STATUS OF STANDARD MODEL CALCULATION OF LEPTON $g - 2$

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The contributions from the standard model interactions to the anomalous magnetic moments of the two lightest charged leptons, the electron and the muon, are reviewed. Comparison with the very accurately measured experimental values is made, using the most recent high-precision determination of the fine structure constant.

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1. Introduction

The anomalous magnetic moments of the electron and of the muon have been experimentally measured with very high accuracies. The latest experiments of the Penning-trap type conducted at Harvard^{1,2} have improved by more than an order of magnitude the relative precision on the value of the magnetic moment g_e of the electron as compared to the value obtained previously³ by the group of H.G. Dehmelt in Seattle. The present best determination² of the electron's magnetic moment at the level of 0.28ppt gives the anomalous magnetic moment $[a_e \equiv (g_e - 2)/2]$

$$a_e^{\text{exp}} = 1\,159\,652\,180.73(0.28) \cdot 10^{-12}, \quad (1.1)$$

at an impressive relative precision of 0.24ppb.

The finite lifetime of the muon, $\tau_\mu \sim 2 \cdot 10^{-6}$ s, precludes the use of similar experimental techniques in order to measure its magnetic moment. Instead, muons are collected in a storage ring where they are accelerated before decaying into electrons and neutrinos. Successive experiments of this type, conducted over several decades,

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first at CERN, and more recently at BNL, have brought the relative precision on the anomalous magnetic moment of the muon below the ppm level. The combination of the results obtained by the E821 experiment at BNL⁴ leads to the value

$$a_{\mu}^{\text{exp}} = 11659208.9(6.3) \cdot 10^{-10} \quad (1.2)$$

which shows a relative precision of 0.54ppm, the total error being dominated by statistics $[\pm 5.4 \cdot 10^{-10}$, vs. $\pm 3.3 \cdot 10^{-10}$ for systematics].

As far as the τ lepton is concerned, the experimental situation is unfortunately far from being close to the one of the electron, or even of the muon, due to its very short lifetime, $\tau_{\tau} \sim 2.9 \cdot 10^{-13}$ s. Only rather crude bounds^{5–7} have been obtained by the LEP and LEP2 experiments. I will therefore only discuss the cases of the electron and of the muon, and refer to Refs. 8 and 9 for the theory of a_{τ} .

The experimental measurements of the anomalous magnetic moments of the electron and of the muon constitute quite impressive achievements, and the natural question that arises is whether theoretical predictions are able to match this precision. At stake is the possibility for an indirect evidence of physics beyond the standard model (for a survey, see Ref. 10), should the comparison between the experimental results and the theoretical calculations reveal a sufficiently significant and robust discrepancy. The present contribution reviews the present status of the theoretical determinations of a_e and of a_{μ} within the standard model.

It has become common practice to decompose the various standard model contributions to a_{ℓ} into parts arising from quantum electrodynamics, from the strong interactions, and finally from the weak interactions,

$$a_{\ell}^{\text{SM}} = a_{\ell}^{\text{QED}} + a_{\ell}^{\text{had}} + a_{\ell}^{\text{weak}}. \quad (1.3)$$

I will adopt this decomposition, and discuss the three contributions in turn.

2. Contributions from quantum electrodynamics

The quantum electrodynamics (QED) contribution, a_{ℓ}^{QED} , to the anomalous magnetic moment of a charged lepton is defined as the sum of the contributions that arise from loops made only of virtual photons and leptons. Due to the smallness of the fine structure constant, a perturbative approach is appropriate:

$$a_{\ell}^{\text{QED}} = \sum_{n \geq 1} A_1^{(2n)} \left(\frac{\alpha}{\pi} \right)^n + \sum_{n \geq 2} A_2^{(2n)} (m_{\ell}/m_{\ell'}) \left(\frac{\alpha}{\pi} \right)^n + \sum_{n \geq 3} A_3^{(2n)} (m_{\ell}/m_{\ell'}, m_{\ell}/m_{\ell''}) \left(\frac{\alpha}{\pi} \right)^n. \quad (2.1)$$

