Lie symmetry analysis of some time fractional partial differential equations

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This paper uses Lie symmetry analysis to reduce the number of independent variables of time fractional partial differential equations. Then symmetry properties have been employed to construct some exact solutions.

Keywords: Lie symmetry method; fractional derivative; time fractional differential equation; similarity variables.

1. Introduction

In recent years, fractional differential equations attracted more attention as they are widely used to describe several phenomena in many fields of sciences such as fluid flow, biology, chemistry, control theory and other areas.¹ Consequently, many powerful and efficient methods for seeking exact and numerical solutions have been developed including the Adomian's decomposition method,² the variational iteration method,^{3,4} transformation methods,⁵ the finite difference method,⁶ the first integral method,⁷ and so on.

Lie symmetry analysis is an important method in studying differential equations with integer order.⁸⁻¹⁰ This method has several applications including construction of new solutions from trivial ones, possibility of some non linear equations to be linearized, construction of integrator factor, reduction of order, reduction of the

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number of independent variables and so on. Concerning symmetry analysis of fractional partial differential equations, the recognized results, in the literature, are very few and the method is still not sufficiently developed. Rigorous study of symmetries, admitted by such fractional equations, was started by Gazizov et al. in¹¹ where they have used two basic fractional derivatives: Riemann-Liouville and Caputo.

We use Lie symmetries with the prolongation formula given by Gazizov et al.¹¹ for the time fractional partial differential equation given by

$$\frac{\partial^\alpha u}{\partial t^\alpha} = x \frac{\partial^2 u}{\partial x^2} + f(x) \frac{\partial u}{\partial x}, \quad (1)$$

where $\frac{\partial^\alpha u}{\partial t^\alpha}$ is the fractional derivative of order α with $0 < \alpha < 1$. The above equation has been extensively studied in the case of $\alpha = 1$ by Craddock¹² with different expressions of the drift function f which satisfies a family of Riccati type differential equations. In this paper, we are interested in two cases for the drift function, the first is $f(x) = \beta \in \mathbb{R}$ and the second is $f(x) = \frac{1+3\sqrt{x}}{2(1+\sqrt{x})}$.

There is no unique definition of fractional derivatives.^{13,14} Here we use the Riemann-Liouville version, defined by

$$D^\alpha u(t, x) = \frac{\partial^\alpha u}{\partial t^\alpha} = \begin{cases} \frac{\partial^n u}{\partial t^n}, & \alpha = n \in \mathbb{N} \\ \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_0^t \frac{u(v, x)}{(t-v)^{\alpha+1-n}} dv, & n-1 < \alpha < n, \quad n \in \mathbb{N}^*, \end{cases}$$

where Γ is the well-known gamma function, and D^α satisfies

$$D^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} t^{\gamma-\alpha}, \quad \alpha > 0, \quad \gamma > -1, \quad t > 0,$$

$$D^\alpha 1 = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \quad \alpha \geq 0, \quad t > 0.$$

This paper is organized as follows. In the second section, a background of symmetry analysis is given and the system of determining equations for symmetry generators corresponding to studied equation is obtained. In the third section, the obtained system of determining equations is solved and infinitesimal generators of each case are given. In the fourth section, some special exact solutions of Eq. (1) are pointed out. Finally, a conclusion is given.

2. Symmetry Analysis of Fractional Partial Differential Equations

Equation (1) is a time fractional partial differential equation of the form

$$\frac{\partial^\alpha u}{\partial t^\alpha} = F(t, x, u, u_x, u_{xx}, \dots), \quad 0 < \alpha < 1,$$

where subscripts denote partial derivatives. Equation (1) is invariant under a one parameter continuous transformations:^{8,9}

$$\begin{aligned} \tilde{t} &= t + \varepsilon \tau(t, x, u) + o(\varepsilon), \\ \tilde{x} &= x + \varepsilon \xi(t, x, u) + o(\varepsilon), \\ \tilde{u} &= u + \varepsilon \varphi(t, x, u) + o(\varepsilon), \end{aligned} \quad (2)$$

with corresponding infinitesimal generator given by

$$X = \tau(t, x, u) \frac{\partial}{\partial t} + \xi(t, x, u) \frac{\partial}{\partial x} + \varphi(t, x, u) \frac{\partial}{\partial u},$$

where τ, ξ and φ are infinitesimals to be determined later. To apply Lie's algorithm, we need to extend the infinitesimal generator X to X^α of the form

$$X^\alpha = X + \varphi^x \frac{\partial}{\partial u_x} + \varphi^{xx} \frac{\partial}{\partial u_{xx}} + \varphi^\alpha \frac{\partial}{\partial u^\alpha},$$

