Low Energy KN Interaction in the Fock-Tani Formalism

Bruna Cesira Folador and Dimiter Hadjimichef

Instituto de Física

Universidade Federal do Rio Grande do Sul

Av. Bento Gonçalves, 9500

Porto Alegre, Rio Grande do Sul, 91501-970, Brazil

bru.folador@gmail.com, dimihadj@gmail.com

Published 15 August 2017

The Fock-Tani formalism is a first principle method to obtain effective interactions from microscopic Hamiltonian. Originally derived for meson-meson or baryon-baryon scattering, we present the corresponding equations for meson-baryon scattering. In particular we shall obtain the low energy cross sections for the $K^- + p \rightarrow \Lambda + \eta$ channel.

Keywords: Fock-Tani Formalism, Non-Relativistic Quark Model, Kaon-Baryon Interaction.

PACS numbers: 12.39.Jh; 12.39.Pn; 13.75.Jz

1. Fock-Tani Formalism

There are many examples of systems in which the internal degrees of freedom of composite particles can not be neglected. For these systems the second quantization formalism is a very difficult mathematical tool to be used. Alternatively, one can map the physical Hilbert space into an ideal Hilbert space, where the bound-states are represented by “ideal particles” in which the elementary creation and annihilation operators, obey canonical commutation rules. The information about its internal structure is transferred to an effective interaction Hamiltonian^{1–3}. This formalism is called the Fock-Tani formalism.

The starting point of the Fock-Tani formalism is the definition of single composite bound states. We write a single-meson state in terms of a meson creation operator as $|\alpha\rangle = M_\alpha^\dagger |0\rangle$, where $|0\rangle$ is the vacuum state. The meson creation operator is written in terms of constituent quark and antiquark creation operators $q^\dagger$

and $\bar{q}^\dagger$,

$$M_\alpha^\dagger = \Phi_\alpha^{\mu\nu} q_\mu^\dagger \bar{q}_\nu^\dagger; \quad (1)$$

here $\Phi_\alpha^{\mu\nu}$ is the meson wave function and $q_\mu |0\rangle = \bar{q}_\nu |0\rangle = 0$. The index α identifies the meson quantum numbers of space, spin and isospin. The indices μ and ν denote the spatial, spin, flavor, and color quantum numbers of the constituent quarks. A sum over repeated indices is implied.

In a similar way, the baryon state can be defined as $|\alpha\rangle = B_\alpha^\dagger |0\rangle$, where the baryon creation operator is

$$B_\alpha^\dagger = \frac{1}{\sqrt{6}} \Psi_\alpha^{\mu_1\mu_2\mu_3} q_{\mu_1}^\dagger q_{\mu_2}^\dagger q_{\mu_3}^\dagger; \quad (2)$$

$\Psi_\alpha^{\mu_1\mu_2\mu_3}$ is the baryon wave function, where the index α identifies the quantum numbers of the baryon, and μ those of the quarks.

The Fock-space amplitude is taken to be orthonormalized,

$$\Phi_\alpha^{*\mu\nu} \Phi_\beta^{\mu\nu} = \delta_{\alpha\beta} \quad ; \quad \Psi_\alpha^{*\mu_1\mu_2\mu_3} \Psi_\beta^{\mu_1\mu_2\mu_3} = \delta_{\alpha\beta}. \quad (3)$$

The quark and antiquark operators satisfy canonical anti-commutation relations,

$$\{q_\mu, q_\nu^\dagger\} = \{\bar{q}_\mu, \bar{q}_\nu^\dagger\} = \delta_{\mu\nu},$$

$$\{q_\mu, q_\nu\} = \{\bar{q}_\mu, \bar{q}_\nu\} = \{q_\mu, \bar{q}_\nu\} = \{q_\mu, \bar{q}_\nu^\dagger\} = 0. \quad (4)$$

