ANALYTICAL SPECTRA OF CMB ANISOTROPIES AND POLARIZATION

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We briefly present the results of our analytical studies on CMB anisotropies and polarizations generated by scalar metric perturbations in synchronous gauge, parallel to the previous work with RGW as a generating source. The analytical the spectra C_l^{XX} are explicitly given and the influences of various cosmological processes are examined.

Keywords: 98.70Vc, 98.80.-k, 98.80.Jk

The temperature anisotropies and polarization of the cosmic microwave background radiation (CMB) are the indispensable cosmological probes.^{1,2} There have been codes of numerical computation for the spectra C_l^{XX} , such as CMBFAST³ and CAMB.⁴ Analytical studies are still useful for understanding how various underlying physical effects give rise to the observed patterns. In particular, the analytical C_l^{XX} are helpful to separate the contribution by the relic gravitational waves (RGW)⁵ from that by the density perturbation.⁶ In the past we systematically derived the analytic C_l^{XX} induced by RGW^{7–9} and their modifications due to reionization.^{10,11} I briefly present the analytic C_l^{XX} induced by the density perturbations in synchronous gauge from our recent studies.

The Boltzmann equation of photons consists of two coupled differential equations for the temperature anisotropies α_k and the polarization β_k for each mode k ,¹²

$$\left(\frac{\partial}{\partial\tau} + q(\tau) + ik\mu\right)\alpha_k = \frac{1}{2}\left(\frac{dH}{d\tau} - \mu^2\frac{dH_l}{d\tau}\right) + q(\tau)\left(\mathcal{I}_1(\tau) + i\mu v_b - \frac{1}{2}P_2(\mu)\mathcal{I}_2(\tau)\right), \quad (1)$$

$$\left(\frac{\partial}{\partial\tau} + q(\tau) + ik\mu\right)\beta_k = \frac{1}{2}q(\tau)(1 - P_2(\mu))\mathcal{I}_2(\tau), \quad (2)$$

where $\mu = \cos\theta$ with θ being the angle between the direction of photon propagation and the wave vector $\mathbf{k}$, $q(\tau)$ is the differential optical depth for recombination, $H(\tau)$ and $H_l(\tau)$ are the two modes of scalar metric perturbations and $\frac{1}{2}(\frac{dH}{d\tau} - \mu^2\frac{dH_l}{d\tau})$ is the Sachs-Wolfe term. $P_2(\mu)$ is the second order Legendre function, and $\mathcal{I}_1(\tau) \equiv \frac{1}{2}\int_{-1}^1 d\mu\alpha_k(\tau, \mu)$, $\mathcal{I}_2(\tau) \equiv \frac{1}{2}\int_{-1}^1 d\mu[(1 - P_2(\mu))\beta_k(\tau, \mu) - P_2(\mu)\alpha_k(\tau, \mu)]$ play a role

of sources. $q(\mathcal{I}_1 + i\mu v_b - \frac{1}{2}P_2\mathcal{I}_2)$ with v_b being the baryon (electron) velocity in Eq. (1) is a new collision term for α_k , which is absent in the case of RGW.

Equations (1) and (2) are analytically solved by time-integration in the long wave-length limit, resulting in explicit expressions of α_k and β_k . Several techniques have been employed in achieving this. The tightly-coupling approximation for the photons during the recombination process has been used. A two-piece half-gaussian function is used for the visibility function during the recombination. The long wave-length approximation for the modes H and H_l during the radiation dominant era is used, and the residual gauge modes have been removed from the solution of H and H_l in the synchronous gauge. A proper connection has been made for H and H_l , respectively, at the radiation-matter equality.

The analytical multipole moment of temperature anisotropies

$$\begin{aligned} a_l^T(k) \simeq & \alpha_0(\tau_d)D_T(k)j_l(k(\tau_0 - \tau_d)) + \alpha_1(\tau_d)D_T(k)\frac{d}{d\zeta}j_l(k(\tau_0 - \tau_d)) \\ & + (H(\tau_0) - H(\tau_d))j_l(k\tau_0) \\ & - \frac{15}{14}\ln\frac{10}{3}\Delta\tau_d M(\tau_d)D_E(k)\left(1 + \frac{d^2}{d\zeta^2}\right)j_l(\zeta)|_{\zeta=k(\tau_0 - \tau_d)}, \end{aligned} \quad (3)$$

where τ_0 is the present time, τ_d is the recombination time, and $\Delta\tau_d$ is the recombination width, and the damping factor $D_T(k) = \frac{1}{2}(e^{-c_T(k\Delta\tau_1)^{b_T}} + e^{-c_T(k\Delta\tau_2)^{b_T}})$ with $c_T \sim 0.65$ and $b_T \sim 0.6$, $D_E(k) = 0.2(e^{-c_E(k\Delta\tau_1)^{b_E}} + e^{-c_E(k\Delta\tau_2)^{b_E}})$ with $c_E \sim 0.27$ and $b_E \sim 2.0$, and $M(\tau) \equiv -\frac{2}{5}\dot{\alpha}_0(\tau) + \frac{2}{5}\dot{H}(\tau)$. In Eq. (3), the monopole α_0 term is dominant, the dipole α_1 term is secondary, the ISW term is smaller than the α_1 term, and the last term is the quadrupole part, which is smaller than the ISW term. We have derived the explicit expressions of $\alpha_0(\tau)$, $\alpha_1(\tau)$, $H(\tau)$, $H_l(\tau)$ which are not written down here for a limited length of paper. The analytical multipole moment of polarization

