Space-fractional Schrödinger equation for a quadrupolar triple Dirac- δ potential: Central Dirac- δ well and barrier cases

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We solve the space-fractional Schrödinger equation for a quadrupolar triple Dirac- δ (QTD- δ) potential for all energies using the momentum-space approach. For the $E < 0$ solution, we consider two cases, i.e., when the strengths of the potential are $V_0 > 0$ (QTD- δ potential with central Dirac- δ well) and $V_0 < 0$ (QTD- δ potential with central Dirac- δ barrier) and derive expressions satisfied by the bound-state energy. For all fractional orders α considered, we find that there is one eigenenergy when $V_0 > 0$, and there are two eigenenergies when $V_0 < 0$. We also obtain both bound- and scattering-state ($E > 0$) wave functions and express them in terms of Fox's H -function.

Keywords: Fractional quantum mechanics; space-fractional Schrödinger equation; quadrupolar triple Dirac- δ potential; Fox's H -function.

1. Introduction

Applications of fractional quantum mechanics (FQM) developed by Laskin^{1,2} via constructing fractional path integral over paths of Lévy flights have gained interest over the past 13 years. The formulation offers generalization of some results obtained in the standard quantum mechanics (SQM). One of its interesting applications is to delta potentials.³ Here we present another application of FQM by considering a quadrupolar triple Dirac- δ (QTD- δ) potential in one dimension, which was first analyzed by Patil⁴ in the framework of SQM.

The time-independent space-fractional Schrödinger equation (TISFSE) in the position representation reads²

$$D_\alpha (-\hbar^2 \Delta)^{\alpha/2} \psi(x) + V(x) \psi(x) = E \psi(x), \quad 1 < \alpha \leq 2, \quad (1)$$

where D_α is the generalized quantum diffusion coefficient [$D_2 = 1/(2m)$ with m being the mass of the particle], $\psi(x)$ is the wave function, $V(x)$ is the potential, E

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is the energy, and $(-\hbar^2\Delta)^{\alpha/2}$ is the quantum Riesz fractional derivative:

$$(-\hbar^2\Delta)^{\alpha/2}\psi(x) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dp e^{ipx/\hbar} |p|^\alpha \int_{-\infty}^{\infty} e^{-ipx'/\hbar} \psi(x') dx'. \quad (2)$$

In the momentum representation the TISFSE can be expressed as³

$$D_\alpha |p|^\alpha \tilde{\psi}(p) + \frac{(\tilde{V} * \tilde{\psi})(p)}{2\pi\hbar} = E \tilde{\psi}(p), \quad (3)$$

where $\tilde{\psi}(p) = \int_{-\infty}^{\infty} e^{-ipx/\hbar} \psi(x) dx$ is the Fourier transform of $\psi(x)$, with the inverse transform $\psi(x) = (2\pi\hbar)^{-1} \int_{-\infty}^{\infty} e^{ipx/\hbar} \tilde{\psi}(p) dp$, and $(\tilde{V} * \tilde{\psi})(p) = \int_{-\infty}^{\infty} \tilde{V}(p-p') \tilde{\psi}(p') dp'$. In the next section we solve Eq. (1) for a QTD- δ potential using the momentum-space approach.

2. QTD- δ Potential and Solutions to the TISFSE

The interaction between an electron and a symmetric linear triatomic molecule can be modeled using the potential⁴ $V(x) = V_0[\delta(x+a) - 2\delta(x) + \delta(x-a)]$, where V_0 is the strength and a is the spacing between the atoms. Its Fourier transform and convolution with $\tilde{\psi}(p)$ are given, respectively, by $\tilde{V}(p) = V_0(e^{iap/\hbar} - 2 + e^{-iap/\hbar})$ and $(\tilde{V} * \tilde{\psi})(p) = V_0[e^{iap/\hbar} C_0(a) - 2C_1(0) + e^{-iap/\hbar} C_2(a)]$, where the constants are defined as $C_0(a) = C_2(-a) = \int_{-\infty}^{\infty} e^{-iap'/\hbar} \tilde{\psi}(p') dp'$ and $C_1(0) = \int_{-\infty}^{\infty} \tilde{\psi}(p') dp'$. Substituting the expression for $(\tilde{V} * \tilde{\psi})(p)$ in Eq. (3), we obtain an expression for $\tilde{\psi}(p)$:

$$\tilde{\psi}(p) = \frac{-\gamma_\alpha}{(|p|^\alpha - E/D_\alpha)} [e^{iap/\hbar} C_0(a) - 2C_1(0) + e^{-iap/\hbar} C_2(a)], \quad (4)$$

with $\gamma_\alpha = V_0/2\pi\hbar D_\alpha$. Below we solve the TISFSE for two separate cases, namely, $E < 0$ and $E > 0$.