Using these quark anti-commutation relations, and the normalization condition of Eq. (3), it is easily shown that the operators satisfy the following non-canonical (anti-) commutation relations

$$[M_\alpha, M_\beta^\dagger] = \delta_{\alpha\beta} - \Delta_{\alpha\beta}^M,$$

$$\{B_\alpha, B_\beta^\dagger\} = \delta_{\alpha\beta} - \Delta_{\alpha\beta}^B, \quad (5)$$

where

$$\Delta_{\alpha\beta}^M = \Phi_\alpha^{*\mu\nu} \Phi_\beta^{\mu\sigma} \bar{q}_\sigma^\dagger \bar{q}_\nu + \Phi_\alpha^{*\mu\nu} \Phi_\beta^{\rho\nu} q_\rho^\dagger q_\mu,$$

$$\Delta_{\alpha\beta}^B = 3\Psi_\alpha^{*\mu_1\mu_2\mu_3} \Psi_\beta^{\mu_1\mu_2\nu_3} q_{\nu_3}^\dagger q_{\mu_3} - \frac{3}{2} \Psi_\alpha^{*\mu_1\mu_2\mu_3} \Psi_\beta^{\mu_1\nu_2\nu_3} q_{\nu_3}^\dagger q_{\nu_2}^\dagger q_{\mu_2} q_{\mu_3}. \quad (6)$$

The presence of the operator $\Delta_{\alpha\beta}$ in Eq. (1) is due to the composite nature of the hadrons. This term enormously complicates the mathematical description of processes that involve the hadron and quark degrees of freedom. The operators M_α , $M_\alpha^\dagger$ and B_α , $B_\alpha^\dagger$ are not convenient dynamical variables to be used.

A transformation is defined such that a single-hadron state $|\alpha\rangle$ is redescribed by an (“ideal”) elementary-hadron state by

$$U_M^{-1} M_\alpha^\dagger |0\rangle = m_\alpha^\dagger |0\rangle \quad ; \quad U_B^{-1} B_\alpha^\dagger |0\rangle = b_\alpha^\dagger |0\rangle, \quad (7)$$

where $m_\alpha^\dagger$ and $b_\alpha^\dagger$ are the ideal meson/baryon creation operators. The ideal particle operators satisfy, by definition, canonical (anti-) commutation relations

$$\begin{aligned} [m_\alpha, m_\beta^\dagger] &= \delta_{\alpha\beta} \quad ; \quad [m_\alpha, m_\beta] = 0, \\ \{b_\alpha, b_\beta^\dagger\} &= \delta_{\alpha\beta} \quad ; \quad \{b_\alpha, b_\beta\} = 0. \end{aligned} \quad (8)$$

In addition, in the new representation the quark and antiquark operators $q^\dagger$, q , $\bar{q}^\dagger$ and $\bar{q}$ are kinematically independent of $m_\alpha^\dagger$, m_α , $b_\alpha^\dagger$ and b_α

$$\begin{aligned} [q_\mu, m_\alpha] &= [q_\mu, m_\alpha^\dagger] = [\bar{q}_\mu, m_\alpha] = [\bar{q}_\mu, m_\alpha^\dagger] = 0, \\ \{q_\mu, b_\alpha\} &= \{q_\mu, b_\alpha^\dagger\} = 0. \end{aligned} \quad (9)$$

The unitary operators in (7) are defined by

$$U_M(t) = \exp[t F_M] \quad ; \quad U_B(t) = \exp[t F_B], \quad (10)$$

where F_M is the generator of the transformation and t a parameter which is set to $-\pi/2$ to implement the mapping. The generators F_M and F_B of the transformations are

$$F_M = m_\alpha^\dagger \tilde{M}_\alpha - \tilde{M}_\alpha^\dagger m_\alpha \quad ; \quad F_B = b_\alpha^\dagger \tilde{B}_\alpha - \tilde{B}_\alpha^\dagger b_\alpha, \quad (11)$$

where

$$\tilde{M}_\alpha = \sum_{i=0}^n \tilde{M}_\alpha^{(i)}, \quad ; \quad \tilde{B}_\alpha = \sum_{i=0}^n \tilde{B}_\alpha^{(i)}. \quad (12)$$

