Sivers Function in the Quasi-Classical Approximation

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In this brief article we summarize the derivation of the Sivers function of a heavy nucleus, which we obtain by generalizing the quasi-classical McLerran-Venugopalan model to incorporate the role of spin and orbital angular momentum. In doing so we obtain a new channel which is capable of generating the Sivers function of the nucleus from the orbital motion of its nucleons. An essential role is played in this channel by the multiple rescatterings on spectator nucleons which screen the distribution function during the initial- or final-state interactions. The combination of orbital angular momentum together with multiple rescattering yields a new interpretation of the sign flip relation between the Sivers function measured in semi-inclusive deep inelastic scattering and the Drell-Yan process.

Keywords: TMD; orbital angular momentum; MV model.

1. Quasi-Classical Factorization in a Heavy Nucleus

In this brief article we summarize the derivation of the Sivers function¹ in the quasi-classical limit, emphasizing the main physical messages and key results. We refer the interested reader to our recent paper [2] for the technical details.

The saturation formalism is a powerful tool to study QCD processes at high energies, yielding considerable theoretical simplifications when the transverse density of color charges in a hadronic system is large. The McLerran-Venugopalan (MV) model of a heavy nucleus³ makes this regime accessible by introducing a large external parameter that controls these high-density effects: the typical number $A^{1/3}$ of nucleons at a given impact parameter. The leading high-density effects must be re-summed when $\alpha_s^2 A^{1/3} \sim \mathcal{O}(1)$, in which regime the interaction with the classical gluon field of the heavy nucleus dominates.⁴ By generalizing the MV model of a heavy nucleus to incorporate spin and orbital angular momentum, we can study

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semi-inclusive deep inelastic scattering (SIDIS) and the Drell-Yan process (DY) in Bjorken kinematics, allowing us to access the transverse-momentum-dependent quark distributions they probe in the quasi-classical limit. Throughout this article we will use the choice of coordinates and notation

$$p^\pm \equiv \frac{1}{\sqrt{2g_{+-}}} (p^0 \pm p^3) \quad \underline{p} \equiv (p_\perp^1, p_\perp^2). \quad (1)$$

We will analyze the case of SIDIS in some detail, with the generalization to DY being straightforward. The quasi-classical approximation results in a “quasi-classical factorization” of the SIDIS cross-section $\gamma^* + A \rightarrow q + X$ which reads²

$$\begin{aligned} \frac{d\sigma^{\gamma^*+A \rightarrow q+X}}{d^2k dy} &= A \int \frac{d^{2+}p db^-}{2(2\pi)^3} \int d^2x d^2y \sum_\sigma W_\sigma(p, b^-, \frac{x+y}{2}) \\ &\times \int \frac{d^2k'}{(2\pi)^2} e^{-i(\underline{k}-\underline{k}') \cdot (\underline{x}-\underline{y})} \frac{d\hat{\sigma}^{\gamma^*+N_\sigma \rightarrow q+X}}{d^2k' dy}(p, q) D_{\underline{x}\underline{y}}[\infty^-, b^-]. \end{aligned} \quad (2)$$

where the Wigner distribution of a nucleon with polarization σ inside the nuclear wave function is given by

$$W_\sigma(\bar{p}, b) \equiv \frac{g_{+-}}{(2\pi)^3} \int d^{2+}(\delta p) e^{-i\delta p \cdot b} \frac{\psi_\sigma(p)}{\sqrt{p^+}} \frac{\psi_\sigma(p')}{\sqrt{p'^+}} \quad (3)$$

with $\bar{p} \equiv \frac{1}{2}(p + p')$ and $\delta p \equiv p - p'$, and

$$D_{\underline{x}\underline{y}}[\infty^-, b^-] \equiv \left\langle \frac{1}{N_c} \text{Tr} \left(V_{\underline{x}}[\infty^-, b^-] V_{\underline{y}}^\dagger[\infty^-, b^-] \right) \right\rangle \quad (4)$$

is a dipole trace of two future-pointing (for SIDIS) semi-infinite Wilson lines in the fundamental representation, averaged over the wave functions of the spectator nucleons. The factorization formula (2) is illustrated in Fig. 1.

