential barrier at different times, providing an anomalous distortion of the Dirac wave function. This distortion occurs in an opposite way for particles and antiparticles. More precisely, the secondary spinor component is delayed compared with the first one for particle states and is anticipated for antiparticle states. All of these results are innovative with respect to those available in the literature. They suggest a way to obtain a source of superluminal massive elementary particles: ultra-relativistic particles scattering through huge potential barriers. As discussed above, the probability of total transmission of the particle/antiparticle through the barrier strongly decreases as its incoming velocity approaches the speed of light. However, we cannot exclude the occurrence of the model used in this study a priori in a cosmological scenario.

References

1. A.M. Steinberg, P.G. Kwiat, R.Y. Chiao, Measurement of the single-photon tunneling time, *Phys. Rev. Lett.*, **71**, 708 (1993).
2. C. Spielmann, R. Szipöcs, A. Stingl, F. Krausz, Tunneling of optical pulses through photonic band gaps, *Phys. Rev. Lett.*, **73**, 2308 (1994).
3. E. Recami, Superluminal waves and objects: an overview of the relevant experiments, *J. of Phys: Conference Series*, **196**, 012020 (2009).
4. R. Ehrlich, Tachyonic neutrinos and the neutrinos masses, *Astropart. Phys.*, **41**, 1 (2013).
5. R.K. Pradhan, L.P. Singh, On reality of tachyon de Broglie waves, (2009) <https://arxiv.org/abs/0908.1365>.
6. I.F. Mirabel, L.F. Rodríguez, A superluminal source in the galaxy, *Nature*, **371**, 46 (1994).
7. O. Smirnova, et al., Interpreting attoclock measurement of tunneling time, *Nature Phys.*, **11**, 503 (2015).

8. N. Camus, et al., Experimental evidence for quantum tunneling time, *Phys. Rev. Lett.*, **119**, 023201 (2017).
9. S. Haessler, et al., Attosecond imaging of molecular electronic wavepackets, *Nature Phys.*, **6**, 200 (2010).
10. R. Landauer, T. Martin, Barrier interaction time in tunneling, *Rev. Modern Phys.*, **66**, 217 (1994).
11. P.C.W. Davies, Quantum tunneling time, *Am. J. Phys.*, **73**, 23 (2005).
12. N. Chuprikov, New Approach to quantum tunneling process: characteristic times for transmission and reflection, *Russian Phys. J.*, **49** (3), 314 (2006).
13. E.H. Hauge, J.A. Stovneng, Tunneling times: a critical review, *Rev. Mod. Phys.*, **61**, 917 (1989).
14. T.E. Hartman, Tunneling of a wave packet, *J. Appl. Phys.*, **33** (12), 3427 (1962).
15. H.G. Winful, Tunneling time, the Hartman effect and superluminality: a proposed resolution of an old paradox, *Phys. Rep.*, **436** (1-2), 1 (2011).
16. E. Recami, Superluminal tunneling through successive barriers: does quantum mechanics predict infinite group velocities?, *J. Mod. Optics*, **51** (6-7), 913 (2004).
17. V. Petrillo, D. Janner, Relativistic analysis of a wave packet interacting with a quantum-mechanical barrier, *Phys. Rev. A*, **67**, 012110 (2003).
18. N. Chuprikov, Tunneling time problem: at the intersection of quantum mechanics, classical probability theory and special relativity, (2014) <https://arxiv.org/abs/1303.6181>.
19. G. Muga, A. Ruschhaupt, A. del Campo, *Time in quantum mechanics*, Vol. 2, (Springer, Berlin, 2009).
20. S. Bandopadhyay, A.M. Jayannavar, Phase time for a tunneling particle, (2005) <https://arxiv.org/abs/quant-ph/0511089>.
21. E.P. Wigner, Lower limit for the energy derivative of the scattering phase shift, *Phys. Rev. B*, **98**, 145 (1955).
22. E. Merzbacher, The early history of quantum tunneling, *Phys. Today*, **55** (8), 44 (2002).
23. L. Nanni, Quantum theory of half-integer spin free particles from the perspective of the Majorana equation, (2016) <https://arxiv.org/abs/1603.05965>.
24. S.T. Park, Propagation of a relativistic electron wave packet in the Dirac equation, *Phys. Rev. A*, **86**, 1 (2012).
25. O. Rosas-Ortiz, S.C. Cruz, N. Fernández-García, Negative time delay for wave reflection from a one-dimensional semi-harmonic well, (2012) <https://arxiv.org/abs/1208.4317>.
26. A.E. Bernardini, Relativistic tunneling and accelerated transmission, (2008) <https://arxiv.org/pdf/0706.3930>.