such that $u^\alpha = \frac{\partial^\alpha u}{\partial t^\alpha}$ and $\varphi^x, \varphi^{xx}, \varphi^\alpha$ are extended infinitesimals of order 1, 2 and α respectively. φ^x and φ^{xx} have the form

$$\varphi^x = \varphi_x + (\varphi_u - \xi_x)u_x - \tau_x u_t - \xi_u u_x^2 - \tau_u u_x u_t, \quad (3)$$

$$\begin{aligned} \varphi^{xx} = & \varphi_{xx} + (2\varphi_{xu} - \xi_{xx})u_x - \tau_{xx}u_t + (\varphi_{uu} - 2\xi_{xu})u_x^2 - 2\tau_{xu}u_x u_t \\ & - \xi_{uu}u_x^3 - \tau_{uu}u_x^2 u_t + (\varphi_u - 2\xi_x)u_{xx} - 2\tau_x u_{xt} - 3\xi_u u_x u_{xx} \\ & - \tau_u u_{xx} u_t - 2\tau_u u_x u_{xt}. \end{aligned} \quad (4)$$

Nevertheless, the α -th extended infinitesimal φ^α related to Riemann-Liouville fractional time derivative has the following form

$$\varphi^\alpha = D_t^\alpha(\varphi) + \xi D_t^\alpha(u_x) - D_t^\alpha(\xi u_x) + D_t^\alpha(D_t(\tau)u) - D_t^{\alpha+1}(\tau u) + \tau D_t^{\alpha+1}(u),$$

where D_t^α denotes the total time fractional derivative. The α -th extended infinitesimal φ^α can be rewritten as^{11, 15, 16}

$$\begin{aligned} \varphi^\alpha = & \frac{\partial^\alpha \varphi}{\partial t^\alpha} + (\varphi_u - \alpha D_t(\tau)) \frac{\partial^\alpha u}{\partial t^\alpha} - u \frac{\partial^\alpha \varphi_u}{\partial t^\alpha} + \mu \\ & + \sum_{n=1}^{+\infty} \left[\binom{\alpha}{n} \frac{\partial^n \varphi_u}{\partial t^n} - \binom{\alpha}{n+1} D_t^{n+1}(\tau) \right] D_t^{\alpha-n}(u) \\ & - \sum_{n=1}^{+\infty} \binom{\alpha}{n} D_t^n(\xi) D_t^{\alpha-n}(u_x). \end{aligned} \quad (5)$$

where

$$\begin{aligned} \mu = & \sum_{n=2}^{+\infty} \sum_{m=2}^n \sum_{k=2}^m \sum_{r=0}^{k-1} \binom{\alpha}{n} \binom{n}{m} \binom{k}{r} \frac{1}{k!} \\ & \times \frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)} [-u]^r \frac{\partial^m}{\partial t^m} (u^{k-r}) \frac{\partial^{n-m+k} \varphi}{\partial t^{n-m} \partial u^k}. \end{aligned} \quad (6)$$

In our case, the term μ vanishes as a consequence of $\varphi_{uu} = 0$. Equation (1) is invariant under transformations (2) if and only if the invariance condition is satisfied, i.e.

$$\varphi^\alpha - (u_{xx} + f'(x)u_x)\xi - f(x)\varphi^x - x\varphi^{xx} = 0, \quad \text{whenever } u^\alpha = xu_{xx} + f(x)u_x, \quad (7)$$

where f is the drift function. To obtain general form of infinitesimals ξ, τ and φ , we need to substitute expressions (3), (4) and (5) into invariance condition (7).

The equation depends on variables $u_x, u_{xx}, u_t, u_{xt}, \dots$ and $D_t^{\alpha-n}u, D_t^{\alpha-n}u_x$ for $n = 1, 2, \dots$. Those variables are considered to be independent variables.

Splitting the defining Eq. (7) with respect to independent variables mentioned above leads to the system of infinitely equations

$$\xi_t = \xi_u = \tau_x = \tau_u = \varphi_{uu} = 0, \quad (8)$$

$$x\varphi_u - \alpha x\tau_t - \xi - x(\varphi_u - 2\xi_x) = 0, \quad (9)$$

$$-\alpha f(x)\tau_t - f'(x)\xi + f(x)\xi_x - x(2\varphi_{xu} - \xi_{xx}) = 0, \quad (10)$$

$$\frac{\partial^\alpha \varphi}{\partial t^\alpha} - u \frac{\partial^\alpha \varphi_u}{\partial t^\alpha} - f(x)\varphi_x - x\varphi_{xx} = 0, \quad (11)$$

$$\binom{\alpha}{n} \frac{\partial^n \varphi_u}{\partial t^n} - \binom{\alpha}{n+1} D_t^{n+1} \tau, \quad n = 0, 1, 2, \dots \quad (12)$$

The above system is known as the system of determining symmetries and solving it leads to infinitesimal generators of symmetries admitted by Eq. (1).

