Integrable systems with unitary invariance from non-stretching geometric curve flows in the Hermitian symmetric space $Sp(n)/U(n)$

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A moving parallel frame method is applied to geometric non-stretching curve flows in the Hermitian symmetric space $Sp(n)/U(n)$ to derive new integrable systems with unitary invariance. These systems consist of a bi-Hamiltonian modified Korteweg-de Vries equation and a Hamiltonian sine-Gordon (SG) equation, involving a scalar variable coupled to a complex vector variable. The Hermitian structure of the symmetric space $Sp(n)/U(n)$ is used in a natural way from the beginning in formulating a complex matrix representation of the tangent space $\mathfrak{sp}(n)/\mathfrak{u}(n)$ and its bracket relations within the symmetric Lie algebra $(\mathfrak{u}(n), \mathfrak{sp}(n))$.

Keywords: Geometric curve flow; Hermitian symmetric space; integrable system; parallel frame; unitary invariance.

1. Introduction

In the theory of integrable systems, the modified Korteweg-de Vries (mKdV) equation and the sine-Gordon (SG) equation are two integrable equations of basic

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importance. They have an elegant geometric origin that arises from the differential invariant of a curve that undergoes certain geometric non-stretching flows (which locally preserve the arclength of the curve) in the Euclidean plane and the sphere, respectively. For generalizing this geometric picture to higher dimensional spaces, it becomes natural to transform from a Frenet frame along the curve which determines the differential invariants of the curve to a parallel frame¹ which determines differential covariants of the curve. This approach has been used in numerous geometric spaces (see Refs. 2, 3, 4, 5, 7, 6, 8, 9, 10, 11, 12, 13, 14).

There is a general construction of a parallel frame¹⁵ for non-stretching curve flows in Riemannian symmetric spaces $M = G/H$. The Cartan structure equations of this frame have the property that they explicitly yield a pair of Hamiltonian and symplectic operators in all symmetric spaces. These operators can be used to derive a hierarchy of group-invariant bi-Hamiltonian mKdV equations and SG equations whose variables are given by the components of the Cartan connection matrix along a curve undergoing particular geometric non-stretching flows.¹⁵

In this paper, we apply this general method¹⁵ to the Hermitian symmetric space $Sp(n)/U(n)$ and get $U(n-1)$ -invariant integrable scalar-vector mKdV and SG equations. These equations and their bi-Hamiltonian integrability structure naturally involve both the Hermitian inner product and the complex structure J on the symmetric Lie algebra $(\mathfrak{u}(n), \mathfrak{sp}(n)/\mathfrak{u}(n))$.

Using this Hermitian structure, we show how to introduce a natural complex matrix representation for the symmetric Lie algebra $(\mathfrak{u}(n), \mathfrak{sp}(n)/\mathfrak{u}(n))$ as well as the Lie bracket relations between $\mathfrak{u}(n)$ and $\mathfrak{sp}(n)/\mathfrak{u}(n)$. In this way, the flow equation can be naturally written in terms of a scalar variable and a complex vector variable that transform properly under the unitary gauge group of the underlying parallel frame. If instead a standard real matrix representation were to be used, there would be a problem of how to translate the flow equation with real variables into a unitarily invariant form. Our approach fully resolves this issue. (See also the examples in Refs. 10, 13, 14.)

More generally, depending on whether the symmetric space is real, Hermitian, or quaternionic, one may use such extra geometric/algebraic structures from the beginning on the tangent space of M as well as on the associated symmetric Lie algebra $(\mathfrak{h}, \mathfrak{g}/\mathfrak{h})$ and its Lie bracket relations.

The outline of the paper is as follows. In Sec. 2, we review the definition of Riemannian symmetric spaces and also Hermitian symmetric spaces using symmetric Lie algebras. For the Hermitian symmetric space $Sp(n)/U(n)$, we decompose the subalgebra $\mathfrak{u}(n)$ and the quotient space $\mathfrak{sp}(n)/\mathfrak{u}(n)$ into parallel and perp subspaces relative to a regular element in the Cartan subspace in $\mathfrak{sp}(n)/\mathfrak{u}(n)$. We give a complex matrix representation of these subspaces and also the bracket relations between them, which will be essential later for setting up a parallel frame for curves in $Sp(n)/U(n)$.

In Sec. 3, we first review the standard frame field and connection formulation of the Riemannian structure common to all symmetric spaces, and then we review the

definition of a moving parallel frame for non-stretching curve flows in symmetric spaces. In Sec. 4, we consider non-stretching curve flows in the Hermitian space $M = Sp(n)/U(n)$. By utilizing a moving parallel frame and then taking the pull back of the torsion and curvature 2-forms from M to the surface swept out by a curve flow, we get Cartan structure equations which encode a pair of Hamiltonian and symplectic operators. These results will be just stated without proof. See Ref. 15 for details and an explicit proof for curve flows in general symmetric spaces. Then we derive a hierarchy of $U(n-1)$ -invariant bi-Hamiltonian equations, starting with a $U(n-1)$ -invariant mKdV equation. All of these equations are integrable systems for an imaginary scalar variable coupled to a complex vector variable. We next derive a $U(n-1)$ -invariant SG equation for the same variables by using the kernel of the symplectic operator and doing an algebraic reduction of the parallel part of the flow. There is a general method behind this reduction which will be explored in a subsequent paper.²⁵

2. Algebraic Structure of $Sp(n)/U(n)$

A general Riemannian symmetric space $M = G/H$ is defined by¹⁶ a simple Lie group G and an involutive compact Lie subgroup H in G . Any linear frame on M provides a soldering identification¹⁷ between the tangent space $T_x M$ at points x and the vector space $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$. There is an orthogonal decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with respect to the Cartan-Killing form with the Lie bracket relations

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}. \quad (1)$$

A Hermitian symmetric space is a Riemannian symmetric space that possesses a covariantly-constant $(1, 1)$ tensor J which acts on each tangent space as a complex structure, $J^2 = -\text{id}$.

Cartan subspaces of $\mathfrak{m}$ are defined as a maximal abelian subspace $\mathfrak{a} \subseteq \mathfrak{m}$ with the property that it is the centralizer of its elements, $\mathfrak{a} = \mathfrak{m} \cap \mathfrak{c}(\mathfrak{a})$. It is well-known¹⁶ that any two Cartan subspaces are isomorphic to one another under some linear transformation in $\text{Ad}(H)$ and that the action of the linear transformation group $\text{Ad}(H)$ on any Cartan subspace $\mathfrak{a}$ generates $\mathfrak{m}$. The dimension of $\mathfrak{a}$ as a vector space is equal to the rank of M .