Eq. (2) is derived in the kinematic limit ($s, Q^2 \gg k_T^2, Q_s^2 \gg \Lambda_{QCD}^2$), for which the formalism of transverse-momentum-dependent (TMD) factorization^{5,6} relates the cross-sections to the TMD correlator

$$\Phi_{ij}^A(x, \underline{k}; P, S) \equiv \frac{g_{+-}}{(2\pi)^3} \int d^2-r e^{i \underline{k} \cdot \underline{r}} \langle A(P, S) | \bar{\psi}_j(0) \mathcal{U}[0, r] \psi_i(r) | A(P, S) \rangle_{r^+=0}, \quad (5)$$

of quarks inside the nucleus, where $\underline{k} \cdot \underline{r} = (xP^+) g_{+-} r^- - \underline{k} \cdot \underline{r}$, $\mathcal{U}$ is the gauge link, and a similar expression holds for the correlator ϕ_{ij}^N of quarks inside the nucleon. This allows us to rewrite (2) as a factorization of the nuclear TMD correlator Φ^A in terms of the nucleonic TMD correlator ϕ^N :

$$\begin{aligned} \Phi_{ij}^A(\bar{x}, \underline{k}; P, S) &= \frac{A}{2(2\pi)^5} \int d^{2+}p db^- d^2x d^2y d^2k' \\ &\times \sum_\sigma W_\sigma(p, b^-, \frac{x+y}{2}) e^{-i(\underline{k}-\underline{k}') \cdot (\underline{x}-\underline{y})} \\ &\times \phi_{ij}^N(x, \underline{k}' - x \underline{p}; p, \sigma) D_{\underline{x}\underline{y}}[\infty^-, b^-]. \end{aligned} \quad (6)$$

2. Nuclear Sivers Function in Semi-Inclusive Deep Inelastic Scattering

$$\frac{1}{9}\text{Tr}[\Phi^A(\bar{x}, \underline{k}; P, S) \gamma^+] = f_1^A(\bar{x}, k_T) + \frac{1}{M_A} (\underline{S} \times \underline{k}) f_{1T}^A(\bar{x}, k_T), \quad (7)$$

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$$\begin{aligned}
(\underline{S} \times \underline{k}) f_{1T}^{\perp A}(\bar{x}, k_T) &= \frac{1}{2} \left(\frac{M_A A}{2(2\pi)^5} \right) \int d^{2+}p db^- d^2x d^2y d^2k' \\
&\times \sum_{\sigma} W_N^{\sigma}(p, b^-, \frac{x+y}{2}) e^{-i(\underline{k}-\underline{k}') \cdot (\underline{x}-\underline{y})} \left[f_1^N(x, |\underline{k}' - x \underline{p}|_T) \right. \\
&+ \frac{1}{m_N} (\underline{\sigma} \times (\underline{k}' - x \underline{p})) f_{1T}^{\perp N}(x, |\underline{k}' - x \underline{p}|_T) \left. \right] \\
&\times D_{\underline{xy}}[\infty^-, b^-] - (\underline{k} \rightarrow -\underline{k}). \tag{8}
\end{aligned}$$

There are thus two channels which may generate the nuclear Siverts function: one which couples to the unpolarized TMD f_1^N of the nucleons and another which couples to the Siverts function $f_{1T}^{\perp N}$ of the nucleons. These couple to different parts of the Wigner distribution, corresponding to unpolarized and transversely-polarized nucleons, respectively:

$$\sum_{\sigma} W_N^{\sigma}(p, b) \equiv W_{unp}(p, b) \tag{9}$$

$$\sum_{\sigma} W_N^{\sigma}(p, b) \underline{\sigma} \equiv \frac{1}{A} W_{trans}(p, b) \underline{S}, \tag{10}$$

