Study of the $\pi\pi$ system in a finite volume

Dan Zhou and Hua-Xing Chen*

*School of Physics and Nuclear Energy Engineering and International Research Center
for Nuclei and Particles in the Cosmos, Beihang University, Beijing 100191, China*
*hxchen@buaa.edu.cn

E. Oset

*Departamento de Física Teórica and IFIC, Centro Mixto Universidad de Valencia-CSIC
Institutos de Investigación de Paterna, Aptdo. 22085, 46071 Valencia, Spain*
oset@ific.uv.es

Published 16 April 2014

We evaluate energy levels of the $\pi\pi$ system in the ρ channel in a finite volume using chiral unitary theory. We investigate $\pi\pi$ phase shifts and ρ meson properties using Lattice QCD data with high precision. We also shortly discuss the πK system in the K^* channel in a finite volume.

Keywords: Multichannel scattering; lattice QCD calculation; meson-meson interaction.

PACS Numbers: 11.80.Gw, 12.38.Gc, 12.39.Fe, 13.75.Lb

1. Introduction

One of the challenging tasks of lattice QCD is to determine hadron spectra. Many efforts have been devoted to it.^{1–12} Particularly, Lüscher's approach is playing gradually a more important role.^{13,14} It has been applied to study the ρ meson and the K^* meson.^{13–23}

In this paper²⁴ we shall investigate the ρ meson using the method proposed in Ref. 25. In this reference Lüscher's approach is simplified and improved by keeping the full relativistic two-body propagator, and is extended to coupled channels. This method has been used to study the κ resonance in the $K\pi$ channel, the DK and ηD_s system, the meson-baryon interaction and the $\pi\rho$ scattering, etc.^{26–35}

To use this method, first we need to use the chiral unitary model to investigate the $\pi\pi$ scattering in P -waves. This has been done in Refs. 36, 37. Based on these results, we shall evaluate energy levels of the $\pi\pi$ system in the ρ channel in a finite

This is an Open Access article published by World Scientific Publishing Company. It is distributed under the terms of the Creative Commons Attribution 3.0 (CC-BY) License. Further distribution of this work is permitted, provided the original work is properly cited.

volume. By doing this we can establish a connection with lattice results which run for heavier pion masses than physical ones.²¹ We shall also discuss the πK scattering in P -waves in a finite volume. The chiral unitary approach has been applied to investigate it in Refs. 36, 38, and the relevant lattice results are shown in Refs. 22, 23.

The paper proceeds as follows: in Sec. 2 we introduce the chiral unitary approach for the $\pi\pi$ scattering in P -waves together with its coupled $K\bar{K}$ channel, both in an infinite space and in a finite volume; in Sec. 3 we evaluate their energy levels and apply Lüscher formalism to get the relevant phase shifts; in Sec. 4 we discuss the πK system in a finite volume; Sec. 5 is a short summary.

2. Formalism for the $\pi\pi$ System

The P -wave $\pi\pi$ scattering amplitude has been obtained in Refs. 37, 39, 40 using the chiral unitary model. They use the coupled channel Bethe-Salpeter equations:

$$\mathbf{T} = (1 - \mathbf{V}\mathbf{G})^{-1}\mathbf{V}. \quad (1)$$

The relevant V matrices for $\pi\pi$ and $K\bar{K}$ scattering have been studied in Ref. 37:

$$\begin{aligned} V_{\pi\pi,\pi\pi}^{I=1} &= T_{\pi^+\pi^-, \pi^+\pi^-} = -\frac{2p_\pi^2}{3f^2} \left(1 + \frac{2G_V^2}{f^2} \frac{s}{M_\rho^2 - s} \right), \\ V_{\pi\pi,K\bar{K}}^{I=1} &= \sqrt{2}T_{\pi^+\pi^-, K^+K^-} = -\frac{\sqrt{2}p_\pi p_K}{3f^2} \left(1 + \frac{2G_V^2}{f^2} \frac{s}{M_\rho^2 - s} \right), \\ V_{K\bar{K},K\bar{K}}^{I=1} &= T_{K^+K^-, K^+K^-} - T_{K^+K^-, K^0\bar{K}^0} = -\frac{p_K^2}{3f^2} \left(1 + \frac{2G_V^2}{f^2} \frac{s}{M_\rho^2 - s} \right). \end{aligned} \quad (2)$$