To define a moving parallel frame on non-stretching curve flows, which will be presented in the next section, we need to choose an element e in the Cartan subspace $\mathfrak{a}$. This element will be identified with the frame components of the tangent vector of a curve. For any choice of e , the corresponding linear operator $\text{ad}(e)$ induces a direct sum decomposition of the vector spaces $\mathfrak{m} = \mathfrak{sp}(n)/\mathfrak{u}(n)$ and $\mathfrak{h} = \mathfrak{u}(n)$ into centralizer spaces $\mathfrak{m}_{\parallel}$ and $\mathfrak{h}_{\parallel}$ and their orthogonal complements (perp spaces) $\mathfrak{m}_{\perp}$ and $\mathfrak{h}_{\perp}$ with respect to the Cartan-Killing form. The Lie bracket relations on $\mathfrak{m}_{\parallel}$, $\mathfrak{m}_{\perp}$, $\mathfrak{h}_{\parallel}$, $\mathfrak{h}_{\perp}$ coming from the structure of $\mathfrak{g}$ as a symmetric Lie algebra (1) are given by

$$[\mathfrak{m}_{\parallel}, \mathfrak{m}_{\parallel}] \subseteq \mathfrak{h}_{\parallel}, \quad [\mathfrak{m}_{\parallel}, \mathfrak{h}_{\parallel}] \subseteq \mathfrak{m}_{\parallel}, \quad [\mathfrak{h}_{\parallel}, \mathfrak{h}_{\parallel}] \subseteq \mathfrak{h}_{\parallel}, \quad (2)$$

$$[\mathfrak{h}_{\parallel}, \mathfrak{m}_{\perp}] \subseteq \mathfrak{m}_{\perp}, \quad [\mathfrak{h}_{\parallel}, \mathfrak{h}_{\perp}] \subseteq \mathfrak{h}_{\perp}, \quad (3)$$

$$[\mathfrak{m}_{\parallel}, \mathfrak{m}_{\perp}] \subseteq \mathfrak{h}_{\perp}, \quad [\mathfrak{m}_{\parallel}, \mathfrak{h}_{\perp}] \subseteq \mathfrak{m}_{\perp}, \quad (4)$$

while the remaining Lie brackets

$$[\mathfrak{m}_{\perp}, \mathfrak{m}_{\perp}], \quad [\mathfrak{h}_{\perp}, \mathfrak{h}_{\perp}], \quad [\mathfrak{m}_{\perp}, \mathfrak{h}_{\perp}] \quad (5)$$

obey the general relations (1).

Through the Lie bracket relations (2)–(4), the operator $\text{ad}(\mathfrak{e})$ maps $\mathfrak{h}_{\perp}$ into $\mathfrak{m}_{\perp}$, and vice versa. Hence $\text{ad}(\mathfrak{e})^2$ is well-defined as a linear mapping of each subspace $\mathfrak{h}_{\perp}$ and $\mathfrak{m}_{\perp}$ into itself.

2.1. Symmetric Lie algebra $(\mathfrak{u}(n), \mathfrak{sp}(n)/\mathfrak{u}(n))$

Hermitian symmetric spaces have been classified by Cartan and others. See Ref. 16 for full details. For the specific Hermitian symmetric space $Sp(n)/U(n)$, we will provide all the algebraic details needed for later use.

The symplectic Lie algebra $\mathfrak{sp}(n)$ consists of all matrices in $\mathfrak{gl}(2n, \mathbb{C})$ satisfying the condition

$$g\theta + \theta g^t = 0, \quad g^t = -\bar{g}, \quad \theta = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \quad (6)$$

An involution σ on $\mathfrak{gl}(2n, \mathbb{C})$ which leaves $\mathfrak{sp}(n)$ invariant is defined by $\sigma(g) = \bar{g}$. The set of matrices in $\mathfrak{sp}(n)$ that is invariant under σ is isomorphic to the unitary Lie algebra $\mathfrak{u}(n)$. Thus the Lie algebra $\mathfrak{sp}(n)$ as a symmetric Lie algebra orthogonally decomposes as $\mathfrak{sp}(n) = \mathfrak{h} \oplus \mathfrak{m}$ into the eigenspaces of σ :

$$\mathfrak{h} := \mathfrak{u}(n) \subset \mathfrak{g}, \quad \sigma(\mathfrak{h}) = \mathfrak{h}, \quad (7)$$

$$\mathfrak{m} := \mathfrak{sp}(n)/\mathfrak{u}(n) \subset \mathfrak{g}, \quad \sigma(\mathfrak{m}) = -\mathfrak{m}. \quad (8)$$

The matrix representation of $\mathfrak{g} = \mathfrak{sp}(n)$ is given as

$$\begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix} \in \mathfrak{sp}(n), \quad A^t = -\bar{A}, \quad B^t = B \quad (9)$$

in which $A, B \in \mathfrak{gl}(n, \mathbb{C})$. The matrix representation of the vector spaces $\mathfrak{m} = \mathfrak{sp}(n)/\mathfrak{u}(n)$ and $\mathfrak{h} = \mathfrak{u}(n)$ is given as

$$(A, B) := i \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \in \mathfrak{m}, \quad A, B \in \mathfrak{gl}(n, \mathbb{R}), \quad A^t = A, \quad B^t = B, \quad (10)$$

$$(C, D) := \begin{pmatrix} C & -D \\ D & C \end{pmatrix} \in \mathfrak{h}, \quad C, D \in \mathfrak{gl}(n, \mathbb{R}), \quad C^t = -C, \quad D^t = D. \quad (11)$$

The complex structure J of the tangent space $T_o M = \mathfrak{m}$ is represented by $J = \frac{1}{2}iI \in \mathfrak{u}(n)$, so then the action of $\text{ad}(J)$ on $\mathfrak{m}$ is simply given by multiplication by i . Therefore one can identify $(A, B) \in \mathfrak{m}$ as a complex symmetric matrix $A + iB \in \mathfrak{s}(n, \mathbb{C})$. The elements (C, D) in $\mathfrak{h}$ can be naturally written as a complex matrix $C + iD \in \mathfrak{u}(n)$.

With this complex representation, the bracket relation (1) is not just a matrix commutator due to the role of complex structure J .

Lemma 2.1.

- (1) *The Hermitian matrix representation of the vector space $\mathfrak{m} = \mathfrak{sp}(n)/\mathfrak{u}(n)$ and Lie subalgebra $\mathfrak{h} = \mathfrak{u}(n)$ is given as*

$$(A_1) \in \mathfrak{m}, \quad A_1 \in \mathfrak{gl}(n, \mathbb{C}), \quad A_1^t - A_1 = 0, \quad (12)$$

$$(C_1) \in \mathfrak{h}, \quad C_1 \in \mathfrak{gl}(n, \mathbb{C}), \quad \overline{C_1}^t + C_1 = 0. \quad (13)$$

- (2) *The Lie bracket relations (1) have the following matrix representation in which $(A_1), (A_2) \in \mathfrak{m}$ and $(C_1), (C_2) \in \mathfrak{h}$:*

$$[A_1, A_2] = A_2 \overline{A_1} - A_1 \overline{A_2} \in \mathfrak{h}, \quad (14)$$

$$[A_1, C_1] = A_1 \overline{C_1} - C_1 A_1 \in \mathfrak{m}, \quad (15)$$

$$[C_1, C_2] = C_1 C_2 - C_2 C_1 \in \mathfrak{h}. \quad (16)$$

- (3) *The restriction of Cartan-Killing form on $\mathfrak{sp}(n)$ to $\mathfrak{m} = \mathfrak{sp}(n)/\mathfrak{u}(n)$ is a negative-definite inner product*

$$\langle A_1, A_2 \rangle = 8(n+1) \operatorname{Re}(\operatorname{tr}(A_1 \overline{A_2})). \quad (17)$$

- (4) *The vector space $\mathfrak{m} = \mathfrak{sp}(n)/\mathfrak{u}(n)$ is of dimension $n^2 + n$ and of rank n . The n -dimensional vector subspace $\mathfrak{a} \subset \mathfrak{m} = \mathfrak{sp}(n)/\mathfrak{u}(n)$ generated by real, diagonal matrices $(E) \in \mathfrak{m}$ is a Cartan subspace.*