which are normalized as

$$\int \frac{d^{2+}p d^{2-}b}{2(2\pi)^3} W_{unp}(p, b) = 1, \quad \int \frac{d^{2+}p d^{2-}b}{2(2\pi)^3} W_{trans}(p, b) = 1, \tag{11}$$

so that $A \int \frac{d^{2+}p d^{2-}b}{2(2\pi)^3} W_{\sigma}(p, b)$ corresponds to the total number of nucleons with polarization σ . By also changing variables $(\underline{k}' \rightarrow -\underline{k}')$, $(\underline{p} \rightarrow -\underline{p})$, $(\underline{x} \leftrightarrow \underline{y})$ in the term which is subtracted, we can impose definite symmetries on each of the factors in the f_1^N and $f_{1T}^{\perp N}$ channels. At the leading order in $A^{-1/3} \ll 1$, the dipole trace representing the multiple scattering is manifestly symmetric:⁷

$$\begin{aligned}
D_{\underline{xy}}[\infty^-, b^-] &\approx S_{\underline{xy}}[\infty^-, b^-] \equiv \frac{1}{2} \left(D_{\underline{xy}}[\infty^-, b^-] + D_{\underline{yx}}[\infty^-, b^-] \right) \\
&\approx \exp \left[-\frac{1}{4} |x - y|_T^2 Q_s^2 \left(\frac{x+y}{2} \right) \left(\frac{R^- - b^-}{2R^-} \right) \ln \frac{1}{|x - y|_T \Lambda} \right], \tag{12}
\end{aligned}$$

where R^- is the radius of the Lorentz-contracted nucleus. The b^- -dependent factor in parentheses indicates that the rescattering occurs on a fraction of nucleons which depends on the depth at which the quark is struck; if the quark is knocked out of the active nucleon at the front of the nucleus ($b^- = -R^-$), a maximal amount of rescattering occurs, but if it is knocked out at the back of the nucleus ($b^- = +R^-$), then no rescattering occurs.

Imposing the (anti)symmetrization described above yields

$$(\underline{S} \times \underline{k}) f_{1T}^{\perp A}(\bar{x}, k_T) = M_A \int \frac{d^{2+}p db^-}{2(2\pi)^5} d^2x d^2y d^2k' e^{-i(\underline{k}-x \underline{p}-\underline{k}') \cdot (\underline{x}-\underline{y})}$$

$$\begin{aligned}
 & \times \left\{ A W_{unp}^{OAM} (p^+, \underline{p}, b^-, \frac{x+y}{2}) f_1^N(x, k'_T) \right. \\
 & \left. + \frac{1}{m_N} (\underline{S} \times \underline{k}') W_{trans}^{symm} (p^+, \underline{p}, b^-, \frac{x+y}{2}) f_{1T}^{\perp N}(x, k'_T) \right\} \\
 & \times S_{\underline{xy}}[\infty^-, b^-]
 \end{aligned} \tag{13}$$

where we have shifted the momentum $\underline{k}' \rightarrow \underline{k}' + x\underline{p}$ and identified the (anti)symmetry under ($\underline{p} \rightarrow -\underline{p}$) of the parts of the Wigner distribution to which f_1^N and $f_{1T}^{\perp N}$ couple:

$$W^{(symm)}(p, b) \equiv \frac{1}{2} [W(p, b) \pm (p \rightarrow -p)]. \tag{14}$$

Having imposed the symmetry constraints explicitly, we can interpret the two channels that can generate the nuclear Sivers function:

- A “Transversity Channel” ($W_{trans}^{symm} \otimes f_{1T}^{\perp N} \otimes S_{\underline{xy}}[\infty^-, b^-]$) that trivially constructs $f_{1T}^{\perp A}$ from $f_{1T}^{\perp N}$. This channel couples to the $\underline{p}$ -symmetric distribution of polarized nucleons in the nucleus, and the multiple scattering broadens the k_T distribution of produced quarks, but plays no essential role.
- An “Orbital Angular Momentum Channel” ($W_{unp}^{OAM} \otimes f_1^N \otimes S_{\underline{xy}}[\infty^-, b^-]$) that mixes the nucleonic f_1^N distribution into the nuclear Sivers function $f_{1T}^{\perp A}$. The transverse momentum asymmetry originates from the net orbital motion of the nucleons in the Wigner distribution (14), and the multiple scattering in this channel is essential to generate the Sivers function.

