Quantum Faraday Effect in the Universe

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We study the Quantum Faraday rotation starting from the photon self-energy in the presence of a constant magnetic field. The Faraday angle is calculated in the non-degenerate regime and for weak field limit. Two physical scenarios, possibly characterized by these conditions, are the recombination epoch and the jets originated in pulsars. We discuss

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the resonant behavior that the Faraday angle exhibits in these scenarios and investigate the possibility of detecting cosmic magnetic fields through this resonant mechanism.

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1. Introduction

Faraday effect ¹ is the well known phenomenon where plane polarized light penetrating in a magnetized transparent charged medium and moving parallel to the magnetic field $\mathbf{B}$ rotates its plane of polarization as a consequence of birefringence: the incoming wave splits in two opposite circularly polarized modes moving with different speeds (and frequencies), and the polarization vector rotates.

Faraday Rotation (FR) is a particular case of the general problem of photon propagation in a charged medium ². Classical and semiclassical approaches can be applied successfully ¹ in non-relativistic media. However, as was pointed out in Refs. ^{3,4}, FR effects can be derived from a quantum-relativistic approach which is more appropriate to describe electromagnetic waves under certain extreme conditions that can be present in the Universe.

In Refs. ^{3,4} we studied the relation between Faraday angle and Hall conductivity, showing their quantized feature. It was done on the basis of the detailed study of general properties of the photon self-energy and the dispersion equations for photons propagating in the medium, parallel and perpendicular to the magnetic field, considering that the photon self-energy satisfies properties of gauge, Lorentz and CPT invariance ². The calculations were done in imaginary time formalism.

The aim of the present paper is to return to the calculations of quantum Faraday rotation in order to further explore the resonance behavior in term of the cyclotron frequency exhibited by the Faraday angle. In particular, we show that in the non-degenerate regime (temperature higher than density) and weak field limit, the resonance is preserved and could be useful to infer the value of the magnetic field. Two relativistic scenarios characterized by these conditions are studied. The first one is a cosmological scenario, where primordial magnetic fields (PMF), possibly present at the recombination epoch, can be estimated in the polarization of the Cosmic Microwave Background (CMB). In this case, we will be able to compare our results with the ones obtained by Planck ⁵. The second one is related to an astrophysical context: the jets originated by pulsars. It is interesting to note that, while with other techniques an upper bound is usually estimated for cosmic magnetic fields, with the method we are proposing here the eventual detection of the resonance would allow us to automatically obtain the value of the magnetic field responsible of the phenomenon.

The paper is organized as follows: in Sec. 2 we briefly summarize the photon self-energy and the Faraday angle calculation in presence of a constant and uniform magnetic field. In section 3 we present the angle in the non-degenerate weak field

limit and discuss the possible application to our cosmological and astrophysical scenarios. Finally, in Sec.4 we state the concluding remarks.

2. Quantum Faraday Rotation Angle

The propagation of photons in a relativistic medium at finite temperature and density, in the presence of a constant magnetic field was studied in Ref. ². The Maxwell equations in Fourier space are

$$[(k_{\perp}^2 + k_{\parallel})g_{\mu\nu} - k_{\mu}k_{\nu} + \Pi_{\mu\nu}(k|A_{\mu}^e)] a_{\nu}(x) = 0, \quad (1)$$

where $k_{\perp}^2 \equiv k_1^2 + k_2^2$ and $k_{\parallel}^2 = k_3^2 + k_4^2$, A_{μ}^e refers to the external magnetic field and $\Pi_{\mu\nu}$ is the photon self-energy which contains all the information related to the interaction with the medium, i.e depends on magnetic field, density and temperature ($\nu, \mu = 1, 2, 3, 4$).