An element e of the Cartan subspace $\mathfrak{a}$ is called a *regular element*^{16,18} if its centralizer subspace $\mathfrak{c}(e)$ in $\mathfrak{g} = \mathfrak{sp}(n)$ is of maximal dimension. Any real, diagonal matrix $E_{ii} \in \mathfrak{gl}(n, \mathbb{R})$ whose only non-zero component is a 1 in its i th row and i th column (with $1 \leq i \leq n$) is regular element $(E_{ii}) \in \mathfrak{a}$. We will choose

$$e := \frac{1}{\sqrt{\chi}}(E_{11}) \in \mathfrak{a} \quad (18)$$

where $\chi \in \mathbb{R}$ is a normalization constant. We choose this constant so that e has unit norm,

$$-1 = \langle e, e \rangle = -8(n+1)/\chi$$

which determines

$$\chi = 8(n+1). \quad (19)$$

In the following lemma, we give an explicit matrix representation of the perp and parallel subspaces of $\mathfrak{m}$ and $\mathfrak{h}$ determined by e . These subspaces are defined by the properties

$$\operatorname{ad}(e)\mathfrak{m}_{\parallel} = 0, \quad \langle \mathfrak{m}_{\perp}, \mathfrak{m}_{\parallel} \rangle = 0$$

and

$$\operatorname{ad}(e)\mathfrak{h}_{\parallel} = 0, \quad \langle \mathfrak{h}_{\perp}, \mathfrak{h}_{\parallel} \rangle = 0.$$

Lemma 2.2.

(1) The matrix representation of $\mathfrak{m}_{\parallel}$ and $\mathfrak{m}_{\perp}$ in $\mathfrak{m} = \mathfrak{sp}(n)/\mathfrak{u}(n)$ is given as

$$(\mathfrak{a}_{\parallel}, \mathbf{A}_{\parallel}) = \begin{pmatrix} \mathfrak{a}_{\parallel} & 0 \\ 0 & \mathbf{A}_{\parallel} \end{pmatrix} \in \mathfrak{m}_{\parallel}, \quad (\mathfrak{a}_{\perp}, \mathbf{a}_{\perp}) := \begin{pmatrix} \mathfrak{a}_{\perp} & \mathbf{a}_{\perp} \\ \mathbf{a}_{\perp}^{\dagger} & 0 \end{pmatrix} \in \mathfrak{m}_{\perp} \quad (20)$$

in which $\mathfrak{a}_{\parallel} \in \mathbb{R}$, $\mathfrak{a}_{\perp} \in i\mathbb{R}$, $\mathbf{a}_{\perp} \in \mathbb{C}^{n-1}$, and $\mathbf{A}_{\parallel} \in \mathfrak{s}(n-1, \mathbb{C})$.

(2) The matrix representation of $\mathfrak{h}_{\parallel}$ and $\mathfrak{h}_{\perp}$ is given as

$$(\mathbf{C}_{\parallel}) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{C}_{\parallel} \end{pmatrix} \in \mathfrak{h}_{\parallel}, \quad (\mathfrak{c}_{\perp}, \mathbf{c}_{\perp}) = \begin{pmatrix} \mathfrak{c}_{\perp} & \mathbf{c}_{\perp} \\ -\bar{\mathbf{c}}_{\perp}^{\dagger} & 0 \end{pmatrix} \in \mathfrak{h}_{\perp} \quad (21)$$

in which $\mathfrak{c}_{\perp} \in i\mathbb{R}$, $\mathbf{c}_{\perp} \in \mathbb{C}^{n-1}$, and $\mathbf{C}_{\parallel} \in \mathfrak{u}(n-1)$.

(3) The dimension of perp and parallel subspaces is given as

$$\begin{aligned} \dim(\mathfrak{m}_{\parallel}) &= n^2 - n + 1, \\ \dim(\mathfrak{h}_{\parallel}) &= n^2 - 2n + 1, \\ \dim(\mathfrak{m}_{\perp}) &= \dim(\mathfrak{h}_{\perp}) = 2n - 1. \end{aligned} \quad (22)$$

(4) The regular element (18) in the Cartan subspace $\mathfrak{a}$ is represented as

$$\mathfrak{e} = \frac{1}{\sqrt{\chi}}(1, 0) \in \mathfrak{m}_{\parallel}. \quad (23)$$

In particular the linear operator $\text{ad}(\mathfrak{e})$ gives an isomorphism of $\mathfrak{m}_{\perp}$ and $\mathfrak{h}_{\perp}$:

$$\begin{aligned} \text{ad}(\mathfrak{e})(\mathfrak{a}_{\perp}, \mathbf{a}_{\perp}) &= \frac{1}{\sqrt{\chi}}(2\mathfrak{a}_{\perp}, -\bar{\mathbf{a}}_{\perp}) \in \mathfrak{h}_{\perp}, \\ \text{ad}(\mathfrak{e})(\mathfrak{c}_{\perp}, \mathbf{c}_{\perp}) &= \frac{1}{\sqrt{\chi}}(-2\mathfrak{c}_{\perp}, \bar{\mathbf{c}}_{\perp}) \in \mathfrak{m}_{\perp}. \end{aligned} \quad (24)$$

To write out the explicit Lie bracket relations on $\mathfrak{m} = \mathfrak{m}_{\parallel} \oplus \mathfrak{m}_{\perp}$ and $\mathfrak{h} = \mathfrak{h}_{\parallel} \oplus \mathfrak{h}_{\perp}$, we will use the following inner products and outer products. For $\mathbf{a}, \mathbf{b} \in \mathbb{C}^{n-1}$, we note

$$\begin{aligned} \bar{\mathbf{a}}\mathbf{b}^{\dagger} + \bar{\mathbf{b}}\mathbf{a}^{\dagger} &= 2\text{Re}\langle \mathbf{a}, \mathbf{b} \rangle \in \mathbb{R}, \\ \bar{\mathbf{a}}\mathbf{b}^{\dagger} - \bar{\mathbf{b}}\mathbf{a}^{\dagger} &= i2\text{Im}\langle \mathbf{a}, \mathbf{b} \rangle \in i\mathbb{R}, \end{aligned} \quad (25)$$

where

$$\langle \mathbf{a}, \mathbf{b} \rangle = \bar{\mathbf{a}}\mathbf{b}^{\dagger} \quad (26)$$

is the Hermitian inner product. Also, we define

$$\begin{aligned} \mathbf{a}^{\dagger}\mathbf{b} - \mathbf{b}^{\dagger}\mathbf{a} &:= \mathbf{a} \wedge \mathbf{b} \in \mathfrak{so}(n-1, \mathbb{C}), \\ \mathbf{a}^{\dagger}\mathbf{b} + \mathbf{b}^{\dagger}\mathbf{a} &:= \mathbf{a} \odot \mathbf{b} \in \mathfrak{s}(n-1, \mathbb{C}), \end{aligned} \quad (27)$$

and

$$\bar{\mathbf{a}}^{\dagger}\mathbf{b} - \bar{\mathbf{b}}^{\dagger}\mathbf{a}^{\dagger} := \mathbf{a} \bar{\wedge} \mathbf{b} \in \mathfrak{u}(n-1), \quad (28)$$

Proposition 2.1.