$$a_l^E(k) \simeq \frac{15}{14}\ln\frac{10}{3}\left(\frac{(l+2)!}{(l-2)!}\right)^{1/2}\frac{\Delta\tau_d}{k^2(\tau_0 - \tau_d)^2}M(\tau_d)D_E(k)j_l(k(\tau_0 - \tau_d)), \quad (4)$$

which is proportional to $M(\tau_d)\Delta\tau_d$. Since $\Delta\tau_d$ is small, the amplitude of a_l^E will be consequently small, in comparison with a_l^T . Notice that the quadrupole part of a_l^T contains the factor $M(\tau_d)\Delta\tau_d$, similar to the polarization a_l^E in Eq. (4).

These new results significantly extend the earlier preliminary works. The analytic expression a_l^T extends those in Refs.13 in Newtonian gauge, and fulfils what was not completed in Ref.12, and, to a great extent, improves what was given in Ref.14. The analytic expressions of polarization a_l^E and the related spectra C_l^{EE} , and C_l^{TE} are what have not been addressed in Refs.13, 14. The spectra are calculated as the integrations over the wavenumber k , $C_l^{TT} = \int |a_l^T|^2 k dk$, $C_l^{TE} = \int a_l^T a_l^E k dk$, $C_l^{EE} = \int |a_l^E|^2 k dk$. Many features are revealed by these analytic expressions, here we only mention a few of them.

Figure 1 shows the analytical spectra $l(l+1)C_l^{XX}$. The first two peaks of C_l^{TT} occur at $l \sim 200$ and $l \sim 500$, while those of C_l^{EE} occur at $l \sim 100$ and $l \sim 400$. This

Fig. 1. The analytic spectra (solid line) compared with the numerical⁴ and the observed² for the scalar index $n_s = 0.96$, the running index $\alpha_s = 0$, and the baryon fraction $\Omega_b = 0.045$.

Fig. 2. The four terms, α_0 , α_1 , the ISW, and $M(\tau_d)\Delta\tau_d$ in Eq. (3) are schematically shown for comparison.

alternating occurrence of the peak locations has been anticipated from our analytic results. The overall profiles of the analytical spectra agree well with the observed result from WMAP5² and numerical result from CAMB⁴ on large angular scales with $l \lesssim 500$. On smaller angular scales, the analytical results deviate considerably from the numerical ones. This has been expected since our calculation is based upon the long wavelength approximation.

Figure 2 demonstrates the contributions by each term to a_l^T in Eq. (3). The contribution by ISW is at most $\sim 10\%$ that of the α_1 term. The contribution by the last term in Eq. (3) is even smaller than the ISW term since $\Delta\tau_d \sim 0.003$ is small, with two low bumps at $l \sim 130$ and at $l \sim 350$. Thus, the overall profile of a_l^T is largely determined by $\alpha_0(\tau_d)$ and $\alpha_1(\tau_d)$.

Fig. 3. A greater $\Delta\tau_d$ yields a higher amplitude of C_l^{EE} , and brings more small-scale damping.

Fig. 4. The two bump locations of C_l^{EE} is determined by $M(\tau_d)$.

Figure 3 tells that a longer recombination process (a greater $\Delta\tau_d$) yields a higher amplitude of polarization, since $a_l^E \propto \Delta\tau_d$ by Eq. (4). It also shows that a longer recombination process brings more damping of C_l^{EE} on small scales. This is because a_l^E in Eq. (4) contains the damping factor $D_E(k) \propto e^{-c_E(k\Delta\tau_d)^2}$. Similarly, this feature also is shared by C_l^{TT} . These properties may help to constrain the recombination width $\Delta\tau_d$ by the observational data.

In Figure 4, we sketch the profile of $l(l+1)C_l^{EE}$ as a function of l , which has two bumps, one at $l \sim 100$, and another at $l \sim 400$. In order to interpret the origin of these two bumps, we also sketch the main factor $D_E(k)M(\tau_d)/[k(\tau_0 - \tau_d)]^2$ of a_l^E in Eq. (4) as a function of k . It is $D_E(k)M(\tau_d)$ with two bumps around $k \sim 100$ and $k \sim 400$ that gives rise to the two bumps of C_l^{EE} .

Acknowledgments

Y. Zhang would like to thank the organizers of the 3rd Galileo - Xu Guangqi meeting. The work is supported by the CNSF No.10773009, SRFPD, and CAS.

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