where the states are defined in the isospin base:

$$\begin{aligned} |\pi\pi\rangle_{I=1} &= \frac{1}{2} |\pi^+\pi^- - \pi^-\pi^+\rangle, \\ |K\bar{K}\rangle_{I=1} &= \frac{1}{\sqrt{2}} |K^+K^- - K^0\bar{K}^0\rangle. \end{aligned} \quad (3)$$

In the chiral unitary model the G function for the two-meson propagator is defined in the infinite space

$$G(p^2) = i \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 - m^2 + i\epsilon} \frac{1}{(p-q)^2 - M^2 + i\epsilon}, \quad (4)$$

and calculated using the dimensional regularization:

$$G_i(s) = \frac{1}{16\pi^2} \left(-2 + d_i + \sigma_i(s) \log \frac{\sigma_i(s) + 1}{\sigma_i(s) - 1} \right). \quad (5)$$

In this paper²⁴ we shall study the ρ meson in a finite volume. To do this we just use the following G function which is defined in the finite box of side L :

$$\tilde{G}(s) - G(s) = \lim_{q_{\max} \rightarrow \infty} \left(\frac{1}{L^3} \sum_{q_i}^{q_{\max}} I(q_i) - \int_{q < q_{\max}} \frac{d^3 q}{(2\pi)^3} I(q) \right), \quad (6)$$

where

$$I(q_i) = \frac{1}{2\omega_1(\vec{q})\omega_2(\vec{q})} \frac{\omega_1(\vec{q}) + \omega_2(\vec{q})}{E^2 - (\omega_1(\vec{q}) + \omega_2(\vec{q}))^2}, \quad (7)$$

and $\omega_{1,2}(\vec{q}) = \sqrt{m_{1,2}^2 + \vec{q}^2}$. The sum is given by $\vec{q} = \frac{2\pi}{L}\vec{n}$ ($\vec{n} \in \mathbb{Z}^3$).

3. Energy Levels for the $\pi\pi$ System

Energy levels for the $\pi\pi$ System in a finite volume are obtained when the determinant of $1 - \mathbf{VG}$ is zero:

$$\text{Det}(1 - \mathbf{VG}) = 1 - V_{11}\tilde{G}_1(s) - V_{22}\tilde{G}_2(s) + (V_{11}V_{22} - V_{12}^2)\tilde{G}_1(s)\tilde{G}_2(s) = 0. \quad (8)$$

We note that they are just the poles of the $\mathbf{T}$ matrix. Through this equation, we can quickly find that energy levels for $\pi\pi$ P -wave scattering are functions of the cubic box size L .

The obtained energy levels are shown in Fig. 1 as functions of the box size L , where the left panel and the right panel are for the coupled channels and only for the single $\pi\pi$ channel, respectively. In these two figures the first and the second energy levels are almost the same, but the higher levels are different. This means the $K\bar{K}$ channel contributes here but it does not contribute much in the low-energy region.

Phase shifts can be obtained using these energy levels. Particularly, the $\pi\pi$ P -wave phase shifts and the T_{11} matrix are related to each other:

$$T_{11}(E) = \frac{-8\pi E}{p \cot \delta(p) - ip}, \quad (9)$$

where p is the pion center-of-mass momentum. We show it in Fig. 2. The solid curve is taken from Ref. 37 using the chiral unitary approach in an infinite space. The

Fig. 1. Energy levels as functions of the box size L . The left panel and the right panel are for the coupled channels and only for the single $\pi\pi$ channel, respectively.