- (1) *The bracket relations (2)–(4) between the perp and parallel subspaces of $\mathfrak{sp}(n)$ are given as:*

$$[\mathfrak{m}_{\parallel}, \mathfrak{m}_{\parallel}] = [(a_{\parallel 1}, \mathbf{A}_{\parallel 1}), (a_{\parallel 2}, \mathbf{A}_{\parallel 2})] = (\mathbf{A}_{\parallel 2} \overline{\mathbf{A}}_{\parallel 1} - \mathbf{A}_{\parallel 1} \overline{\mathbf{A}}_{\parallel 2}) \in \mathfrak{h}_{\parallel} \quad (29a)$$

$$[\mathfrak{m}_{\parallel}, \mathfrak{h}_{\parallel}] = [(a_{\parallel}, \mathbf{A}_{\parallel}), (\mathbf{C}_{\parallel})] = (0, \mathbf{A}_{\parallel} \overline{\mathbf{C}}_{\parallel} - \mathbf{C}_{\parallel} \mathbf{A}_{\parallel}) \in \mathfrak{m}_{\parallel} \quad (29b)$$

$$[\mathfrak{h}_{\parallel}, \mathfrak{h}_{\parallel}] = [(\mathbf{C}_{\parallel 1}), (\mathbf{C}_{\parallel 2})] = (\mathbf{C}_{\parallel 1} \mathbf{C}_{\parallel 2} - \mathbf{C}_{\parallel 2} \mathbf{C}_{\parallel 1}) \in \mathfrak{h}_{\parallel} \quad (29c)$$

$$[\mathfrak{m}_{\parallel}, \mathfrak{m}_{\perp}] = [(a_{\parallel}, \mathbf{A}_{\parallel}), (a_{\perp}, \mathbf{a}_{\perp})] = (2a_{\parallel} a_{\perp}, \mathbf{a}_{\perp} \overline{\mathbf{A}}_{\parallel} - a_{\parallel} \overline{\mathbf{a}}_{\perp}) \in \mathfrak{h}_{\perp} \quad (29d)$$

$$[\mathfrak{m}_{\parallel}, \mathfrak{h}_{\perp}] = [(a_{\parallel}, \mathbf{A}_{\parallel}), (c_{\perp}, \mathbf{c}_{\perp})] = (-2a_{\parallel} c_{\perp}, a_{\parallel} \overline{\mathbf{c}}_{\perp} - \mathbf{c}_{\perp} \mathbf{A}_{\parallel}) \in \mathfrak{m}_{\perp} \quad (29e)$$

$$[\mathfrak{m}_{\perp}, \mathfrak{h}_{\parallel}] = [(a_{\perp}, \mathbf{a}_{\perp}), (\mathbf{C}_{\parallel})] = (0, \mathbf{a}_{\perp} \mathbf{C}_{\parallel}) \in \mathfrak{m}_{\perp} \quad (29f)$$

$$[\mathfrak{h}_{\parallel}, \mathfrak{h}_{\perp}] = [(\mathbf{C}_{\parallel}), (c_{\perp}, \mathbf{c}_{\perp})] = (0, -\mathbf{c}_{\perp} \mathbf{C}_{\parallel}) \in \mathfrak{h}_{\perp} \quad (29g)$$

- (2) *The remaining bracket relations (5) between perp spaces are given as:*

$$\begin{aligned} [\mathfrak{m}_{\perp}, \mathfrak{m}_{\perp}]_{\mathfrak{h}_{\parallel}} &= [(a_{\perp 1}, \mathbf{a}_{\perp 1}), (a_{\perp 2}, \mathbf{a}_{\perp 2})]_{\mathfrak{h}_{\parallel}} \\ &= (\overline{\mathbf{a}}_{\perp 2}^t \overline{\mathbf{a}}_{\perp 1}) \in \mathfrak{h}_{\parallel} \end{aligned} \quad (30a)$$

$$\begin{aligned} [\mathfrak{m}_{\perp}, \mathfrak{m}_{\perp}]_{\mathfrak{h}_{\perp}} &= [(a_{\perp 1}, \mathbf{a}_{\perp 1}), (a_{\perp 2}, \mathbf{a}_{\perp 2})]_{\mathfrak{h}_{\perp}} \\ &= (2i \operatorname{Im} \langle \mathbf{a}_{\perp 1}, \mathbf{a}_{\perp 2} \rangle, a_{\perp 1} \overline{\mathbf{a}}_{\perp 2} - a_{\perp 2} \overline{\mathbf{a}}_{\perp 1}) \in \mathfrak{h}_{\perp} \end{aligned} \quad (30b)$$

$$\begin{aligned} [\mathfrak{m}_{\perp}, \mathfrak{h}_{\perp}]_{\mathfrak{m}_{\parallel}} &= [(a_{\perp}, \mathbf{a}_{\perp}), (c_{\perp}, \mathbf{c}_{\perp})]_{\mathfrak{m}_{\parallel}} \\ &= (-2 \operatorname{Re} \langle \mathbf{a}_{\perp}, \overline{\mathbf{c}}_{\perp} \rangle - 2a_{\perp} c_{\perp}, \mathbf{a}_{\perp} \odot \overline{\mathbf{c}}_{\perp}) \in \mathfrak{m}_{\parallel} \end{aligned} \quad (30c)$$

$$\begin{aligned} [\mathfrak{m}_{\perp}, \mathfrak{h}_{\perp}]_{\mathfrak{m}_{\perp}} &= [(a_{\perp}, \mathbf{a}_{\perp}), (c_{\perp}, \mathbf{c}_{\perp})]_{\mathfrak{m}_{\perp}} \\ &= (2i \operatorname{Im} \langle \mathbf{a}_{\perp}, \overline{\mathbf{c}}_{\perp} \rangle, a_{\perp} \overline{\mathbf{c}}_{\perp} - c_{\perp} \mathbf{a}_{\perp}) \in \mathfrak{m}_{\perp} \end{aligned} \quad (30d)$$

$$\begin{aligned} [\mathfrak{h}_{\perp}, \mathfrak{h}_{\perp}]_{\mathfrak{h}_{\parallel}} &= [(c_{\perp 1}, \mathbf{c}_{\perp 1}), (c_{\perp 2}, \mathbf{c}_{\perp 2})]_{\mathfrak{h}_{\parallel}} \\ &= (\mathbf{c}_{\perp 2} \overline{\mathbf{c}}_{\perp 1}) \in \mathfrak{h}_{\parallel} \end{aligned} \quad (30e)$$

$$\begin{aligned} [\mathfrak{h}_{\perp}, \mathfrak{h}_{\perp}]_{\mathfrak{h}_{\perp}} &= [(c_{\perp 1}, \mathbf{c}_{\perp 1}), (c_{\perp 2}, \mathbf{c}_{\perp 2})]_{\mathfrak{h}_{\perp}} \\ &= (2i \operatorname{Im} \langle \mathbf{c}_{\perp 1}, \mathbf{c}_{\perp 2} \rangle, c_{\perp 1} \mathbf{c}_{\perp 2} - c_{\perp 2} \mathbf{c}_{\perp 1}) \in \mathfrak{h}_{\perp} \end{aligned} \quad (30f)$$

- (3) *The Cartan-Killing form on $\mathfrak{m}_{\perp}$ is given as*

$$\langle (a_{\perp 1}, \mathbf{a}_{\perp 1}), (a_{\perp 2}, \mathbf{a}_{\perp 2}) \rangle = 8(n+1) \operatorname{Re}(-a_{\perp 1} a_{\perp 2} + 2 \langle \mathbf{a}_{\perp 1}, \mathbf{a}_{\perp 2} \rangle). \quad (31)$$