As was discussed in Refs. ^{2,3} the polarization tensor in the case of photons propagating parallel to the magnetic field can be expressed in terms of three independent scalars, I_t , I_r and I_s

$$\Pi_{\mu\nu} = \begin{bmatrix} I_t & iI_r & 0 \\ -iI_r & I_t & 0 \\ 0 & 0 & I_s \end{bmatrix}. \quad (2)$$

To solve Eq (1) we have to diagonalize (2). The eigenvectors of $\Pi_{\mu\nu}$ satisfy the secular equation $\Pi_{\mu\nu}b_{\nu}^i = \kappa_i b_{\nu}^i$. In the particular case of propagation along the magnetic field there are three non vanishing eigenvalues. Two correspond to transverse modes $b_{\mu}' = b_1 \pm b_2$ which describe circularly polarized waves in the plane perpendicular to B with eigenvalues

$$\kappa_{1,2} = I_t \pm I_r. \quad (3)$$

There is a third mode b_3 corresponding to a longitudinal wave which propagates along the magnetic field and $\kappa_3 = I_s$ is the corresponding eigenvalue. The existence of the former two eigenvectors/eigenvalues is the key of the Faraday effect.

The scalars I_r and I_t in the case of propagation parallel to magnetic field ($k_{\perp} = 0$) can be written as

$$I_r = \frac{e^3 B \omega}{2\pi^2} \sum_n \int_{-\infty}^{\infty} dp_3 \frac{k_{\parallel}^2 + 2eB(2n+1)}{(k_{\parallel}^2 + 2eB)^2 - 4\omega^2 \varepsilon_n^2} - \frac{k_{\parallel}^2 + 2eB(2n-1)}{(k_{\parallel}^2 - 2eB)^2 - 4\omega^2 \varepsilon_n^2} \\ \times (n_e(\varepsilon_n) - n_p(\varepsilon_n)), \quad (4)$$

$$I_t = -\frac{e^3 B}{4\pi^2} \sum_n \int_{-\infty}^{\infty} \frac{dp_3}{\varepsilon_n} \left(\left(1 - \frac{(k_{\parallel}^2 + 2eB)(k_{\parallel}^2 + 2eB(2n+1))}{(k_{\parallel}^2 + 2eB)^2 - 4\omega^2 \varepsilon_n^2} \right) \right. \\ \left. + \left(1 - \frac{(k_{\parallel}^2 - 2eB)(k_{\parallel}^2 + 2eB(2n-1))}{(k_{\parallel}^2 - 2eB)^2 - 4\omega^2 \varepsilon_n^2} \right) \right) \times (n_e(\varepsilon_n) + n_p(\varepsilon_n)), \quad (5)$$

where

$$\varepsilon_n = \sqrt{p_3^2 + m^2 + 2enB} \quad (6)$$

and n are the Landau numbers, n_e and n_p are the distribution of electrons and positrons.

The solution of the dispersion relations for the eigenmodes is

$$k_{\pm} = \sqrt{\omega^2 + \kappa_{1,2}}, \quad (7)$$

and the electric field associated to the wave is given by

$$\mathbf{E} = \left[\frac{i\omega}{\sqrt{2}} A e^{i(k_+ x_3 - \omega t)} \mathbf{e}^+ + \frac{i\omega}{\sqrt{2}} A e^{i(k_- x_3 - \omega t)} \mathbf{e}^- \right], \quad (8)$$

where $\mathbf{e}^{\pm} = (\mathbf{e}_1 \mp i\mathbf{e}_2)/\sqrt{2}$ are the polarization vectors of the right and left circularly polarized waves, respectively. So, the superposition of both modes leads to an elliptically polarized wave, whose principal axes rotate.

We can define the refraction indices $n_{\pm}$ of the right and left circularly polarized waves, respectively as ²

$$n_{\pm}(\omega, k_3) = \left(1 + \frac{\kappa_{1,2}(\omega, k_3)}{\omega^2} \right)^{1/2}. \quad (9)$$

The amount of the FR angle, after traveling a distance L in the medium is

$$\theta_F = \frac{1}{2} \omega (n_+ - n_-) L = \frac{1}{2} (Re[k_-] - Re[k_+]) L, \quad (10)$$

where we have used the relation $k = n\omega$. Considering expressions (8)-(9) we obtain

$$Re[k_{\pm}] = \frac{1}{\sqrt{2}} \left[\sqrt{(\omega^2 + Re[\kappa_{1,2}])^2 + Im[\kappa_{1,2}]^2} + (\omega^2 + Re[\kappa_{1,2}]) \right]^{1/2}.$$