Fig. 2. The $\pi\pi$ scattering P -wave phase shift calculated using the chiral unitary approach, in the infinite space (solid curve) and in the finite space (dashed curve).

dashed curve is the phase shift extracted from the first energy level of the right figure of Fig. 1, obtained using only the $\pi\pi$ channel. The experimental data are taken from Ref. 41.

Phase shifts can be used to fit the physical quantities, such as m_ρ , $g_{\rho\pi\pi}$ and Γ_ρ , through the following two equations:

$$\cot \delta(s) = \frac{m_\rho^2 - s}{\sqrt{s}\Gamma_\rho(s)}, \quad \text{and} \quad \Gamma_\rho(s) = \frac{p^3}{s} \frac{g_{\rho\pi\pi}^2}{8\pi}. \quad (10)$$

We note that the factor 8π in the second equation is our normalization, while in Refs. 21 the authors use 6π . The results from fitting the $\pi\pi$ channel are

$$m_\rho = 768.6 \text{ MeV}, g_{\rho\pi\pi} = 6.59, \Gamma_\rho = 135 \text{ MeV}, \quad (11)$$

while those from fitting the coupled channels are

$$m_\rho = 769.6 \text{ MeV}, g_{\rho\pi\pi} = 6.79, \Gamma_\rho = 144 \text{ MeV}. \quad (12)$$

To compare our results with the lattice QCD results,²¹ we study the mass and decay width of the ρ meson using different pion masses. We define m_π^0 to be the physical π mass, and now m_π is a free parameter. It changes from m_π^0 to $3m_\pi^0$ in our study. When changing m_π , other parameters in our previous formulae also change. For the parameter f (f_π) we find a good fit to results obtained in Refs. 42, 43, 44, 45, 46:

$$\frac{f_\pi(m_\pi)}{f_\pi(m_\pi^0)} = 1 + 0.035 \left(\left(\frac{m_\pi}{m_\pi^0} \right)^2 - 1 \right), \quad (13)$$

with $f_\pi(m_\pi^0) = 87.4 \text{ MeV}$ as we needed in our fit to the ρ data. The coupling G_V with the vector meson dominance assumption of Refs. 47, 48 is related to f_π ,⁴⁹ as

$G_V = f_\pi/\sqrt{2}$. Therefore, we still take the value $G_V = 53$ MeV of Refs. 37 but assume it to be proportional to f_π as a function of m_π . On the other hand, the bare ρ mass, M_ρ in Eqs. (15), provides the link of the theory to a genuine component of the ρ meson, not related to the pion cloud, and we assume it to be m_π independent.

We compare our results with the Lattice results of Ref. 21:

Our Input and Result	Input and Result in Ref. 21
$m_\pi = 276\text{MeV}, L = 2.75m_\pi^{-1} = 1.96\text{fm}$	$m_\pi = 266\text{MeV}, L = 1.98\text{fm}$
$E_1 = 813.2^{+7.2}_{-7.3}\text{MeV},$ $E_2 = 1390.5^{+1.0}_{-1.0}\text{MeV}$	$E_1 = 813.4 \pm 6.3\text{MeV},$ $E_2 = 1433.7 \pm 16.1\text{MeV}$
$\delta_1 = 136.3^{+1.3}_{-1.4}^\circ, \delta_2 = 175.6^{+0.3}_{-0.3}^\circ$	$\delta_1 = 130.56^\circ \pm 1.37^\circ, \delta_2 = 146.03^\circ \pm 6.58^\circ$

We find that the first (lowest) energy level and the extracted phase shift from the two approaches are very similar, while those from the second energy level have small differences.

4. Energy Levels for the πK System

In this section we discuss the πK system in a finite volume. The scattering amplitude has been obtained in Refs. 36, 38 using the chiral unitary model. They use the single channel Bethe-Salpeter equation:

$$t(s) = \frac{v(s)}{1 - v(s)G(s)}. \quad (14)$$

The relevant v matrix for πK scattering has been studied in Ref. 38:

$$v(s) = -\frac{1}{2f^2} \left(1 + \frac{2G_V^2}{f^2} \frac{s}{M_{K^*}^2 - s} \right).$$

Here the loop function $G(s)$ is given by

$$G(s) = \int_{|\vec{q}| < q_{\max}} \frac{d^3\vec{q}}{(2\pi)^3} \frac{|\vec{q}|^2}{s - (\omega_1 + \omega_2)^2 + i\epsilon} \frac{\omega_1 + \omega_2}{2\omega_1\omega_2}, \quad (15)$$

where $\omega_i = \sqrt{m_i^2 + \vec{q}^2}$ and $i = \pi$ and K .