The adjoint action of the Lie subalgebra $\mathfrak{h}_{\parallel} \subset \mathfrak{h} = \mathfrak{u}(n)$ on $\mathfrak{g} = \mathfrak{sp}(n)$ generates the linear transformation group $H_{\parallel}^* \subset H^* = \operatorname{Ad}(H)$, leaving invariant the element e in the Cartan subspace $\mathfrak{a} \subset \mathfrak{m} = \mathfrak{sp}(n)/\mathfrak{u}(n)$. The group H^* is given by the unitary group $U(n-1) \subset U(n)$ which has the matrix representation

$$\begin{pmatrix} 1 & 0 \\ 0 & C_{\parallel} \end{pmatrix} \in U(n-1) \simeq H_{\parallel}^*, \quad C_{\parallel} \in U(n-1). \quad (32)$$

This unitary group $U(n-1)$ acts on the subspace $\mathfrak{m}_\perp$ as

$$\text{Ad}(C_\parallel)(\mathbf{a}_\perp, \mathbf{a}_\perp) = (\mathbf{a}_\perp, \mathbf{a}_\perp C_\parallel^t) \in \mathfrak{m}_\perp. \quad (33)$$

The action of $H_\parallel^* \simeq U(n-1)$ on $\mathfrak{m}_\parallel$ is given by

$$\text{Ad}(C_\parallel)(\mathbf{a}_\parallel, \mathbf{A}_\parallel) = (\mathbf{a}_\parallel, C_\parallel \mathbf{A}_\parallel C_\parallel^t) \in \mathfrak{m}_\parallel. \quad (34)$$

Notice that these actions leave the scalar component unchanged. This observation will help us to give a geometric interpretation of the algebraic reduction used later to derive the SG flow.

Proposition 2.2. *The linear map $\text{ad}(\mathbf{e})^2$ on $\mathfrak{m}_\perp, \mathfrak{h}_\perp \simeq \mathbb{R} \oplus \mathbb{C}^{n-1}$ is given by*

$$\text{ad}(\mathbf{e})^2(\mathbf{a}_\perp, \mathbf{a}_\perp) = -\chi^{-1}(4\mathbf{a}_\perp, \mathbf{a}_\perp) \in \mathfrak{m}_\perp,$$

$$\text{ad}(\mathbf{e})^2(\mathbf{c}_\perp, \mathbf{c}_\perp) = -\chi^{-1}(4\mathbf{c}_\perp, \mathbf{c}_\perp) \in \mathfrak{h}_\perp.$$

The irreducible subspaces of $\mathfrak{m}_\perp$ and $\mathfrak{h}_\perp$ in this representation are $(\mathbf{a}_\perp, 0) \in \mathbb{R}$ and $(0, \mathbf{a}_\perp) \in \mathbb{C}^{n-1}$ on which $\text{ad}(\mathbf{e})^2$ has respective eigenvalues $-4/\chi$ and $-1/\chi$.

3. Moving Parallel Frames for Non-Stretching Curve Flows in Riemannian Symmetric Spaces

The Riemannian structure of the space $M = G/H$ is most naturally described^{17,19} in terms of a $\mathfrak{m}$ -valued linear coframe e and a $\mathfrak{h}$ -valued linear connection ω whose torsion and curvature

$$\mathfrak{T} := de + [\omega, e], \quad \mathfrak{R} := d\omega + \frac{1}{2}[\omega, \omega] \quad (35)$$

are 2-forms with respective values in $\mathfrak{m}$ and $\mathfrak{h}$, given by the following Cartan structure equations

$$\mathfrak{T} = 0, \quad \mathfrak{R} = -\frac{1}{2}[e, e]. \quad (36)$$

Here the brackets denote the wedge product combined with the Lie bracket.

The underlying Riemannian metric on the space $M = G/H$ is given by

$$g = -\langle e, e \rangle \quad (37)$$

in terms of the Cartan-Killing form restricted to $\mathfrak{m}$.

Now consider any smooth flow $\gamma(t, x)$ of a curve in M . The flow is called *non-stretching* if it preserves the G -invariant arclength $ds = |\gamma_x|_g dx$, with $|\gamma_x|_g^2 = g(\gamma_x, \gamma_x)$, in which case we can put $|\gamma_x|_g = 1$ without loss of generality (whereby x is the arclength parameter).

For flows that are transverse to the curve (such that γ_x and γ_t are linearly independent), $\gamma(t, x)$ will describe a smooth two-dimensional surface in M . The pullback of the torsion and curvature equations (36) to this surface yields

$$D_x e_t - D_t e_x + [\omega_x, e_t] - [\omega_t, e_x] = 0, \quad (38)$$

$$D_x \omega_t - D_t \omega_x + [\omega_x, \omega_t] = -[e_x, e_t], \quad (39)$$

with

$$e_x := e|_{\gamma_x}, \quad e_t := e|_{\gamma_t}, \quad \omega_x := \omega|_{\gamma_x}, \quad \omega_t := \omega|_{\gamma_t}, \quad (40)$$

where D_x, D_t denote total derivatives with respect to x, t .

For any non-stretching curve flow, these structure equations (38)–(40) encode an explicit pair of Hamiltonian and symplectic operators once a specific choice of frame along $\gamma(t, x)$ is made. For the case of curve flows in two-dimensional and three-dimensional Riemannian spaces, see Refs. 3, 6. A proof for the general case of curve flows in a general Riemannian symmetric space $M = G/H$ is given in Ref. 15. See Refs. 20, 21, 22, 23 for a related, more abstract formulation.

As shown in Ref. 15, a natural moving parallel frame can be defined by the following two properties which are a direct algebraic generalization of a moving parallel frame in Euclidean geometry:¹

- (i) e_x is a constant unit-norm element e belonging to a Cartan subspace $\mathfrak{a} \subset \mathfrak{m}$, i.e. $D_x e_x = D_t e_x = 0$, $\langle e_x, e_x \rangle = -1$.
- (ii) ω_x belongs to the perp space $\mathfrak{h}_\perp$ of the Lie subalgebra $\mathfrak{h}_\parallel \subset \mathfrak{h}$ of the linear isotropy group $H_\parallel^* \subset H^* = \text{Ad}(H)$ that preserves e_x , i.e. $\langle \mathfrak{h}_\parallel, \omega_x \rangle = 0$.

A moving frame satisfying properties (i) and (ii) is called *H-parallel* and its existence can be established by constructing¹⁵ a suitable gauge transformation on an arbitrary frame at each point x along the curve.

Through property (i), the set of curve flows $\gamma(t, x)$ in $M = G/H$ can be divided into algebraic equivalence classes defined by the orbit of the element $e_x = e$ in $\mathfrak{a} \subset \mathfrak{m}$ under the action of the gauge group $H^* = \text{Ad}(H)$.

4. Bi-Hamiltonian Structure and a Hierarchy of $U(n-1)$ -Invariant mKdV and SG Flows

In a general symmetric space $M = G/H$, the Cartan structure equations (38)–(40) yield a flow equation on the components of the Cartan connection along the curve, where the flow is specified by the perp component of γ_t . To write down the Hamiltonian and symplectic operators appearing in the flow equation, we use the following notation:

$$e_x = e \in \mathfrak{a} \subset \mathfrak{m}_\parallel, \quad (41)$$

$$e_t = h_\parallel + h_\perp \in \mathfrak{m}_\parallel \oplus \mathfrak{m}_\perp, \quad (42)$$

$$\omega_t = \varpi^\parallel + \varpi^\perp \in \mathfrak{h}_\parallel \oplus \mathfrak{h}_\perp, \quad (43)$$

$$\omega_x = u \in \mathfrak{h}_\perp. \quad (44)$$

Also we write

$$h^\perp = \text{ad}(e_x)h_\perp \in \mathfrak{h}_\perp. \quad (45)$$

Then we have the following theorem from Ref. 15.