If

$$Im[\kappa_{1,2}] \ll \omega^2 + Re[\kappa_{1,2}], \quad (11)$$

and

$$Re[\kappa_{1,2}] \ll \omega^2, \quad (12)$$

in the leading order approximation

$$Re[k_{\pm}] \approx \omega + \frac{Re[\kappa_{1,2}]}{2\omega}, \quad (13)$$

and, according to the relation given in (10), we obtain for the rotated Faraday angle per unit length^a

$$\frac{\theta_F}{L} \approx \frac{I_r(\hbar\omega + i\Gamma, B)}{2\hbar(\omega + i\Gamma)c}. \quad (14)$$

Γ is the probability of absorption of the eigenmodes which remains finite at the threshold and depend only on the value of the magnetic field and has been added

^aWe have returned to the units $\hbar$ and c for obtaining this result.

to the total photon energy in the expression obtained for the Faraday angle. It is possible to estimate this correction by using the equation $\Gamma = \frac{Im\kappa_i}{2Re\omega}$. Then, in order to estimate the amount of Faraday rotation we must evaluate the expression (14).

3. Faraday Angle for Non-Degenerate and Weak Field Limit

In the weak field limit ($eB \ll m^2$) the distance between Landau levels is small ($\sim eB$). Therefore, we can consider the discrete spectra as a continuum and replace the sum over n in Eq. (4) by an integral. If we consider the non-degenerate limit

$$n_{e,p} \ll 1, \quad n_{e,p} \rightarrow e^{-(\varepsilon \mp \mu)\beta}, \quad (15)$$

with relativistic spectrum for the particles

$$\varepsilon = \sqrt{\mathbf{p}^2 + m^2}. \quad (16)$$

Then

$$n_e - n_p \simeq 2e^{-\varepsilon\beta} \sinh(\mu\beta). \quad (17)$$

In the limit of long wave $k_3 \rightarrow 0$, after some algebra we find for I_r the following expression:

$$I_r = A(\mu, \beta) \left(\frac{J}{4} \left(\frac{2eB + 2\omega^2}{3\omega^2} \right) F_2(z) - \frac{J'}{4} \left(\frac{-2eB + 2\omega^2}{3\omega^2} \right) F_2(z') \right), \quad (18)$$

where

$$\begin{aligned} J &= -\omega^2 + 2eB; \quad J' = -\omega^2 - 2eB, \\ z &= \frac{J\beta}{2\omega}; \quad z' = \frac{J'\beta}{2\omega}, \end{aligned} \quad (19)$$

and

$$A = \frac{-2e^2 \sinh(\mu\beta)}{\pi^2 \omega^2 \beta}. \quad (20)$$

We have used the following definition

$$F_2(z) = \int_0^\infty dx \frac{x^2 e^{-x}}{x^2 - z^2}, \quad (21)$$

and consider two limits:

$$(1) \quad |z, z'| \gg 1, \text{ which means}$$

$$\beta(2eB \mp \omega^2) \gg 2\omega. \quad (22)$$

In this limit, we can approximately consider that the singularity at $x = z$ is out of the integration interval so $F_2(z)$ turns into $F_2(z) \simeq \frac{2!}{1-z^2}$. Then θ_F takes the form

$$\frac{\theta_F}{L}(\omega, B, \mu, T) \approx \frac{1}{2c} \frac{16e^2}{3\pi^2} \frac{\sinh(\mu\beta)}{\beta^3} \frac{eB + (\omega + i\Gamma)^2}{4e^2 B^2 - (\omega + i\Gamma)^4}. \quad (23)$$

Expression (23) shows a resonant behavior at $\omega^2 \simeq 2eB$ which could be useful to obtain a value of the magnetic field for fixed frequencies, considering the

bound (22). In such way the condition to be fulfilled is $\omega \gg T$. We analyze below two scenarios, a cosmological one and an astrophysical one, where this approximation makes sense.