3. Admitted Symmetries of Studied Equation

Examining equations (8) and (9) in the above system, we readily obtain

$$\xi(x) = a\alpha x + m\sqrt{x}, \quad \text{and} \quad \tau(t) = at + b,$$

with a, b and m are arbitrary constants. In virtue of $\varphi_{uu} = 0$, φ must be linear in u . Thus

$$\varphi = g(t, x)u + h(t, x),$$

for some functions $g(t, x)$ and $h(t, x)$. On the other hand, Eq. (12) requires that

$$\frac{\partial \varphi_u}{\partial t} = 0, \quad \text{then} \quad g = g(x).$$

We substitute these in Eq. (11) to derive that the function h is an arbitrary solution of the original fractional differential Eq. (1). Substituting of ξ and τ their expressions into Eq. (10) leads to the equation

$$g'(x) = \frac{1}{4x\sqrt{x}}mf(x) - \frac{1}{2}(a\alpha + \frac{\sqrt{x}}{x}m)f'(x) - \frac{m}{8x\sqrt{x}}. \quad (13)$$

Differentiation of the Eq. (11) over u leads to

$$xg''(x) + f(x)g'(x) = 0. \quad (14)$$

Differentiation of Eq. (13) with respect the independent variable x and substituting this into Eq. (14), leads

$$-\frac{a\alpha}{2}x\frac{d}{dx}\mathfrak{L}f + 16^{-1}mx^{-\frac{1}{2}}[3 + 8\mathfrak{L}f - 8x\frac{d}{dx}\mathfrak{L}f] = 0. \quad (15)$$

with $\mathfrak{L}f = xf' - f + \frac{f^2}{2}$. Finally, the choice of a function $y(x)$ leads to a drift function f as a solution of the Ricatti type equation $xf' - f + \frac{f^2}{2} = y(x)$ and hence Eq. (15) determines m and a .

Case 1: In this first case, we consider the fractional differential equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = x \frac{\partial^2 u}{\partial x^2} + \beta \frac{\partial u}{\partial x}, \quad (16)$$

where β is an arbitrary constant. In this case we have $xf'' + ff' = 0$. Using Eqs. (15) and (13) we obtain

- Subcase 1. $\beta = \frac{1}{2}$. In this case a basis for Lie algebra of symmetries admitted by Eq. (16) is

$$\begin{aligned} X_1 &= \alpha x \frac{\partial}{\partial x} + t \frac{\partial}{\partial t}, & X_2 &= \sqrt{x} \frac{\partial}{\partial x} \\ X_3 &= \frac{\partial}{\partial t}, & X_4 &= u \frac{\partial}{\partial u}, & X_h &= h(t, x) \frac{\partial}{\partial u}, \end{aligned}$$

with h an arbitrary solution of Eq. (16).

- Subcase 2. $\beta = \frac{3}{2}$, for this choice of β , the symmetry Lie algebra is spanned by X_1, X_3, X_4, X_h and the infinitesimal generator X_5 given by

$$X_5 = \sqrt{x} \frac{\partial}{\partial x} - \frac{1}{2\sqrt{x}} u \frac{\partial}{\partial u}.$$

- Subcase 3. Therefore, if the drift does not belongs to $\{\frac{1}{2}, \frac{3}{2}\}$ the only possible symmetries are X_1, X_3, X_4 and X_h .

Case 2: $f(x) = \frac{1+3\sqrt{x}}{2(1+\sqrt{x})}$. In this case the drift function f satisfies the Ricatti equation $xf' - f + \frac{1}{2}f^2 = -\frac{3}{8}$ and the equation arising from f is

$$\frac{\partial^\alpha u}{\partial t^\alpha} = x \frac{\partial^2 u}{\partial x^2} + \frac{1+3\sqrt{x}}{2(1+\sqrt{x})} \frac{\partial u}{\partial x}. \quad (17)$$

Thus from Eq. (15) we obtain that a and m are arbitrary. Finally, the symmetry algebra admitted by Eq. (17) is spanned by X_3, X_4, X_h and the infinitesimal generators X_6 and X_7 given by

$$\begin{aligned} X_6 &= \alpha x \frac{\partial}{\partial x} + t \frac{\partial}{\partial t} + \frac{\alpha}{2(1+\sqrt{x})} u \frac{\partial}{\partial u}, \\ X_7 &= \sqrt{x} \frac{\partial}{\partial x} - \frac{1}{2(1+\sqrt{x})} u \frac{\partial}{\partial u}. \end{aligned}$$

4. Exact Solutions

In this section we use the obtained symmetries to construct exact solutions.