Theorem 4.1. *The Cartan structure equations (38)–(39) for any H -parallel linear coframe e and linear connection ω pulled back to the two-dimensional surface $\gamma(t, x)$ in $M = G/H$ yield the flow equation*

$$u_t = \mathcal{H}(\varpi^\perp) + h^\perp, \quad \varpi^\perp = \mathcal{J}(h^\perp) \quad (46)$$

where

$$\mathcal{H} = \mathcal{K}|_{\mathfrak{h}_\perp}, \quad \mathcal{J} = -\text{ad}(\mathbf{e})^{-1} \mathcal{K}|_{\mathfrak{m}_\perp} \text{ad}(\mathbf{e})^{-1} \quad (47)$$

are compatible Hamiltonian and symplectic operators that act on $\mathfrak{h}_\perp$ -valued functions and that are invariant under $H_\parallel^*$, as defined in terms of the linear operator

$$\mathcal{K} := D_x + [u, \cdot]_\perp - [u, D_x^{-1}[u, \cdot]_\parallel]. \quad (48)$$

These operators arise directly from projecting the Cartan structure equations (38)–(40) into the parallel and perp subspaces of $\mathfrak{h}$ and $\mathfrak{m}$, which yields

$$D_x h_\parallel + [u, h_\perp]_\parallel = 0, \quad (49)$$

$$D_x h_\perp + [u, h_\parallel] + [u, h_\perp]_\perp - [\varpi^\perp, \mathbf{e}] = 0, \quad (50)$$

and

$$D_x \varpi^\parallel + [u, \varpi^\perp]_\parallel = 0, \quad (51)$$

$$D_x \varpi^\perp - u_t + [u, \varpi^\parallel] + [u, \varpi^\perp]_\perp + h^\perp = 0. \quad (52)$$

We will essentially use these equations (49)–(52) throughout the paper.

As shown in Ref. 15, there are two natural flows that each give rise to a group-invariant integrable system. One flow is generated by the x -translation symmetry of the Hamiltonian and symplectic operators, which yields a group-invariant mKdV equation. The other flow is defined by the kernel of symplectic operator, which produces a group-invariant SG equation. These two equations are at the bottom of a hierarchy of higher-order integrable systems.

Theorem 4.2. *Composition of the operators $\mathcal{H}$ and $\mathcal{J}$ yields a recursion operator $\mathcal{R} = \mathcal{H}\mathcal{J}$ that produces a hierarchy of $H_\parallel^*$ -invariant flows (46) on u starting from the flow*

$$h^\perp = u_x \quad (53)$$

which gives an integrable group-invariant mKdV equation. The kernel of the recursion operator $\mathcal{R}$ yields a further $H_\parallel^*$ -invariant flow (46) on u defined by

$$\mathcal{J}(h^\perp) = 0. \quad (54)$$

This flow gives an integrable group-invariant SG equation after an algebraic reduction is made.

The integrability structure of these two flows is shown in detail in Ref. 15.

5. Bi-Hamiltonian Flow Equations in $Sp(n)/U(n)$

We will now derive the $U(n-1)$ -invariant mKdV flow and $U(n-1)$ -invariant SG flow in the Hermitian symmetric space $M = Sp(n)/U(n)$. Employing the notation and preliminaries in Sec. 2.1, we consider a non-stretching curve flow $\gamma(t, x)$ that has a $U(n)$ -parallel framing along γ given as

$$e = \frac{1}{\sqrt{\chi}}(1, 0) \in \mathbb{R} \oplus \mathfrak{s}(n-1, \mathbb{C}) \simeq \mathfrak{m}_{\parallel}, \quad \chi = 8(n+1) \quad (55)$$

$$u = (u, \mathbf{u}) \in i\mathbb{R} \oplus \mathbb{C}^{n-1} \simeq \mathfrak{h}_{\perp}, \quad (56)$$

and

$$h_{\parallel} = (h_{\parallel}, \mathbf{H}_{\parallel}) \in \mathbb{R} \oplus \mathfrak{s}(n-1, \mathbb{C}) \simeq \mathfrak{m}_{\parallel}, \quad (57)$$

$$h_{\perp} = (h_{\perp}, \mathbf{h}_{\perp}) \in i\mathbb{R} \oplus \mathbb{C}^{n-1} \simeq \mathfrak{m}_{\perp}, \quad (58)$$

$$\varpi^{\parallel} = (\overline{\mathbf{W}}^{\parallel}) \in \mathfrak{u}(n-1) \simeq \mathfrak{h}_{\parallel}, \quad (59)$$

$$\varpi^{\perp} = (w^{\perp}, \mathbf{w}^{\perp}) \in i\mathbb{R} \oplus \mathbb{C}^{n-1} \simeq \mathfrak{h}_{\perp}, \quad (60)$$

as well as

$$h^{\perp} = (h^{\perp}, \mathbf{h}^{\perp}) = \text{ad}(e)h_{\perp} = \frac{1}{\sqrt{\chi}}(2h_{\perp}, -\overline{\mathbf{h}}_{\perp}) \in i\mathbb{R} \oplus \mathbb{C}^{n-1} \simeq \mathfrak{h}_{\perp} \quad (61)$$

using the matrix identifications (20)–(21), where $u, h_{\perp}, w^{\perp}, h^{\perp} \in i\mathbb{R}$ are imaginary (complex) scalar variables, $h_{\parallel} \in \mathbb{R}$ is a real scalar variable, $\mathbf{u}, \mathbf{h}_{\perp}, \mathbf{w}^{\perp}, \mathbf{h}^{\perp} \in \mathbb{C}^{n-1} \in \mathbb{C}^{n-1}$ are complex vector variables, $\mathbf{H}_{\parallel} \in \mathfrak{s}(n-1, \mathbb{C})$ is complex symmetric matrix variable.

We remark that, since the rank of the space $M = Sp(n)/U(n)$ is n , then for $n \geq 2$ the element (55) belonging to the Cartan subspace of $\mathfrak{m} = \mathfrak{sp}(n)/\mathfrak{u}(n)$ determines one particular algebraic equivalence class of non-stretching curve flows in which the tangent vector γ_x of the curve is identified with the orbit of this element under the action of the gauge group $H_{\parallel}^* = \text{Ad}(H)$ of the $U(n)$ -parallel frame.