To understand the identification of the transverse motion of the nucleons with orbital angular momentum (OAM), consider the distribution W_{unp} of unpolarized nucleons in the rest frame. Because of parity and time-reversal invariance, the Wigner distribution satisfies

$$W_{unp}(\vec{p}, \vec{b}, \vec{S}) = W_{unp}(\vec{p}^2, \vec{b}^2, (\vec{p} \cdot \vec{b})^2, (\vec{b} \times \vec{p}) \cdot \vec{S}). \tag{15}$$

The part of W_{unp} which is antisymmetric under $\underline{p} \rightarrow -\underline{p}$ is therefore proportional to $b_z(\underline{p} \times \underline{S})$ and arises from the coupling of the orbital angular momentum $\vec{L} = \vec{b} \times \vec{p}$ to the nuclear spin $\vec{S}$. However, this also implies that the preferred direction of $\underline{p}$ arising from OAM reverses under $b_z \rightarrow -b_z$ ($b^- \rightarrow -b^-$ in the boosted frame). Thus, without multiple scattering, the net preferred direction coming from OAM at the front of the nucleus is exactly cancelled by the preferred direction from OAM at the back of the nucleus:

$$\int db^- W_{unp}^{OAM}(p, b) = 0; \tag{16}$$

this feature is consistent with the well-known fact^{8–9} that the Sivers function vanishes when the gauge link representing multiple rescattering is neglected. However, when the multiple scattering factor (12) is included, the depth (b^-) dependence

Fig. 2. Illustration of the OAM channel for generating the nuclear Siverts function. Spin-orbit coupling ($\vec{L} \cdot \vec{S}$) generates orbital motion of the nucleons, but the net asymmetry of produced quarks cancels if the knockout is equally likely to occur at any depth in the nucleus. Multiple scattering breaks this front/back symmetry, making SIDIS most sensitive to orbital motion near the back of the nucleus.

breaks this front-back symmetry. For a given $k_T \sim 1/|x-y|_T$, the closer to the front of the nucleus ($b^- \rightarrow -R^-$) the knockout occurs, the greater the suppression of the quark distribution; this reflects the fact that a quark knocked out from the front of the nucleus experiences the maximum amount of multiple scattering. Therefore multiple scattering in SIDIS biases the knockout to occur closer to the *back* of the nucleus, making it most sensitive to the nucleon orbital motion there and generating a net preferred direction for the nuclear Siverts function. These features are illustrated in Fig. 2.

3. Extension to Drell-Yan and Other Applications

In the same way as was done for SIDIS in (2) and (6), we can derive an analogous quasi-classical factorization formula for the Drell-Yan process. Taking the projectile as a single antiquark, this equation decomposes the $\bar{q} + A \rightarrow \gamma^* + X$ cross-section on a heavy nucleus in terms of the $\bar{q} + N \rightarrow \gamma^* + X$ cross-section on one of its nucleons. Equivalently, this can be formulated in the language of TMD factorization in terms of the corresponding TMD quark correlators,

$$\begin{aligned} \Phi_{ij}^A(\bar{x}, \underline{q}; P, S) &= \frac{A}{2(2\pi)^5} \int d^2+ p \, db^- \, d^2 x \, d^2 y \, d^2 k' \, e^{-i \underline{k}' \cdot (\underline{x} - \underline{y})} \sum_{\sigma} W_{\sigma}(p, b^-, \frac{x+y}{2}) \\ &\times \phi_{ij}^N(x, \underline{q} - \underline{k}' - x \underline{p}; p, \sigma) D_{\underline{y} \underline{x}}[b^-, -\infty^-], \end{aligned} \quad (17)$$

and is visualized in Fig. 3. Repeating the same steps which led to (13) for SIDIS, we obtain for DY

$$(\underline{S} \times \underline{q}) f_{1T}^{\perp A}(\bar{x}, q_T) = M_A \int \frac{d^2+ p \, db^-}{2(2\pi)^5} d^2 x \, d^2 y \, d^2 k' \, e^{-i(\underline{k}' + \underline{q} - x \underline{p}) \cdot (\underline{x} - \underline{y})}$$