To study the K^* meson in a finite volume, we just need to change it with the G function defined in the finite box of side L . Energy levels for the πK System in the finite volume are then obtained when $1 - v(s)G(s)$ is zero. The detailed analyses will be shown in our future work.

5. Conclusion

We have made a study of the $\pi\pi$ interaction in P -waves in a finite volume using the chiral unitary approach. We obtained energy energies of the $\pi\pi$ system in the ρ channel in a finite volume, and evaluated the relevant phase shifts. These results are compared with the Lattice results of Ref. 21, and we found that the first (lowest)

energy level and the extracted phase shift from the two approaches are very similar, while those from the second energy level have small differences. We also discussed the πK system in the K^* channel in a finite volume.

Acknowledgments

This work is partly supported by DGICYT contract FIS2011-28853-C02-01, the Generalitat Valenciana in the program Prometeo, 2009/090, the EU Integrated Infrastructure Initiative Hadron Physics 3 Project under Grant Agreement no. 283286, the National Natural Science Foundation of China under Grant No. 11205011, and the Fundamental Research Funds for the Central Universities.

References

1. Y. Nakahara, M. Asakawa and T. Hatsuda, Phys. Rev. D **60**, 091503 (1999).
2. N. Mathur, A. Alexandru, Y. Chen, S. J. Dong, T. Draper, I. Horvath, F. X. Lee and K. F. Liu *et al.*, Phys. Rev. D **76**, 114505 (2007).
3. S. Basak, R. G. Edwards, G. T. Fleming, K. J. Juge, A. Lichtl, C. Morningstar, D. G. Richards and I. Sato *et al.*, Phys. Rev. D **76**, 074504 (2007).
4. M. G. Alford and R. L. Jaffe, Nucl. Phys. B **578**, 367 (2000).
5. A. Hart *et al.* [UKQCD Collaboration], Phys. Rev. D **74**, 114504 (2006).
6. H. Wada, T. Kunihiro, S. Muroya, A. Nakamura, C. Nonaka and M. Sekiguchi, Phys. Lett. B **652**, 250 (2007).
7. S. Prelovsek, C. Dawson, T. Izubuchi, K. Orginos and A. Soni, Phys. Rev. D **70**, 094503 (2004).
8. H. -W. Lin *et al.* [Hadron Spectrum Collaboration], Phys. Rev. D **79**, 034502 (2009).
9. C. Gatttringer, C. Hagen, C. B. Lang, M. Limmer, D. Mohler and A. Schafer, Phys. Rev. D **79**, 054501 (2009).
10. G. P. Engel *et al.* [BGR [Bern-Graz-Regensburg] Collaboration], Phys. Rev. D **82**, 034505 (2010).
11. R. G. Edwards, J. J. Dudek, D. G. Richards and S. J. Wallace, Phys. Rev. D **84**, 074508 (2011).
12. C. McNeile *et al.* [UKQCD Collaboration], Phys. Lett. B **556**, 177 (2003).
13. M. Lüscher, Commun. Math. Phys. **105** (1986) 153.
14. M. Lüscher, Nucl. Phys. B **354** (1991) 531.
15. V. Bernard, M. Lage, U. -G. Meissner, A. Rusetsky, JHEP **0808**, 024 (2008).
16. X. Feng *et al.* [ETM Collaboration], PoS LAT **2010**, 104 (2010).
17. S. Aoki *et al.* [CP-PACS Collaboration], Phys. Rev. D **76**, 094506 (2007).
18. M. Gockeler *et al.* [QCDSF Collaboration], PoS LATTICE **2008**, 136 (2008).
19. S. Aoki *et al.* [PACS-CS Collaboration], PoS LATTICE **2010**, 108 (2010).
20. J. Frison *et al.* [Budapest-Marseille-Wuppertal Collaboration], PoS LATTICE **2010**, 139 (2010).
21. C. B. Lang, D. Mohler, S. Prelovsek and M. Vidmar, Phys. Rev. D **84**, 054503 (2011).
22. C. B. Lang, L. Leskovec, D. Mohler and S. Prelovsek, Phys. Rev. D **86**, 054508 (2012).
23. Z. Fu and K. Fu, Phys. Rev. D **86**, 094507 (2012).
24. H. -X. Chen and E. Oset, Phys. Rev. D **87**, 016014 (2013).
25. M. Doring, U. -G. Meissner, E. Oset and A. Rusetsky, Eur. Phys. J. A **47**, 139 (2011).
26. M. Doring, J. Haidenbauer, U. -G. Meissner, A. Rusetsky, Eur. Phys. J. A **47**, 163 (2011).