We need a transformation such that a single-hadron state $|\alpha\rangle$ is redescribed by an (“ideal”) elementary-hadron state. In the present calculation we obtain a mixed meson-baryon transformation defined by

$$O_\alpha(t, t') = U_B^{-1} U_M^{-1} O_\alpha U_M U_B. \quad (13)$$

Once a microscopic interaction Hamiltonian H is defined, at the quark level, a new transformed Hamiltonian can be obtained. The transformed Fock-Tani Hamiltonian is a result of the application of the unitary transformation on the microscopic Hamiltonian

$$H_{\text{FT}} = U_B^{-1} U_M^{-1} H U_M U_B. \quad (14)$$

The transformed Hamiltonian H_{FT} describes all possible processes involving mesons, baryons and quarks. After the applying the Fock-Tani transformation we obtain the

following general meson-baryon potential

$$V_{\text{mb}}(\alpha\beta; \delta\gamma) = \sum_{i=1}^4 V_i(\alpha\beta; \delta\gamma) m_{\alpha}^{\dagger} b_{\beta}^{\dagger} m_{\gamma} b_{\delta}, \quad (15)$$

where

$$V_1(\alpha\beta; \delta\gamma) = -3V_{qq}(\mu\nu; \sigma\rho) \Phi_{\alpha}^{*\mu\nu_2} \Psi_{\beta}^{*\nu\mu_2\mu_3} \Phi_{\gamma}^{\rho\nu_2} \Psi_{\delta}^{\sigma\mu_2\mu_3};$$

$$V_2(\alpha\beta; \delta\gamma) = -3V_{q\bar{q}}(\mu\nu; \sigma\rho) \Phi_{\alpha}^{*\mu_1\nu} \Psi_{\beta}^{*\mu\mu_2\mu_3} \Phi_{\gamma}^{\sigma\rho} \Psi_{\delta}^{\mu_1\mu_2\mu_3};$$

$$V_3(\alpha\beta; \delta\gamma) = -3V_{qq}(\mu\nu; \sigma\rho) \Phi_{\alpha}^{*\mu\nu_2} \Psi_{\beta}^{*\mu_1\nu\mu_3} \Phi_{\gamma}^{\mu_1\nu_2} \Psi_{\delta}^{\sigma\rho\mu_3};$$

$$V_4(\alpha\beta; \delta\gamma) = -6V_{q\bar{q}}(\mu\nu; \sigma\rho) \Phi_{\alpha}^{*\nu_1\nu} \Psi_{\beta}^{*\mu_1\mu\mu_3} \Phi_{\gamma}^{\mu_1\rho} \Psi_{\delta}^{\nu_1\sigma\mu_3}.$$

Fig. 1. Scattering diagrams V_1 , V_2 , V_3 and V_4 respectively.

2. Differential Cross-Section

The interaction potential between quarks is

$$V_{qq} = \sum_{ij} \left[\frac{\alpha_s}{r_{ij}} - \frac{3}{4} b r_{ij} - \frac{8\alpha_s}{3\sqrt{\pi} m_i m_j} \sigma_1^3 e^{-\sigma_1^2 r_{ij}^2} (\vec{S}_i \cdot \vec{S}_j) - \frac{\pi\alpha_s}{2} \sigma_2^3 e^{-\sigma_2^2 r_{ij}^2} \left(\frac{1}{m_i^2} - \frac{1}{m_j^2} \right) \right] (\vec{T}_i \cdot \vec{T}_j). \quad (16)$$

In the Born approximation, the matrix element $\langle \alpha\beta | V_{\text{mb}} | \gamma\delta \rangle$ between ideal states results in the following S-matrix

$$S_{fi} = \delta_{fi} - 2\pi i \delta(E_f - E_i) V_{\text{mb}}(\alpha\beta; \delta\gamma). \quad (17)$$

Due to translational invariance, one can write V_{mb} in (17) as

$$V_{\text{mb}}(\alpha\beta; \delta\gamma) = \delta(P_f - P_i) h_{fi}, \quad (18)$$

and the scattering amplitude h_{fi} is defined by

$$h_{fi} = \sum_{k=1}^4 V_k. \quad (19)$$

The diagrams V_k can be seen in figure 1 and can be written as

$$V_k = \omega_k I_k^e, \quad (20)$$

where I_k^e are the spatial integrals and ω_k the spin-flavor-color part. These amplitudes can be related to the cross section in the center of mass system ⁴