The lowest-order contribution $A_1^{(2)}$ arising from the vertex correction at one-loop is known¹¹ since long time, $A_1^{(2)} = 1/2$. At next-to-leading order, there are seven graphs contributing to $A_1^{(4)}$, and the corresponding expression, also known analytically, was first given correctly in Refs. 12 and 13:

$$A_1^{(4)} = \frac{197}{144} + \left(\frac{1}{2} - 3 \ln 2 \right) \zeta(2) + \frac{3}{4} \zeta(3) = -0.328\,478\,965\,579\,193\dots \quad (2.2)$$

At this same order, there also appears a mass-dependent contribution. It arises from the insertion of a vacuum polarization loop coming from the lepton ℓ' into the photon propagator of the one-loop vertex correction of the lepton ℓ . Its expression reads

$$\begin{aligned} A_2^{(4)}(m_\ell/m_{\ell'}) &= \frac{1}{3} \int_{4m_{\ell'}^2}^{\infty} ds \sqrt{1 - \frac{4m_{\ell'}^2}{s}} \frac{s + 2m_{\ell'}^2}{s^2} \int_0^1 dx \frac{x^2(1-x)}{x^2 + (1-x)\frac{s}{m_\ell^2}} \\ &= \frac{1}{3} \int_{4m_{\ell'}^2}^{\infty} \frac{ds}{s} K(s) R^{\ell'^+ \ell'^-}(s), \end{aligned} \quad (2.3)$$

and is equivalent to the formulas given in Refs. 14 and 15 after changes of the integration variables. The function $K(s)$ in the second expression is defined by the second integral in the first expression, and $R^{\ell'^+ \ell'^-}(s)$ stands for the ratio $\sigma(\ell^+ \ell^- \rightarrow \ell'^+ \ell'^-)/\sigma^{\text{pt}}(s)$, with $\sigma^{\text{pt}}(s) = 4\pi\alpha^2/(3s)$, at tree level. The integrations in Eq. (2.3) can be done explicitly in order to obtain analytical expressions, which were given in Ref. 16, for $m_\ell/m_{\ell'} > 1$ and, in a more compact way, in Refs. 17 and 18 for the general case. In practice, one may also use series expansions^{19–22} in the mass ratios, which can be carried out to any desired order, the only limitation in precision coming from the accuracy of our knowledge of the mass ratios they involve. Using the values given by the CODATA compilation²³ one finds^{24, 25}

$$\begin{aligned} A_2^{(4)}(m_e/m_\mu) &= 5.197\,386\,67(26) \cdot 10^{-7}, \quad A_2^{(4)}(m_e/m_\tau) = 1.837\,98(34) \cdot 10^{-9}, \\ A_2^{(4)}(m_\mu/m_e) &= 1.094\,258\,312\,0(83), \quad A_2^{(4)}(m_\mu/m_\tau) = 7.807\,9(15) \cdot 10^{-5}. \end{aligned} \quad (2.4)$$

At order $\mathcal{O}(\alpha^3)$, there are 72 diagrams to consider. A quite impressive series of computational achievements²⁶ led to the following analytical expression for the sixth-order mass-independent contribution

$$\begin{aligned} A_1^{(6)} &= \frac{87}{72} \pi^2 \zeta(3) - \frac{215}{24} \zeta(5) + \frac{100}{3} \left[\left(a_4 + \frac{1}{24} \ln^4 2 \right) - \frac{1}{24} \pi^2 \ln^2 2 \right] - \frac{239}{2160} \pi^4 \\ &\quad + \frac{139}{18} \zeta(3) - \frac{298}{9} \pi^2 \ln 2 + \frac{17101}{810} \pi^2 + \frac{28259}{5184} \left[a_p = \sum_{n=1}^{\infty} 1/(2^n n^p) \right] \\ &= 1.181\,241\,456\dots \end{aligned} \quad (2.5)$$

The mass-dependent contributions at sixth order are also known analytically and were obtained in Refs. 27 and 28. The corresponding numerical values read:^{24, 25}

$$A_2^{(6)}(m_e/m_\mu) = -7.373\,941\,55(27) \cdot 10^{-6}, \quad A_2^{(6)}(m_e/m_\tau) = -6.5830(11) \cdot 10^{-8},$$

