LOCALIZATION AND KNOTS

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We study the $1/2$ BPS condition of Wilson loop on squashed three-sphere, and find that the solution becomes torus knot or unknot. We also calculate the expectation value of $1/2$ BPS Wilson loops by using localization technique with the gauge group $U(2)$ and Wilson loop as fundamental representation. And it completely matches with known results.

Keywords: Localization; knots.

1. Introduction

As well known, the expectation values of fundamental-representation Wilson loops with pure $SU(2)$ Chern-Simons theory on the three-sphere turn to be celebrated Jones polynomials.¹ On the other hand, one can perform exact calculation of $\mathcal{N} = 2$ SUSY Chern-Simons theory's path integral by using localization technique. We can check the consistency between these two exact results through the knot theory.

2. $1/2$ BPS Condition

We use localization procedure on squashed three-sphere.² The metric is

$$ds^2 = f(\theta)^2 d\theta^2 + l^2 \cos^2 \theta d\phi^2 + \tilde{l}^2 \sin^2 \theta d\chi^2, \quad f(\theta) = \sqrt{l^2 \sin^2 \theta + \tilde{l}^2 \cos^2 \theta}. \quad (1)$$

We calculate following supersymmetric Wilson loop

$$W_S = \text{Tr}_R \mathcal{P} \exp \left(\oint_C d\tau (i A_\mu \dot{x}^\mu + \sigma |\dot{x}|) \right) \quad (2)$$

weighted by $\exp(ik S_{SCS})$ where

$$S_{SCS} = \int d^3x \sqrt{g} \text{Tr} \left[\frac{1}{\sqrt{g}} \epsilon^{\mu\nu\lambda} (A_\mu \partial_\nu A_\lambda - \frac{2i}{3} A_\mu A_\nu A_\lambda) - \bar{\lambda} \lambda + 2D\sigma \right]. \quad (3)$$

In order to apply localization technique, Wilson loop contour $x^\mu(t) = (\theta(t), \phi(t), \chi(t))$ must satisfy following $1/2$ BPS condition³

$$\dot{x}^\mu \frac{\partial}{\partial x^\mu} = \begin{cases} \frac{1}{l} \frac{\partial}{\partial \phi} - \frac{1}{\tilde{l}} \frac{\partial}{\partial \chi} & (\theta \neq 0, \frac{\pi}{2}) \\ \frac{1}{l} \frac{\partial}{\partial \phi} & (\theta = 0) \\ -\frac{1}{\tilde{l}} \frac{\partial}{\partial \chi} & (\theta = \frac{\pi}{2}) \end{cases}. \quad (4)$$

3. Integral Curves and Results of $U(2)$ Gauge Theory

We summarize the results on the following table.

	$\theta = 0$	$\theta \neq 0, \pi/2$	$\theta = \pi/2$
1/2 BPS Wilson loop on S^3
1/2 BPS Wilson loop on $T^2 \subset S^3$
$\frac{\langle W_S \rangle}{\langle 1 \rangle}$	$e^{-\frac{l}{\tilde{l}} \frac{2\pi i}{k}} \frac{q-q^{-1}}{q^{1/2}-q^{-1/2}}$	$e^{-l\tilde{l} \frac{2\pi i}{k}} q^{\frac{1}{2}(l-1)(\tilde{l}-1)} \times \frac{1-q^2}{(1+q^{l+\tilde{l}}-q^{l+1}-q^{\tilde{l}+1})} \frac{q-q^{-1}}{q^{1/2}-q^{-1/2}}$	$e^{-\frac{\tilde{l}}{l} \frac{2\pi i}{k}} \frac{q-q^{-1}}{q^{1/2}-q^{-1/2}}$

For simplicity, we draw $(\theta \neq 0, \pi/2)$ picture by substituting $(l = 2, \tilde{l} = 3)$. Here we performed the calculation with gauge group $U(2)$ and Wilson loop's representation R as fundamental-representation. And we denote $e^{\frac{2\pi i}{k}}$ as q . These results correspond to the level $(k - 2)$ pure Chern-Simons theory.

4. Remark

Polynomials on the table completely match with $(l, \tilde{l})$ -torus knot Jones polynomial and unknot Jones polynomial respectively up to overall phase factor. This phase comes from the point-splitting regularization of Wilson loop. It is also possible to calculate the correlation function of many 1/2 BPS Wilson loops by inserting 1/2 BPS Wilson loops into the integrand.³

References

1. E. Witten, "Quantum Field Theory and the Jones Polynomial," *Commun. Math. Phys.* **121** (1989) 351.
2. N. Hama, K. Hosomichi, and S. Lee, "SUSY Gauge Theories on Squashed Three-Spheres," *JHEP* **05** (2011) 014, [arXiv:1102.4716 \[hep-th\]](#).
3. A. Tanaka, "Comments on knotted 1/2 BPS Wilson loops," [arXiv:1204.5975 \[hep-th\]](#).