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \frac{|\vec{p}_f|}{|\vec{P}_k|} |\mathcal{M}|^2, \quad (21)$$

where $\vec{P}_k$ and $\vec{p}_f$ are the initial and final state momenta. The invariant amplitude $\mathcal{M}$ is related to h_{fi} ⁵

$$\mathcal{M} = 4(2\pi)^3 \sqrt{E_A E_B E_C E_D} h_{fi}. \quad (22)$$

Using (22) in (21) in the laboratory system, one obtains

$$\frac{d\sigma}{d\Omega} = \frac{16\pi^2 E_{K^-} E_p E_\Lambda E_\eta}{(E_{K^-} + E_p)(E_\Lambda + E_\eta)} \frac{|\vec{p}_f|}{|\vec{P}_k|} |h_{fi}|^2, \quad (23)$$

where E_k are the energy particles and $m_{K^-} = 0.493$ GeV, $m_p = 0.938$ GeV, $m_\eta = 0.547$ GeV, $m_\Lambda = 1.11$ GeV.

3. Results

Using Eq.(21) with $\sigma_1 = 3.5$ GeV, $\sigma_2 = 0.2$ GeV, $\alpha_s = 0.7$ GeV, $\alpha_\lambda = 0.6$ GeV, $x = 0.7$, $\beta = 0.7$ GeV and $b = 0.2$ GeV², we have found

Fig. 2. Differential cross sections of the $K^- + p \rightarrow \Lambda + \eta$ compared with the data from Ref. ⁶. The y-axis units are mb/sr.

The solid bold curves are for the model calculations and the dots are the data results. The overall fit is relatively close to experimental model even in the naive version of the model where the annihilation channels are not including. Our next step to improve our results is calculate the total cross-section and the differential cross sections of the $K^- + p \rightarrow \Lambda + \eta$ including annihilation. We also want to calculate DN and D^*N cross-sections.

Acknowledgments

This work was supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) and Universidade Federal do Rio Grande do Sul (UFRGS).

Appendix A. Wave-Functions

The general meson wave-function is

$$\Phi_{\alpha}^{\mu\nu} = \delta(\vec{p}_{\alpha} - \vec{p}_{\mu} - \vec{p}_{\nu}) \xi_{\alpha}^{f_{\mu} f_{\nu}} \Upsilon^{c_{\mu} c_{\nu}} \varphi(\vec{p}_{\mu}, \vec{p}_{\nu}), \quad (\text{A.1})$$

where $\xi_{\alpha}^{f_{\mu} f_{\nu}}$ is the spin-flavor part; $\Upsilon^{c_{\mu} c_{\nu}}$ is the color part given by

$$\Upsilon^{c_{\mu} c_{\nu}} = \frac{\delta^{c_{\mu} c_{\nu}}}{\sqrt{3}}, \quad (\text{A.2})$$

and $\varphi(\vec{p}_{\mu}, \vec{p}_{\nu})$ is the spatial contribution, which is chosen

$$\varphi(\vec{p}_q, \vec{p}_{\bar{q}}) = (\pi\beta^2)^{-\frac{3}{4}} \exp \left[-\frac{(m_1 \vec{p}_q - m_2 \vec{p}_{\bar{q}})^2}{8\beta^2} \right], \quad (\text{A.3})$$

with

$$m_1 = \frac{2m_{\bar{q}}}{m_q + m_{\bar{q}}}; \quad m_2 = \frac{2m_q}{m_q + m_{\bar{q}}}. \quad (\text{A.4})$$

The baryon wave-function has a similar structure

$$\Psi_{\alpha}^{\mu_1 \mu_2 \mu_3} = \delta \left(\vec{P}_{\alpha} - \sum_{i=1}^3 \vec{p}_i \right) \xi_{\alpha}^{f_1 f_2 f_3} \Upsilon^{c_1 c_2 c_3} \Psi_{\vec{P}_{\alpha}}^{\vec{p}_1^{\mu_1} \vec{p}_2^{\mu_2} \vec{p}_3^{\mu_3}}, \quad (\text{A.5})$$

where $\xi_{\alpha}^{f_1 f_2 f_3}$ is the spin-flavor part; $\Upsilon^{c_1 c_2 c_3}$ is the color part given by

$$\Upsilon^{c_1 c_2 c_3} = \frac{\varepsilon^{c_1 c_2 c_3}}{\sqrt{6}}. \quad (\text{A.6})$$