Case I: $E < 0$. For this case, let us first consider $V_0 > 0$ (central Dirac- δ well case) and define $\lambda^\alpha = -E/D_\alpha$ ($\lambda > 0$). Equation (4) then becomes $\tilde{\psi}(p) = -\gamma_\alpha(|p|^\alpha + \lambda^\alpha)^{-1} [e^{iap/\hbar} C_0(a) - 2C_1(0) + e^{-iap/\hbar} C_2(a)]$. When we use this expression to find C_0, C_1, C_2 , an equation satisfied by bound-state energy can be derived, namely, $R\epsilon^{\alpha-1}A(\epsilon) = R^3\epsilon^{3(\alpha-1)} - B(\epsilon)$, where $\epsilon = a\lambda/\hbar$, $R = 2\pi\hbar^\alpha D_\alpha/(a^{\alpha-1}V_0)$, $A(\epsilon) = 3T^2(0) - 4T^2(\epsilon) + T^2(2\epsilon)$, $B(\epsilon) = 2\{T(0)[T^2(0) - 2T^2(\epsilon) - T^2(2\epsilon)] + 2T^2(\epsilon)T(2\epsilon)\}$, and $T(y) = 2 \int_0^\infty \cos(yq)(q^\alpha + 1)^{-1} dq$ ($q = p/\lambda$). Figure 1 shows plots of the functions $f(\alpha, \epsilon) = R\epsilon^{\alpha-1}A(\epsilon)$ and $h(\alpha, \epsilon) = R^3\epsilon^{3(\alpha-1)} - B(\epsilon)$ for $R = 2$ and some values of α . The corresponding eigenvalue for each α can be identified as the ϵ -coordinate of the point of intersection of the two curves. For the central Dirac- δ barrier case, $V_0 < 0$ [letting $V_0 = -g$ ($g > 0$)], the following energy equations can be derived: (i) $T(2\epsilon) = T(0) - Q\epsilon^{\alpha-1}$ and (ii) $T(0) - T(2\epsilon) + 2Q\epsilon^{\alpha-1} = \pm[9T^2(0) + 6T(2\epsilon)T(0) + T^2(2\epsilon) - 16T^2(\epsilon)]^{1/2}$, where $Q = 2\pi\hbar^\alpha D_\alpha/(a^{\alpha-1}g)$. The energy equations (i) and (ii) are plotted in the left and right panels of Fig. 2, respectively, for $Q = 2$.

Fig. 1. Plots of the functions $f(\alpha, \epsilon)$ and $h(\alpha, \epsilon)$ for the case $V_0 > 0$ with $R = 2$ and $\alpha = 1.6$ (dotted curve), $\alpha = 1.8$ (dash-dotted curve), and $\alpha = 2.0$ (solid curve).

Fig. 2. Plots of the energy equations for the case $V_0 < 0$ with $Q = 2$ and $\alpha = 1.6$ (dotted curves), $\alpha = 1.8$ (dash-dotted curves), and $\alpha = 2.0$ (solid curves).

The wave function $\psi(x)$ can be obtained by inverse Fourier transforming $\tilde{\psi}(p)$. Furthermore, by a suitable choice of phase and application of Parseval's theorem,³ $\int_{-\infty}^{\infty} \psi^*(x)\psi(x)dx = (2\pi\hbar)^{-1} \int_{-\infty}^{\infty} \tilde{\psi}^*(p)\tilde{\psi}(p)dp$, the normalized wave function for the case $V_0 > 0$ can be expressed as $\Psi(x) = N_\alpha[W\phi(x+a) - \phi(x) + Z\phi(x-a)]$, where $N_\alpha = \sqrt{\pi\lambda/\hbar F(\alpha, W, Z)}$ is the normalization constant, with $F(\alpha, W, Z) = [\alpha^{-2}(\alpha-1)\pi(W^2 + Z^2 + 1)\csc(\pi/\alpha) - 4(W+Z)I(a\lambda/\hbar) + 2WZ I(2a\lambda/\hbar)]^{-1/2}$ from which we define $W = C_0(a)/2C_1(0)$, $Z = C_2(a)/2C_1(0)$, and $I(y) = \int_0^\infty \cos(yq)(q^2 + 1)^{-2}dq$ ($q = p/\lambda$), and the ϕ 's are expressed in terms of Fox's H -function:⁵

$$\phi(y) = H_{2,3}^{2,1} \left[(\lambda/\hbar)^\alpha |y|^\alpha \left| \begin{matrix} (1-1/\alpha, 1), (1/2, \alpha/2) \\ (0, \alpha), (1-1/\alpha, 1), (1/2, \alpha/2) \end{matrix} \right. \right]. \quad (5)$$

Figure 3 plots this bound-state wave function for $W = Z$.

