Jefimenko's Gravito-Electromagnetic Equations as an Implication of the Helmholtz Theorem, Adapted to Time-Varying Vector Fields

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Using the Helmholtz theorem (modified to time-varying vector fields) the Jefimenko's gravito-electromagnetic equations are deduced. Newton's gravitational field emerges as a particular case when an action at a distance theory is considered.

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1. Helmholtz Theorem in the Lorenz Gauge

One of the most important theorems in classical field theories is the Helmholtz theorem for vector fields,¹ which states that any sufficiently smooth, rapidly decaying vector field in three dimensions can be written as the sum of two parts: an irrotational vector field represented by the gradient of a scalar potential $\nabla\phi$ and another one, the solenoidal part, represented by a curl of a vector potential $\nabla \times \mathbf{A}$.

The Helmholtz theorem is usually applied to write and solve the field equations in electrostatics and magnetostatics. However, in order to write the field equations in

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the presence of a time-dependent vector field is necessary to consider two auxiliary functions as

$$\phi(\mathbf{r}, t) = \int \frac{f(\mathbf{r}', t - \frac{|\mathbf{r} - \mathbf{r}'|}{c_g})}{4\pi|\mathbf{r} - \mathbf{r}'|} d^3V', \quad (1)$$

and

$$\mathbf{A}(\mathbf{r}, t) = \int \frac{\mathbf{\Gamma}(\mathbf{r}', t - \frac{|\mathbf{r} - \mathbf{r}'|}{c_g})}{4\pi|\mathbf{r} - \mathbf{r}'|} d^3V', \quad (2)$$

that represents the retarded potentials to the sources $f(\mathbf{r}, t)$ (scalar source) and $\mathbf{\Gamma}(\mathbf{r}, t)$ (vector source) in close analogy with the electromagnetism. Here, c_g represents the field propagation speed in free space.

From the functions represented in Eqs.(1) and (2) is possible to show that:

$$\left(\nabla^2 - \frac{1}{c_g^2} \frac{\partial^2}{\partial t^2} \right) \phi(\mathbf{r}, t) = -f(\mathbf{r}, t), \quad (3)$$

and

$$\left(\nabla^2 - \frac{1}{c_g^2} \frac{\partial^2}{\partial t^2} \right) \mathbf{A}(\mathbf{r}, t) = -\mathbf{\Gamma}(\mathbf{r}, t), \quad (4)$$

which is of course the wavelike solutions to the sources. The associated sources are defined as

$$f(\mathbf{r}, t) = -4\pi G\rho,$$

and

$$\mathbf{\Gamma}(\mathbf{r}, t) = -4\pi G\mathbf{J}/c_g.$$

Considering the double curl vector identity for the vector potential as follows

$$\nabla \times \nabla \times \mathbf{A} = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}, \quad (5)$$

and applying some modifications

$$\nabla \times \nabla \times \mathbf{A} = \nabla \left(-\frac{1}{c_g} \frac{\partial \phi}{\partial t} \right) - \left(\frac{4\pi G}{c_g} \mathbf{J} + \frac{1}{c_g^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} \right), \quad (6)$$

then Eq.(6) can be conveniently rewrite as

$$-\frac{4\pi G}{c_g} \mathbf{J} = \frac{1}{c_g} \frac{\partial}{\partial t} \left(\nabla \phi + \frac{1}{c_g} \frac{\partial \mathbf{A}}{\partial t} \right) + \nabla \times \nabla \times \mathbf{A}, \quad (7)$$

which is nothing more than of the Helmholtz theorem, modified for time-varying vector fields.³ Eq.(7) is obtained using the Lorenz gauge represent by

$$\frac{1}{c_g} \frac{\partial \phi(\mathbf{r}, t)}{\partial t} = -\nabla \cdot \mathbf{A}(\mathbf{r}, t). \quad (8)$$

This step ensures the Lorenz gauge condition within the Helmholtz theorem^a.