Thus the wave-function is

$$\Psi_{\vec{P}}^{\vec{p}_1 \vec{p}_2 \vec{p}_3} = \delta(\vec{P} - \vec{p}_1 - \vec{p}_2 - \vec{p}_3) \left(\frac{\sqrt{3}}{\pi \alpha_\Lambda \alpha_\rho} \right)^{3/2} \times \exp \left[-\frac{\alpha_\Lambda^2 (\vec{p}_1 - \vec{p}_2)^2 + 3\alpha_\rho^2 [M_2 \vec{p}_1 + M_2 \vec{p}_2 - 2M_1 \vec{p}_3]^2}{4\alpha_\Lambda^2 \alpha_\rho^2} \right], \quad (\text{A.7})$$

or equivalently defining $x = \alpha_\rho / \alpha_\Lambda$

$$\Psi_{\vec{P}}^{\vec{p}_1 \vec{p}_2 \vec{p}_3} = \delta(\vec{P} - \vec{p}_1 - \vec{p}_2 - \vec{p}_3) \left(\frac{\sqrt{3}}{\pi x \alpha_\Lambda^2} \right)^{3/2} \times \exp \left[-\frac{(\vec{p}_1 - \vec{p}_2)^2 + 3x^2 [M_2 \vec{p}_1 + M_2 \vec{p}_2 - 2M_1 \vec{p}_3]^2}{4\alpha_\Lambda^2 x^2} \right]. \quad (\text{A.8})$$

A simplification that can be introduced is to consider $x = 1$,

$$\alpha_\lambda = \alpha_\rho \equiv \alpha. \quad (\text{A.9})$$

On that way the wave-function (A.8) becomes

$$\Psi_{\vec{P}}^{\vec{p}_1 \vec{p}_2 \vec{p}_3} = \delta(\vec{P} - \vec{p}_1 - \vec{p}_2 - \vec{p}_3) \left(\frac{\sqrt{3}}{\pi \alpha^2} \right)^{3/2} \times \exp \left[-\frac{(\vec{p}_1 - \vec{p}_2)^2 + 3 [M_2 \vec{p}_1 + M_2 \vec{p}_2 - 2M_1 \vec{p}_3]^2}{4\alpha^2} \right]. \quad (\text{A.10})$$

On the nucleons particular case,

$$m_l = m_p \rightarrow M_1 = M_2 = \frac{1}{3}, \quad (\text{A.11})$$

the wave-function (A.10) reduces to the well-known result of the Quark Model

$$\Psi_{\vec{P}}^{\vec{p}_1 \vec{p}_2 \vec{p}_3} = \delta(\vec{P} - \vec{p}_1 - \vec{p}_2 - \vec{p}_3) \left(\frac{\sqrt{3}}{\pi \alpha^2} \right)^{3/2} \times \exp \left[\frac{-p_1^2 - p_2^2 - p_3^2 + \vec{p}_1 \cdot \vec{p}_2 + \vec{p}_1 \cdot \vec{p}_3 + \vec{p}_2 \cdot \vec{p}_3}{3\alpha^2} \right]. \quad (\text{A.12})$$

References

1. D. Hadjimichef, G. Krein, S. Szpigel, and J. S. da Veiga, *Ann. of Phys.* **268** n.1, 105 (1998).
2. D. Hadjimichef, J. Haidenbauer, and G. Krein, *Phys.Rev.* **C63**, 035204 (2001); e-Print Archive: nucl-th/0010044(2000).
3. D. Hadjimichef, J. Haidenbauer, and G. Krein, *Phys.Rev.* **C66**, 055214 (2002).
4. K. A. Olive et al. (Particle Data Group), *Chin. Phys. C* **38**, 090001 (2014).
5. T. Barnes and E. S. Swanson, *Phys. Rev. D* **46**, 131 (1992).
6. A. Starostin et al, *Phys. Rev.* **C64**, 055205 (2001).
7. L. Y. Xiao and X. H. Zhong, *Phys. Rev.* **C88**, 065201 (2013).