TOWARDS THE PROOF OF AGT-W RELATION: TODA 3-POINT FUNCTION AND $\mathcal{W}_{1+\infty}$ ALGEBRA

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The AGT-W relation is a conjecture of the nontrivial duality between 4-dim quiver gauge theory and 2-dim conformal field theory. We verify a part of this conjecture for all the cases of quiver gauge groups by studying on the property of 3-point correlation function of conformal theory. We also mention the relation to $\mathcal{W}_{1+\infty}$ algebra as one of the promising direction towards the proof of the remaining part.

Keywords: String duality; p -branes; conformal field theory.

1. Introduction

The AGT-W relation is a nontrivial correspondence between the partition function of 4-dim $\mathcal{N} = 2$ $SU(N)$ quiver gauge theory and the correlation function of 2-dim A_{N-1} Toda conformal field theory. First, this relation was conjectured in $SU(2)$ case,^{1,2} and soon after that it was generalized to $SU(N)$ case.³ For a general case of $SU(d_1) \times \cdots \times SU(d_n)$ quiver gauge theory, satisfying $(d_i - d_{i+1}) - (d_{i-1} - d_i) \geq 0$ for all i in order to be conformal, the AGT-W relation is of the form

$$\begin{aligned} & \left\langle V_{\vec{\beta}_\infty}(\infty) V_{\vec{\beta}_0}(1) V_{\vec{\beta}_1}(q_1) \cdots V_{\vec{\beta}_n}(q_1 \cdots q_n) V_{\vec{\beta}_{n+1}}(0) \right\rangle \\ &= \int \prod_i d\vec{a}_i \left| Z_{\vec{\beta}_\infty}^{\vec{\beta}_0} \vec{a}_0^{\vec{\beta}_1} \vec{a}_1^{\vec{\beta}_2} \cdots \vec{a}_n^{\vec{\beta}_{n+1}}(q_1, \dots, q_n) \right|^2 \end{aligned} \quad (1)$$

up to so-called the $U(1)$ factor. Here, the left hand side is the $(n+3)$ -point correlation function of Toda theory, and Z in the right hand side is the partition function of gauge theory. $V_{\vec{\beta}}(z) = e^{\vec{\beta} \cdot \vec{\varphi}(z)}$ is a Toda vertex, and $\vec{a}$ is a Cartan of the VEV of the adjoint scalar field in gauge theory. q_i is a coupling constant for each $SU(d_i)$ gauge theory, which is related to the position of Toda vertices as in Eq. (1). Toda momentum $\vec{\beta}$ corresponds to the mass of the matter fields in gauge theory.

Towards the proof of this relation, we must clearly understand *i)* the concrete form of Toda vertex in AGT-W relation, *ii)* how to calculate the conformal blocks of

Toda theory, and *iii*) the concrete form of $U(1)$ factor. We have already conjectured the concrete form of Toda vertex,⁴ and checked it in $SU(3)$ quiver case.⁵ Then we try to verify our conjecture for all the cases of general $SU(N)$ quiver gauge groups.

2. Results and Future Directions

In our conjecture,⁴ Toda momentum $\vec{\beta}$ is always of the form

$$\vec{\beta} = (\underbrace{\beta_{(1)}, \dots, \beta_{(1)}}_{l_1}, \dots, \underbrace{\beta_{(s)}, \dots, \beta_{(s)}}_{l_s}) \quad (2)$$

where $[l_1, \dots, l_s]$ can be a Young tableau with N boxes. The 3-point correlation function of A_{N-1} Toda theory has been obtained⁶ only if one of the three vertices satisfies $l_{1 \text{ or } 2} = N - 1$. In AGT-W relation, fortunately, all the 3-point functions obtained by the pants decomposition of the whole correlation function satisfy this condition. Then by using the property of this function, we can show the primary part of the correlation function of Toda theory, which is just the product of these 3-point functions, corresponds to the 1-loop part of the partition function of quiver gauge theory under the certain correspondence of parameters in both theories:

$$\langle V \cdots V \rangle_{\text{primary}} = |Z_{1\text{-loop}}|^2 \quad (3)$$

up to a divergent factor. The degree of this divergence is interpreted as the number of Hanany-Witten transition in the corresponding D4/NS5/D6-brane system.

The remaining part is the correspondence between the descendant part of Toda correlation function and the instanton part of partition function of gauge theory. The proof is very difficult because of the last two problems *ii*) and *iii*) listed above. One of the promising direction is to consider the $\mathcal{W}_{1+\infty}$ algebra:⁸ This algebra has a $U(1)$ generator besides the W_N generators of A_{N-1} Toda theory, so it should naturally derive the concrete form of $U(1)$ factor. Moreover, this algebra is linear, so the recursion relation to calculate the conformal blocks may be more simple.

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