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$$A_2^{(6)}(m_\mu/m_e) = 22.868\,380\,04(23), \quad A_2^{(6)}(m_\mu/m_\tau) = 36.070(13) \cdot 10^{-5}, \quad (2.6)$$

and^{22, 24, 25}

$$A_3^{(6)}(m_e/m_\mu, m_e/m_\tau) = 0.1909(1) \cdot 10^{-12}, \quad A_3^{(6)}(m_\mu/m_e, m_\mu/m_\tau) = 5.2776(11) \cdot 10^{-4}. \quad (2.7)$$

The eighth-order contribution consists of 891 Feynman diagrams. Its value is known only from a numerical integration of the corresponding Feynman-parametrized loop integrals. The latest result of the mass-independent part reads²⁴

$$A_1^{(8)} = -1.9106(20). \quad (2.8)$$

It is interesting to follow the evolution of this numerical determination during time, as displayed in Eq. (9) of Ref. 29. The discussion following it shows that these numerical evaluations present tough difficulties that in some cases have led to erroneous values. Having such delicate calculations done by a single group can be potentially problematic. An independent evaluation of these contribution through different techniques and by a different group³⁰ can thus only be welcome.

The contribution at order $\mathcal{O}(\alpha^5)$ to a_ℓ arises from 12672 diagrams. Its numerical evaluation was completed only recently^{29, 31–24, 39} The mass-independent part amounts to

$$A_1^{(10)} = 9.168(571), \quad (2.9)$$

whereas the values of the relevant mass-dependent contributions are

$$A_2^{(10)}(m_e/m_\mu) = -0.00382(39), \quad (2.10)$$

and

$$A_2^{(10)}(m_\mu/m_e) = 742.18(87), \quad A_2^{(10)}(m_\mu/m_\tau) = -0.068(5), \quad (2.11)$$

$$A_3^{(10)}(m_\mu/m_e, m_\mu/m_\tau) = 2.011(10). \quad (2.12)$$

Summing up all the contributions at a given order, one arrives at the perturbative expansion

$$a_\ell^{\text{QED}} = C_\ell^{(2)} \left(\frac{\alpha}{\pi}\right) + C_\ell^{(4)} \left(\frac{\alpha}{\pi}\right)^2 + C_\ell^{(6)} \left(\frac{\alpha}{\pi}\right)^3 + C_\ell^{(8)} \left(\frac{\alpha}{\pi}\right)^4 + C_\ell^{(10)} \left(\frac{\alpha}{\pi}\right)^5 + \dots \quad (2.13)$$

with the values^{24, 25} of the coefficients $C_\ell^{(n)}$ gathered in the following table:

	$\ell = e$	$\ell = \mu$
$C_\ell^{(2)}$	0.5	0.5
$C_\ell^{(4)}$	-0.328 478 444 00	0.765 857 425(17)
$C_\ell^{(6)}$	1.181 234 017	24.050 509 96(32)
$C_\ell^{(8)}$	-1.9144(35)	130.879 6(63)
$C_\ell^{(10)}$	9.16(58)	753.29(1.04)

Fig. 1. Lowest-order hadronic corrections to a_ℓ (left). The shaded blob represents the hadronic vacuum polarization function. The contribution on the right, although of higher order, is also included in $a_\ell^{\text{HVP-LO}}$.

Since $(\alpha/\pi)^2 \sim 6.76 \cdot 10^{-14}$, we see that the tenth-order QED contributions are relevant at the level of the experimental precision on a_e and on a_μ , in the latter case because of the large value of the coefficients $C_\mu^{(10)}$.