In terms of the variables (55)–(61), the Cartan structure equations (49)–(50) and (51)–(52) are respectively given by

$$h_{\parallel x} + 2h_{\perp}u + \mathbf{h}_{\perp}u^t + \overline{\mathbf{h}}_{\perp}\overline{u}^t = 0, \quad (62)$$

$$\mathbf{H}_{\parallel x} - \mathbf{h}_{\perp}^t\overline{u} - \overline{u}^t\mathbf{h}_{\perp} = 0, \quad (63)$$

$$h_{\perp x} + 2uh_{\parallel} + \mathbf{h}_{\perp}u^t - \overline{\mathbf{h}}_{\perp}\overline{u}^t - 2w^{\perp} = 0, \quad (64)$$

$$\mathbf{h}_{\perp x} + u\mathbf{H}_{\parallel} - h_{\parallel}\overline{u} + u_{\perp}\mathbf{h}_{\perp} - h_{\perp}\overline{u} + \overline{\mathbf{w}}^{\perp} = 0, \quad (65)$$

and

$$\mathbf{W}_x^{\parallel} + \overline{\mathbf{w}}^{\perp t}u - \overline{u}^t\mathbf{w}^{\perp} = 0, \quad (66)$$

$$w_x^{\perp} - u_t - u\overline{\mathbf{w}}^{\perp t} + \mathbf{w}^{\perp}\overline{u}^t + 2h_{\perp} = 0, \quad (67)$$

$$\mathbf{w}_x^{\perp} - u_t + u\mathbf{W}^{\parallel} + u\mathbf{w}^{\perp} - \mathbf{w}^{\perp}u - \overline{\mathbf{h}}_{\perp} = 0. \quad (68)$$

Writing these equations (62)–(68) in the operator form (46), we obtain the flow equation

$$\begin{pmatrix} \mathbf{u} \\ \mathbf{u} \end{pmatrix}_t = \mathcal{H} \begin{pmatrix} \mathbf{w}^\perp \\ \mathbf{w}^\perp \end{pmatrix} + \begin{pmatrix} \mathbf{h}^\perp \\ \mathbf{h}^\perp \end{pmatrix}, \quad \begin{pmatrix} \mathbf{w}^\perp \\ \mathbf{w}^\perp \end{pmatrix} = \mathcal{J} \begin{pmatrix} \mathbf{h}^\perp \\ \mathbf{h}^\perp \end{pmatrix}, \quad (69)$$

in terms of the Hamiltonian operator

$$\mathcal{H} = \begin{pmatrix} D_x & i2\text{Im}\langle \mathbf{u}, \rangle \\ -\mathbf{u} & D_x + \mathbf{u} + \mathbf{u}D_x^{-1}\mathbf{u}\overline{\wedge} \end{pmatrix} \quad (70)$$

and the symplectic operator

$$\mathcal{J} = \begin{pmatrix} \frac{1}{4}D_x - \mathbf{u}D_x^{-1}\mathbf{u} & i\text{Im}\langle \mathbf{u}, \rangle + 2\mathbf{u}D_x^{-1}\text{Re}\langle \mathbf{u}, \rangle \\ -\frac{1}{2}\mathbf{u} - \mathbf{u}D_x^{-1}\mathbf{u} & D_x - \mathbf{u} + 2\mathbf{u}D_x^{-1}\text{Re}\langle \mathbf{u}, \rangle + \overline{\mathbf{u}}D_x^{-1}\overline{\mathbf{u}}\odot \end{pmatrix}. \quad (71)$$

5.1. mKdV flow

The mKdV flow is produced by the x -translation generator

$$\begin{pmatrix} \mathbf{h}^\perp \\ \mathbf{h}^\perp \end{pmatrix} = \begin{pmatrix} \mathbf{u}_x \\ \mathbf{u}_x \end{pmatrix} = \frac{1}{\sqrt{\chi}} \begin{pmatrix} 2\mathbf{h}_\perp \\ -\overline{\mathbf{h}}_\perp \end{pmatrix}. \quad (72)$$

Substitution of this expression into the flow equation (69) yields an integrable mKdV system for the variables $(\mathbf{u}, \mathbf{u})$:

$$\begin{aligned} \mathbf{u}_t - \chi^{-1}\mathbf{u}_x &= \frac{1}{4}\mathbf{u}_{xxx} - \frac{3}{2}\mathbf{u}^2\mathbf{u}_x + i3\text{Im}\langle \mathbf{u}, \mathbf{u}_{xx} \rangle, \\ \mathbf{u}_t - \chi^{-1}\mathbf{u}_x &= \mathbf{u}_{xxx} + \frac{3}{2}(|\mathbf{u}|^2 - \frac{1}{2}\mathbf{u}^2)_x\mathbf{u} + 3(|\mathbf{u}|^2 - \frac{1}{2}\mathbf{u}^2)\mathbf{u}_x - \frac{3}{2}\mathbf{u}_x\mathbf{u}_x - \frac{3}{4}\mathbf{u}_{xx}\mathbf{u}, \end{aligned} \quad (73)$$

where we have rescaled $t \rightarrow t/\chi$, for convenience.

This system (73) has a bi-Hamiltonian structure and exhibits invariance under the unitary group $U(n-1)$ acting on $\mathbf{u}$ and $\mathbf{u}$ by the transformations $\text{Ad}(R)(\mathbf{u}, \mathbf{u}) = (\mathbf{u}, \mathbf{u}R^{-1})$ for all matrices $R \in H_\parallel = U(n-1)$.

5.2. SG flow

The SG flow is defined by

$$0 = \begin{pmatrix} \mathbf{w}^\perp \\ \mathbf{w}^\perp \end{pmatrix} = \mathcal{J} \begin{pmatrix} \mathbf{h}^\perp \\ \mathbf{h}^\perp \end{pmatrix}. \quad (74)$$

This yields the flow equation

$$\begin{pmatrix} \mathbf{u}_t \\ \mathbf{u}_t \end{pmatrix} = \begin{pmatrix} \mathbf{h}^\perp \\ \mathbf{h}^\perp \end{pmatrix} = \frac{1}{\sqrt{\chi}} \begin{pmatrix} 2\mathbf{h}_\perp \\ -\overline{\mathbf{h}}_\perp \end{pmatrix} \quad (75)$$

with $(\mathbf{h}_\perp, \mathbf{h}_\perp)$ satisfying

$$\mathbf{h}_{\perp x} + 2\mathbf{u}\mathbf{h}_\parallel + \mathbf{h}_\perp\mathbf{u}^t - \overline{\mathbf{h}}_\perp\overline{\mathbf{u}}^t = 0, \quad (76)$$

$$\mathbf{h}_{\perp x} + \mathbf{u} \mathbf{H}_{\parallel} - \mathbf{h}_{\parallel} \bar{\mathbf{u}} + \mathbf{u}_{\perp} \mathbf{h}_{\perp} - \mathbf{h}_{\perp} \bar{\mathbf{u}} = 0, \quad (77)$$

where $\mathbf{h}_{\parallel}$ and $\mathbf{H}_{\parallel}$ are determined by equations (62)–(63).

Similarly to the method^{9, 10, 13, 14, 24} used to derive SG flows in other symmetric spaces, we seek a local expression for $\mathbf{h}_{\parallel}$ and $\mathbf{H}_{\parallel}$ through an algebraic reduction of the forms

$$\mathbf{H}_{\parallel} = \alpha \mathbf{h}_{\perp}^{\mathbf{t}} \mathbf{h}_{\perp} \in \mathfrak{s}(n-1, \mathbb{C}), \quad (78)$$

with some coefficient $\alpha(\mathbf{h}_{\parallel}, \mathbf{h}_{\perp}) \in \mathbb{C}$. Notice that, under gauge transformations (34), it is precisely the components $\mathbf{H}_{\parallel}$ and $\mathbf{h}_{\perp}$ that change while the components $\mathbf{h}_{\parallel}$ and $\mathbf{h}_{\perp}$ are invariant. This observation provides a geometrical motivation for the form of the reduction (78).