^aFor a detailed deduction of the gravito-electromagnetic equations see *Gravito-electromagnetic equations from the Helmholtz theorem, adapted to time-varying vector fields*.²

From Eqs. (3) and (4) and its sources in the retarded time is possible to write the scalar potential and the vector potential respectively as

$$\phi(\mathbf{r}, t) = -G \int \frac{\rho(\mathbf{r}', t - \frac{|\mathbf{r} - \mathbf{r}'|}{c_g})}{|\mathbf{r} - \mathbf{r}'|} dV', \quad (9)$$

and

$$\mathbf{A}(\mathbf{r}, t) = -\frac{G}{c_g} \int \frac{\mathbf{J}(\mathbf{r}', t - \frac{|\mathbf{r} - \mathbf{r}'|}{c_g})}{|\mathbf{r} - \mathbf{r}'|} dV'. \quad (10)$$

Note that the scalar source ($f(\mathbf{r}, t)$) is the only one experimental input. All the others equations are consequence of the approach.

2. Jefimenko's Gravito-Electromagnetic Equations

From Eq.(7), the time-dependent gravitational field $\mathbf{g}(\mathbf{r}, t)$ and the time-dependent co-gravitational field $\mathbf{K}(\mathbf{r}, t)$, as named by Jefimenko,⁴ can be defined in analogy with the electromagnetism as follows:

$$-\nabla\phi - \frac{1}{c_g} \frac{\partial \mathbf{A}}{\partial t} = \mathbf{g}(\mathbf{r}, t), \quad (11)$$

$$\nabla \times \mathbf{A} = \mathbf{K}(\mathbf{r}, t). \quad (12)$$

The Eqs.(11) and (12) for the potentials yields the Jefimenko's equations for fields $\mathbf{g}(\mathbf{r}, t)$ and $\mathbf{K}(\mathbf{r}, t)$. In order to obtain the first one in terms of their sources and its time variations, the retarded scalar potential Eq.(9) can be rewrite as

$$\phi(\mathbf{r}, t) = -G \int \frac{\rho(\mathbf{r}', \tau)}{R} dV', \quad (13)$$

where $|\mathbf{r} - \mathbf{r}'| \equiv R$ and $t - R/c_g \equiv \tau$ are conveniently defined as done for the electromagnetic case in the reference.³

Using the definition given by Eq.(11) the field equation becomes

$$\mathbf{g}(\mathbf{r}, t) + \frac{1}{c_g} \frac{\partial \mathbf{A}(\mathbf{r}, t)}{\partial t} = G \int \nabla \left(\frac{\rho(\mathbf{r}', \tau)}{R} \right) dV'. \quad (14)$$

It is easy to show that

$$\nabla \rho = -\frac{1}{c_g} \frac{\partial \rho}{\partial \tau} \hat{e}_R, \quad (15)$$

and

$$\nabla \left(\frac{1}{R} \right) = -\frac{1}{R^2} \hat{e}_R. \quad (16)$$

Finally the Jefimenko's equation to the time-dependent gravitational field is

$$\mathbf{g}(\mathbf{r}, t) = -G \int \left(\frac{\rho}{R^2} + \frac{1}{c_g R} \frac{\partial \rho}{\partial \tau} \right) \hat{e}_R dV' + \frac{G}{c_g^2} \int \frac{1}{R} \frac{\partial \mathbf{J}}{\partial \tau} dV'. \quad (17)$$

Similarly to performed for the $\mathbf{g}(\mathbf{r}, t)$ field and knowing that the $\mathbf{K}(\mathbf{r}, t)$ field is solenoidal is possible to write from the vector potential Eq.10

$$\mathbf{K}(\mathbf{r}, t) = -\frac{G}{c_g} \int \left(\frac{\mathbf{J}}{R^2} + \frac{1}{R c_g} \frac{\partial \mathbf{J}}{\partial \tau} \right) \times \hat{e}_R dV'. \quad (18)$$

In your book *Gravitation and Co-gravitation*,⁴ Jefimenko starts his generalization of Newton's theory from the Eqs.(17) and (18). Newton's Gravitational field can be obtained as a particular case when the Eqs.(17) and (18) are placed in the scenario of an action at a distance theory ($c_g \rightarrow \infty$),

$$\mathbf{g}(\mathbf{r}) = -G \int \frac{\rho}{R^2} \hat{e}_R dV'. \quad (19)$$

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