27. A. Martinez Torres, L. R. Dai, C. Koren, D. Jido and E. Oset, Phys. Rev. D **85**, 014027 (2012).
28. M. Doring, U. G. Meissner, JHEP **1201**, 009 (2012).
29. L. Roca and E. Oset, Phys. Rev. D **85**, 054507 (2012).
30. M. Albaladejo, J. A. Oller, E. Oset, G. Rios and L. Roca, JHEP **1208**, 071 (2012).
31. M. Albaladejo, C. Hidalgo-Duque, J. Nieves and E. Oset, Phys. Rev. D **88**, 014510 (2013).
32. M. Albaladejo, G. Rios, J. A. Oller and L. Roca, arXiv:1307.5169 [hep-lat].
33. E. J. Garzon, R. Molina, A. Hosaka and E. Oset, arXiv:1310.0972 [hep-lat].
34. J. -J. Xie and E. Oset, Eur. Phys. J. A **48**, 146 (2012).
35. F. Aceti and E. Oset, Phys. Rev. D **86**, 014012 (2012).
36. J. A. Oller and E. Oset, Phys. Rev. D **60**, 074023 (1999).
37. J. A. Oller, E. Oset and J. E. Palomar, Phys. Rev. D **63**, 114009 (2001).
38. C. W. Xiao, F. Aceti and M. Bayar, Eur. Phys. J. A **49**, 22 (2013).
39. J. Gasser and H. Leutwyler, Nucl. Phys. B **250**, 465 (1985).
40. G. Ecker, J. Gasser, A. Pich and E. de Rafael, Nucl. Phys. B **321**, 311 (1989).
41. P. Estabrooks and A. D. Martin, Nucl. Phys. B **79**, 301 (1974).
42. J. R. Pelaez and G. Rios, Phys. Rev. D **82**, 114002 (2010).
43. C. Hanhart, J. R. Pelaez and G. Rios, Phys. Rev. Lett. **100**, 152001 (2008).
44. P. Boucaud *et al.* [ETM Collaboration], Phys. Lett. B **650**, 304 (2007).
45. S. R. Beane, T. C. Luu, K. Orginos, A. Parreno, M. J. Savage, A. Torok and A. Walker-Loud, Phys. Rev. D **77**, 014505 (2008).
46. J. Noaki, S. Aoki, T. W. Chiu, H. Fukaya, S. Hashimoto, T. H. Hsieh, T. Kaneko and H. Matsufuru *et al.*, PoS LATTICE **2008**, 107 (2008).
47. J.J. Sakurai, Currents and mesons (University of Chicago Press, Chicago Il 1969).
48. M. Bando, T. Kugo and K. Yamawaki, Phys. Rept. **164**, 217 (1988).
49. G. Ecker, J. Gasser, H. Leutwyler, A. Pich and E. de Rafael, Phys. Lett. B **223**, 425 (1989).