3. Contributions from quantum chromodynamics

Formally speaking, the contributions from the strong interactions arise from loops involving, in addition to the leptons and photon, also the quarks and gluons as described by quantum chromodynamics (QCD). Unfortunately, the kinematical regime relevant for static quantities, such as the anomalous magnetic moments of the leptons, corresponds to large distances, where a perturbative approach to the QCD effects is not useful. The first QCD contribution arises at order $\mathcal{O}(\alpha^2)$, through a hadronic vacuum polarization insertion into the one-loop vertex diagram, see Fig. 1. This contribution, $a_\ell^{\text{HVP-LO}}$, is evaluated using the second expression in Eq. (2.3), with the same function $K(s)$, but with $R^{\ell'\ell''}(s)$ replaced by the hadronic ratio $R^{\text{had}}(s)$ ^{40–42} [and the range of integration starts at $4M_{\pi^\pm}^2$], that can be extracted from available data for the cross-section of $e^+e^- \rightarrow \text{hadrons}$. Note that $K(s)$ is positive, so that $a_\ell^{\text{HVP-LO}}$ is positive. Furthermore, $K(s) \sim m_\ell^2/(3s)$ as $s \rightarrow \infty$, so that the low-energy region dominates. Two recent determinations in the case of the muon give very close results

$$a_\mu^{\text{HVP-LO}} = 692.3(4.2) \cdot 10^{-12} \text{ [43]}, \quad a_\mu^{\text{HVP-LO}} = 694.9(4.3) \cdot 10^{-12} \text{ [44]}. \quad (3.1)$$

For a somewhat different approach to this issue, see Ref. 45. For a critical review, see Ref. 46.

At order $\mathcal{O}(\alpha^3)$, there are further hadronic corrections,^{47,48} shown in Fig. 2, which involve the same R ratio, but convoluted with a different function $K^{(2)}(s)$,

Fig. 2. Representative Feynman graphs for the higher-order hadronic corrections $a_\ell^{\text{HVP-NLO}}$.

Fig. 3. Hadronic Light-by-Light contribution to a_ℓ .

which is no longer positive definite. One finds for this NLO hadronic vacuum polarization contribution⁴⁴

$$a_\mu^{\text{HVP-NLO}} = -9.84(7) \cdot 10^{-12}, \quad (3.2)$$

At order $\mathcal{O}(\alpha^3)$, there is also another hadronic contribution arising from the diagram in Fig. 3. This hadronic (virtual) light-by-light scattering contribution cannot be related to available data, and has thus to be evaluated by other means. For an update on this issue, see Ref. 49. For reference, I quote the “best estimate” of Ref. 50

$$a_\mu^{\text{HLxL}} = 10.5(2.6) \cdot 10^{-10}, \quad (3.3)$$

and the somewhat more conservative value⁵¹

$$a_\mu^{\text{HLxL}} = 11.5(4.0) \cdot 10^{-10}. \quad (3.4)$$

In the case of the electron, these hadronic contributions are suppressed by approximatively a factor $(m_e/m_\mu)^2$ as compared to the muon, and the most recent values are⁵²

$$a_e^{\text{HVP-LO}} = 1.866(11) \cdot 10^{-12}, \quad a_e^{\text{HVP-NLO}} = -0.2234(14) \cdot 10^{-12}, \quad (3.5)$$

and

$$a_e^{\text{HLxL}} = 0.035(10) \cdot 10^{-12} \text{ [53]}, \quad a_e^{\text{HLxL}} = 0.039(13) \cdot 10^{-12} \text{ [51]}. \quad (3.6)$$

4. Contributions from weak interactions

The one-loop contributions to a_ℓ from weak interactions have been worked out more than forty years ago by several groups^{54–58} and the result reads [$X = Z, H$]

$$a_\ell^{\text{weak}(1)} = \frac{G_F}{\sqrt{2}} \frac{m_\ell^2}{8\pi^3} \left[\frac{5}{3} + \frac{1}{3} (1 - 4s_w^2)^2 + \mathcal{O} \left(\frac{m_\ell^2}{M_X^2} \ln \frac{M_X^2}{m_\ell^2} \right) \right] = 19.48 \cdot 10^{-10}. \quad (4.1)$$

At the two-loop level, one conveniently distinguishes between contributions without internal fermion loops (called bosonic), and those containing internal fermion loops (called fermionic). The former were evaluated in Refs. 59–61, and the latter were treated in Refs. 62–65. A recent reanalysis⁶⁶ gives, for the sum of one-loop and two-loop corrections, the total value

$$a_\mu^{\text{weak}} = 15.36(10) \cdot 10^{-10}. \quad (4.2)$$

The corresponding value for the electron case is expected to be $\sim 40\,000$ times smaller,

$$a_e^{\text{weak}} = 0.0297(5) \cdot 10^{-12}, \quad (4.3)$$

which lies one order of magnitude below the current experimental precision.