(2) $|z, z'| \ll 1$, $F_2(z) \simeq 1$ and the expression for the Faraday angle becomes

$$\frac{\theta_F}{L}(\omega, B, \mu, T) \approx \frac{8e^3 B \mu}{\pi^2 \omega^2 \beta}, \quad (24)$$

where we have considered $\sinh(\mu\beta) \sim \mu\beta$. Expression (24) is the well-known semi-classical limit for the Faraday angle ^{6,7}, proportional to the magnetic field and inverse to the frequency.

3.1. *Primordial magnetic fields and cosmic microwave background radiation*

As is well known, CMB is fundamental for observational cosmology because it is the oldest light in the universe, dating to the epoch of recombination. On the other hand, magnetic fields are present at all scales in the universe, from compact objects⁸, up to clusters of galaxies⁹ and superclusters¹⁰. The mechanism of generation of these cosmic fields is an open problem and one of the hypothesis is that they could be a remnant of a primordial magnetic field. If this is the case and they were generated at some pre-decoupling epoch, they would have left an imprint on the CMB spectrum, both in temperature and polarization⁵. Among their effects, we are interested in the fact that they induce a Faraday rotation¹¹. Different methods are exploited in the Planck report in order to establish constraints on PMF, from which the more stringent are at the level of nanogauss. On the other hand, Faraday rotation of CMB polarization, with Planck power spectra in EE and BB at 70GHz, give a less restrictive bound: $1.38\mu G$. We are then interested in exploring the values that can be pinpointed by (23). The temperature of the CMB radiation being $2.73K$, the condition of $\omega \gg T$ is satisfied for frequencies at least higher than $55GHz$, which means fields corresponding to $B > 4.5 \times 10^{-6}G$. In particular, for $70GHz$ we obtain a value of the magnetic field $7.1 \times 10^{-6}G$ which is of the same order than the bound imposed by Faraday rotation from Plank through the semi-classical expression, far from the cyclotron frequency. Unfortunately, the fact that this method obtains values 3 orders higher than the constraint obtained through other mechanisms, renders its application not very suitable for CMB. The point that the cyclotron and the observational frequencies are far apart when studying CMB was already noticed by Ref. ¹². So, we will now turn to another scenario, where both frequencies can be more coincident.

3.2. *Jets from radio-pulsars*

Relativistic pulsar jets is a phenomenon where streams of matter are emitted almost at the speed of light along the axis of rotation of a compact object. The mechanism

responsible for pulsar jet formation is a matter of debate but it is widely believed that magnetic fields play an important role. The temperature of a typical radio-pulsar is $10^6 K \sim 10^2 eV$. If the emission is in the X-ray band, as is the case of Crab [13] and Vela [14] pulsars, frequencies are in the range of $3 \times 10^{16} - 10^{19} Hz$ which admits the condition $\omega \gg T$ imposed by (22). Hence, the Faraday angle (23) for this radiation would be able to detect magnetic field strengths in the range of $10^6 - 10^9 G$. These are plausible values for magnetic fields in jets since the strengths on radio pulsar surfaces are of order of $10^{12} - 10^{13} G$ (see Ref. ¹⁵). We think that it would be important to study the polarization of the radiation received from the jet-pulsar, as is done for AGN-jets ¹⁶, since the detection of the resonance would contribute to build the model for the generation of jets.

4. Conclusions

We have studied the quantum Faraday angle for weak field limit and non degenerate gas. The expression has a singularity that allows to estimate precise values of magnetic fields in physical scenarios. In particular, we have considered the possibility of detecting primordial magnetic fields in the CMB and astrophysical magnetic fields present in jets of radio pulsars. In the first case, the frequencies at which the CMB spectrum was observed by the Planck satellite would only allow to detect strengths of primordial fields already excluded by other methods. In the second scenario, observing the polarization of jet radiation in the X-ray band would select magnetic fields in a range of values that could effectively be present in the jet.

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