Fig. 3. Plot of the wave function Ψ/N_α as a function of $x' = x\lambda/\hbar$ for $\alpha = 1.6$ (dotted curve), $\alpha = 1.8$ (dash-dotted curve), and $\alpha = 2.0$ (solid curve) with $W = Z = 2$.

Case II: $E > 0$. For this case we let $\lambda^\alpha = E/D_\alpha$ ($\lambda > 0$). Using the property $f(x)\delta(x) = f(0)\delta(x)$ of the delta function (see also Ref. 3 for the treatment of single and double delta potentials for the case $E > 0$ using this property), we write $\tilde{\psi}(p)$ as $\tilde{\psi}(p) = A_1\delta(p-\lambda) + A_2\delta(p+\lambda) - \gamma_\alpha(|p|^\alpha - \lambda^\alpha)^{-1}[e^{iap/\hbar}C_0(a) - 2C_1(0) + e^{-iap/\hbar}C_2(a)]$. Using this to find C_0, C_1, C_2 we obtain

$$UC_0(a) = \frac{\mu(2\rho l_1 - 3l_2^2 + 1)}{1 - l_2}M_0(a\lambda/\hbar) + 2\mu\rho M_1(0) + \frac{2\mu(l_2 - \rho l_1)}{1 - l_2}M_2(a\lambda/\hbar), \quad (6)$$

$$UC_1(0) = -\mu\rho M_0(a\lambda/\hbar) + \nu(1 + l_2)M_1(0) - \mu\rho M_2(a\lambda/\hbar), \quad (7)$$

$$UC_2(a) = \frac{2\mu(l_2 - \rho l_1)}{1 - l_2}M_0(a\lambda/\hbar) + \mu\rho M_1(0) + \frac{2\mu(2\rho l_1 + 1)}{1 - l_2}M_2(a\lambda/\hbar), \quad (8)$$

where we denote $\mu = [1 + \Lambda_\alpha^{-1}S(0)]^{-1}$, $\nu = [1 - 2\Lambda_\alpha^{-1}S(0)]^{-1}$, $\rho = \Lambda_\alpha^{-1}\nu S(a\lambda/\hbar)$, $l_j = \Lambda_\alpha^{-1}\mu S(ja\lambda/\hbar)$ ($j = 1, 2$), $U = 4\rho l_1 + l_2 + 1$, $M_0(a\lambda/\hbar) = M_2(-a\lambda/\hbar) = A_1e^{-ia\lambda/\hbar} + A_2e^{ia\lambda/\hbar}$, $M_1(0) = A_1 + A_2$, and $S(y) = 2\int_0^\infty \cos(yq)(q^\alpha - 1)^{-1}dq$ ($q = p/\lambda$); note from these definitions that $\nu l_1 = \mu\rho$.

The wave function $\psi(x)$ after inverse Fourier transforming $\tilde{\psi}(p)$ is $\psi(x) = A'_1e^{i\lambda x/\hbar} + A'_2e^{-i\lambda x/\hbar} + (2\hbar\Lambda_\alpha)^{-1}[C_0(a)\zeta(x+a) - 2C_1(0)\zeta(x) + C_2(a)\zeta(x-a)]$, where $A'_j = A_j/2\pi\hbar$ ($j = 1, 2$), the C_n 's are those enumerated in Eqs. (6)–(8), and the ζ 's

are the following Fox's H -functions:

$$\zeta(y) = H_{2,3}^{2,1} \left[(\lambda/\hbar)^\alpha |y|^\alpha \right]_{(0,\alpha),(1-1/\alpha,1),(1-[2+\alpha]/2\alpha,[2+\alpha]/2)}^{(1-1/\alpha,1),(1-[2+\alpha]/2\alpha,[2+\alpha]/2)} - H_{2,3}^{2,1} \left[(\lambda/\hbar)^\alpha |y|^\alpha \right]_{(0,\alpha),(1-1/\alpha,1),(1-[2-\alpha]/2\alpha,[2-\alpha]/2)}^{(1-1/\alpha,1),(1-[2-\alpha]/2\alpha,[2-\alpha]/2)}. \quad (9)$$

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