5. Determination of the fine-structure constant

In order to answer the question raised at the end of Sec. 1, it is necessary to know the value of the fine-structure constant at the level of precision that matches the experimental accuracy. In the case of the electron, the precision required is thus $\Delta\alpha/\alpha \sim 0.24\text{ppb}$. Recently, a series of measurements of the recoil velocity through photon absorption of Cesium⁶⁷ and Rubidium^{68–70} atoms has led⁷⁰ to the value

$$\alpha^{-1}[Rb11] = 137.035\,999\,037(91), \quad (5.1)$$

with a relative error of only 0.66ppb, i.e. quite close to what is actually required. The comparison²⁴ between the experimental value of $a_e[HV08]$ obtained in 2008² and the prediction using the value $\alpha^{-1}[Rb11]$ gives the quite impressive agreement

$$a_e[HV08] - a_e[\text{theory}] = -1.05(0.82) \cdot 10^{-12}. \quad (5.2)$$

Conversely, using the theoretical calculations in order to extract α from the value of a_e measured by the latest Harvard experiment yields

$$\alpha^{-1}[HV08] = 137.035\,999\,172\,7(68)_{\alpha^4(46)}\alpha^5(19)_{\text{had+weak}}(331)_{a_e[HV08]}, \quad (5.3)$$

which corresponds to a relative uncertainty of 0.25ppb, more than twice smaller than the relative error on $\alpha^{-1}[Rb11]$, and with the largest contribution to the error coming from the experimental uncertainty on $a_e[HV08]$.

6. Conclusion and perspectives for the future

The anomalous magnetic moments of the electron and of the muon have been measured very precisely (to 0.24ppb and to 0.54ppm, respectively), improving previous measurements by several factors. Recent high-precision measurements of the fine structure constant in atomic physics allow to reach the level of accuracy required in order to test QED with a_e , and even to become sensitive to the contributions from the strong interactions $a_e^{\text{HVP-LO}}$. Although a_e to date still provides the most precise determination of the fine structure constant, it is certainly of interest and worthwhile to pursue the efforts to measure α with the accuracy required to test the prediction for a_e at the level where this quantity has been measured.

The anomalous magnetic moments of the muon probes all the interactions of the standard model, and perhaps even beyond. Indeed, there is a persistent discrepancy between the measured value and the SM prediction at the level of 3 to 3.6 σ ,

$$a_{\mu}^{\text{exp}} - a_{\mu}^{\text{SM}} = (28.7 \pm 8.0) \cdot 10^{-10} [3.6\sigma], \quad (6.1)$$

with the values from Refs. 43 and 50. The discrepancy shrinks somewhat below the 3σ level if one takes Refs. 44 and 51

$$a_{\mu}^{\text{exp}} - a_{\mu}^{\text{SM}} = (25.0 \pm 8.6) \cdot 10^{-10} [2.9\sigma]. \quad (6.2)$$

Whether this discrepancy is real or not will be probed soon by two forthcoming experiments, at FNAL⁷¹ and at J-PARC,⁷² which aim at measuring a_{μ} with a relative precision of 0.14ppm.

The interpretation of these future experiments requires further theoretical improvement on the evaluation of the hadronic contributions. As far as hadronic vacuum polarization is concerned, improvements can be expected from forthcoming data (see Ref. 46 and the various contributions to the R^{had} measurement session at this conference). A more refined theoretical understanding of the hadronic light-by-light scattering contribution, which remains a difficult challenge, would also be welcome. Finally, simulations of QCD on the lattice could provide valuable alternative determinations of these hadronic contributions, provided they can meet the required precision. For prospects in this direction, see Ref. 73.

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