To find the coefficient α , we substitute expression (78) into equation (63) and use equations (76) and (77) to eliminate x derivatives of $\mathbf{h}_{\perp}$, $\mathbf{h}_{\perp}$. Then since the matrix $\mathbf{H}_{\parallel}$ is symmetric, we expand equation (63) as a linear combination of the symmetric matrices $\mathbf{h}_{\perp}^{\mathbf{t}} \mathbf{h}_{\perp}$ and $\bar{\mathbf{u}}^{\mathbf{t}} \mathbf{h}_{\perp} + \mathbf{h}_{\perp}^{\mathbf{t}} \bar{\mathbf{u}}$. Putting their coefficients to zero, we obtain

$$\alpha_x - 2\alpha^2(\mathbf{h}_{\perp} \mathbf{u}^{\mathbf{t}}) - 2\alpha \mathbf{u} = 0, \quad (79)$$

$$\alpha \mathbf{h}_{\parallel} + \alpha \mathbf{h}_{\perp} - 1 = 0. \quad (80)$$

From equation (80) we obtain

$$\alpha = \frac{1}{\mathbf{h}_{\parallel} + \mathbf{h}_{\perp}} = \frac{\mathbf{h}_{\parallel} - \mathbf{h}_{\perp}}{\mathbf{h}_{\parallel}^2 + |\mathbf{h}_{\perp}|^2} \in \mathbb{C}. \quad (81)$$

The remaining equation (79) is then just a consistency condition between equation (81) for α and the equations (76)–(77), which holds identically.

Hence we have found an expression for $\mathbf{H}_{\parallel}$ as a function of $\mathbf{h}_{\parallel}$, $\mathbf{h}_{\perp}$:

$$\mathbf{H}_{\parallel} = \frac{1}{\mathbf{h}_{\parallel} + \mathbf{h}_{\perp}} \mathbf{h}_{\perp}^{\mathbf{t}} \mathbf{h}_{\perp} = \frac{\mathbf{h}_{\parallel} - \mathbf{h}_{\perp}}{\mathbf{h}_{\parallel}^2 + |\mathbf{h}_{\perp}|^2} \mathbf{h}_{\perp}^{\mathbf{t}} \mathbf{h}_{\perp}. \quad (82)$$

Next we find $\mathbf{h}_{\parallel}$ as a function of $\mathbf{h}_{\perp}$, $\mathbf{h}_{\perp}$. To do that we use the conservation law

$$D_x(\mathbf{h}_{\parallel}^2 + |\mathbf{H}_{\parallel}|^2 - \mathbf{h}_{\perp}^2 + 2|\mathbf{h}_{\perp}|^2) = 0 \quad (83)$$

which is admitted by the system of equations (76)–(77) and (62)–(63), where

$$|\mathbf{H}_{\parallel}|^2 := \text{tr}(\mathbf{H}_{\parallel} \bar{\mathbf{H}}_{\parallel}) = |\alpha|^2 |\mathbf{h}_{\perp}|^4 = \frac{|\mathbf{h}_{\perp}|^4}{\mathbf{h}_{\parallel}^2 + |\mathbf{h}_{\perp}|^2}, \quad (84)$$

$$|\alpha|^2 = \alpha \bar{\alpha} = \frac{1}{|\mathbf{h}_{\parallel} + \mathbf{h}_{\perp}|^2} = \frac{1}{\mathbf{h}_{\parallel}^2 + |\mathbf{h}_{\perp}|^2}. \quad (85)$$

This conservation law is also given by $D_x(\langle h, h \rangle) = 0$ in which $h = \mathbf{h}_{\parallel} + \mathbf{h}_{\perp}$ is given by equations (57) and (58).

Substitution of expressions (84) and (85) into the conservation law (83) gives

$$|\alpha|^{-2} + |\alpha|^2 |\mathbf{h}_\perp|^4 + 2|\mathbf{h}_\perp|^2 = 1. \quad (86)$$

Solving this quadratic equation for $|\alpha|^2$, we obtain

$$|\alpha|^{-2} = -(|\mathbf{h}_\perp|^2 - \frac{1}{2}) \pm \frac{1}{2} \sqrt{1 - 4|\mathbf{h}_\perp|^2}. \quad (87)$$

Then from equations (81) and (85) we derive the expression

$$h_\parallel = \pm \sqrt{|\alpha|^{-2} - |\mathbf{h}_\perp|^2}. \quad (88)$$

Finally, we take the x derivative of the flow equation (75) and substitute $h_\parallel$ and $\mathbf{H}_\parallel$ given by the equations (88) and (82). Thus we get the following hyperbolic system for the variables $(\mathbf{u}, \mathbf{u})$:

$$\begin{aligned} \mathbf{u}_{tx} &= -4\mathbf{u}A + i4\text{Im}\langle \mathbf{u}_t, \mathbf{u} \rangle, \\ \mathbf{u}_{tx} &= -|\mathbf{u}|^2 \frac{A - \frac{1}{2}\mathbf{u}_t}{A^2 + \frac{1}{4}|\mathbf{u}_t|^2} \mathbf{u}_t + A\mathbf{u} - \mathbf{u}\mathbf{u}_t - \frac{1}{2}\mathbf{u}_t\mathbf{u}, \end{aligned} \quad (89)$$

in which

$$A := h_\parallel = \pm \frac{1}{\sqrt{2}} \sqrt{1 - 2|\mathbf{u}_t|^2 + \frac{1}{4}\mathbf{u}_t^2 \pm \sqrt{1 - 4|\mathbf{u}_t|^2}}. \quad (90)$$

This system (89) is invariant under the unitary group $U(n-1)$, which acts on $\mathbf{u}$ and $\mathbf{u}$ by the transformations $\text{Ad}(R)(\mathbf{u}, \mathbf{u}) = (\mathbf{u}, \mathbf{u}R^{-1})$ for all matrices $R \in H_\parallel = U(n-1)$.

6. Conclusion

The Hermitian symmetric space $Sp(n)/U(n)$ is one of few such spaces listed in the classification of symmetric spaces.¹⁶ By adapting the moving parallel frame method developed in Ref. 15 to derive group-invariant bi-Hamiltonian integrable systems from non-stretching geometric curve flows in Riemannian symmetric spaces, we have obtained new integrable mKdV and SG systems for coupled scalar-complex vector variables. These systems are invariant under the unitary group $U(n-1)$ and have a bi-Hamiltonian structure.

Our derivation makes use of the complex structure of the symmetric space $Sp(n)/U(n)$ in an essential way to formulate a natural complex matrix representation for the spaces $\mathfrak{sp}(n)$ and $\mathfrak{u}(n)$ as well as for their Lie bracket relations. This approach resolves the problem of how to express the usual real matrix representation for $\mathfrak{sp}(n)$ and $\mathfrak{u}(n)$ in a coupled complex form that transforms properly under the unitary gauge group $U(n)$. The same method can be applied to quaternionic symmetric spaces as well.

Another problem that we have addressed is how to carry out systematically the algebraic reduction needed to get a SG system from the underlying non-stretching curve flow.^{9, 10, 13, 14, 24} We show that this reduction has a very simple geometrical formulation by using the projection of the parallel part of the curve flow into the

complement of tangent direction along the curve. This method is motivated by our observation that the components given by this projection are invariant under gauge transformations on the parallel frame along the curve. We are currently extending this geometrical reduction for SG curve flows to general symmetric spaces.²⁵

In a different direction, we plan to look at how to extend the work in Ref. 15 to derive nonlinear Schrodinger systems (complex and quaternionic) from non-stretching curve flows in symmetric spaces with Hermitian or quaternionic structures. Work is underway using examples of low-dimensional symmetric spaces.²⁶

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