4.1. Reduction in the case $f(x) = \frac{1}{2}$ with $X_2 + X_4$

Invariants of the operator $X_2 + X_4$ are $z = t$, and $I = ue^{-2\sqrt{x}}$. The corresponding invariant solution in this case has the form $u(t, x) = \psi(t)e^{2\sqrt{x}}$, and the reduced

equation is

$$D_t^\alpha(\psi(t)) = \psi(t).$$

Let us consider that the above equation is a fractional order differential equation of order $\frac{2}{q}$ with q a positive integer greater than 2. Then a solution of the above reduced equation is given by¹⁴

$$\psi(t) = \frac{1}{2} \sum_{k=0}^{q-1} \{E_t(-k\alpha, 1) - (-1)^{q-k-1} E_t(-k\alpha, (-1)^q)\},$$

where the function E_t is defined by

$$E_t(\nu, a) = t^\nu \sum_{k=0}^{\infty} \frac{(at)^k}{\Gamma(\nu + 1 + k)}.$$

Consequently, the exact solution of original equation in this case is given by

$$u(t, x) = \frac{1}{2} e^{2\sqrt{x}} \sum_{k=0}^{q-1} \left\{ E_t\left(-k\frac{\alpha}{2}, 1\right) - (-1)^{q-k-1} E_t\left(-k\frac{\alpha}{2}, (-1)^q\right) \right\}.$$

4.2. Reduction in the case $f(x) = \frac{3}{2}$ with X_5

The expression of this operator is

$$X_5 = \sqrt{x} \frac{\partial}{\partial x} - \frac{1}{2\sqrt{x}} u \frac{\partial}{\partial u}.$$

The corresponding invariants of X_5 are

$$z = t, \quad \text{and} \quad I = ux^{\frac{1}{2}}.$$

with invariant solution of the form

$$u(t, x) = x^{-\frac{1}{2}} \psi(t),$$

and the reduced equation is

$$D_t^\alpha(\psi(z)) = 0.$$

Thus the invariant solution becomes

$$u(t, x) = cx^{-\frac{1}{2}} t^{\alpha-1},$$

with c an arbitrary constant.

4.3. Reduction in the case $f(x) = \frac{1+3\sqrt{x}}{2(1+\sqrt{x})}$ with X_7

This operator has the form $X_7 = \sqrt{x} \frac{\partial}{\partial x} - \frac{1}{2(1+\sqrt{x})} u \frac{\partial}{\partial u}$. Corresponding invariants of X_8 are $z = t$, and $I = u \frac{1}{1+\sqrt{x}} = \psi(z)$. Invariant solution in this case has the form

$$u(t, x) = \frac{1}{1+\sqrt{x}} \psi(t),$$

and the reduced equation of (17) is

$$D_t^\alpha \psi(t) = 0.$$

The exact solution constructed in this case is given by

$$u(t, x) = \frac{c}{1 + \sqrt{x}} t^{\alpha-1},$$

where c is an arbitrary constant.

Remark 4.1. To obtain the group transformation generated by infinitesimal generator X_6 , we solve the system of first order ordinary differential equations,

$$\begin{aligned} \frac{d\tilde{t}}{d\varepsilon} &= \xi(\tilde{t}, \tilde{x}, \tilde{u}), \\ \frac{d\tilde{x}}{d\varepsilon} &= \varphi(\tilde{t}, \tilde{x}, \tilde{u}), \\ \frac{d\tilde{u}}{d\varepsilon} &= \psi(\tilde{t}, \tilde{x}, \tilde{u}), \end{aligned}$$

subject to the initial conditions

$$\tilde{t}(0) = t, \quad \tilde{x}(0) = x, \quad \tilde{u}(0) = u.$$

The one-parameter groups G generated by X_6 is given as follows.⁸ The entries gives the transformed point $\exp(\varepsilon X_6)(t, x, u) = (\tilde{t}, \tilde{x}, \tilde{u})$:

$$G : (xe^{\alpha\varepsilon}, te^\varepsilon, \frac{1 + \sqrt{x}}{\sqrt{x} + e^{-\frac{\alpha}{2}\varepsilon}}u).$$

As G is a symmetry group, so if $u(t, x)$ is a solution of equation (1), then is the transformed functions $\varepsilon.u(t, x)$ which is given by

$$\varepsilon.u(t, x) = \frac{\sqrt{x} + e^{\frac{\alpha}{2}\varepsilon}}{\sqrt{x} + 1} u(e^{-\varepsilon}t, e^{-\alpha\varepsilon}x),$$

where ε is sufficiently small real number. Using this invariance property, we can derive new exact solutions of equation (1) from the known one.

5. Conclusion

Using Lie point symmetry analysis we see that the studied time fractional partial differential equation can be transformed into fractional differential equations with a single independent variable. Consequently, some exact solutions are found and an example of a one parameter group of transformations is constructed.

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