Fig. 3. Illustration of the quasi-classical factorization formula (17) for DY. As in Fig. 1, this decomposes the TMD quark distribution of the nucleus into the Wigner distribution of nucleons in the nucleus, the TMD quark distribution of the nucleons, and the multiple rescattering. In DY, the multiple scattering of the antiquark occurs in the initial state, as opposed to SIDIS, where it occurred in the final state.

$$\begin{aligned}
 & \times \left\{ A W_{unp}^{OAM} \left(p^+, \underline{p}, b^-, \frac{x+y}{2} \right) f_1^N(x, k_T') \right. \\
 & \left. - \frac{1}{m_N} (\underline{S} \times \underline{k}') W_{trans}^{symm} \left(p^+, \underline{p}, b^-, \frac{x+y}{2} \right) f_{1T}^{\perp N}(x, k_T') \right\} \\
 & \times S_{y\bar{y}}[b^-, -\infty^-],
 \end{aligned} \tag{18}$$

which reflects the decomposition of $f_{1T}^{\perp A}$ into the transversity and OAM channels for DY kinematics. The channels are the same as for SIDIS, with one difference: the initial-state interactions of DY bias the quark/antiquark annihilation to occur closer to the *front* of the nucleus, as in Fig. 4.

Fig. 4. Illustration of the OAM channel for the nuclear Sivers function in the Drell-Yan process. The mechanism is the same as in SIDIS, except that the initial-state interactions make the asymmetry most sensitive to the orbital motion of the nucleons near the front of the nucleus.

This change in sensitivity to orbital motion in the back vs. front of the nucleus from SIDIS to DY is the origin of the famous sign-flip relation⁹ for the OAM channel. This novel mechanism of generating the sign flip for the OAM channel is fundamentally different from the “QCD lensing” interpretation of Refs. [10–12]; that mechanism is driven by the color-correlated rescattering between the active quark and the spectator system, while the OAM mechanism presented here is driven by rescatterings which are explicitly color-*uncorrelated*.

The formalism presented here is potentially very powerful, providing a new tool to calculate the TMD’s from first principles. The general quasi-classical factorization formulas (6) and (17) arise from a resummation of the parameter $\alpha_s^2 A^{1/3}$ in QCD; this one assumption of a high-density target justifies the dominance of multiple scattering on spectator nucleons, the emergence of a hard saturation scale Q_s , and the application of perturbation theory to the process.⁴ As such, the model dependence has been relegated to the light-cone wave functions of nucleons in the nucleus, which are encoded in the Wigner distribution (3). One application of these general results which can be used for phenomenology is to replace the TMD’s of the nucleons with those of the “quark target model” of Ref. [13]; together with a simple model or parameterization of the Wigner distributions, these can be used to explicitly calculate the TMD’s of the nucleus. One such calculation in which the nucleus is modelled as a rigid rotator was explored in Ref. [2].

Finally, we note that the treatment explored here for the nuclear Sivers function $f_{1T}^{\perp A}$ can apply equally well for the projection of (6) or (17) onto any of the other TMD’s. The combination of spin-orbit coupling and multiple scattering provides a very general mechanism for the mixing of nuclear and nucleonic TMD’s with opposite PT -parity, such as the nuclear Sivers function and the nucleonic f_1 distributions considered here. Extending this treatment to the complete set of TMD’s will reveal a detailed picture of spin-orbit structure in the dense limit which is grounded in a systematic resummation of QCD.

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