KILLING–YANO SYMMETRY IN SUPERGRAVITY THEORIES

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Killing–Yano symmetry has played an important role in the study of black hole physics. In supergravity theories, Killing–Yano symmetry is deformed by the presence of the fluxes which can be identified with skew-symmetric torsion. Therefore, we attempt to classify spacetimes admitting Killing–Yano symmetry with torsion. In particular, the classification problem of metrics admitting a principal Killing–Yano tensor with torsion is discussed.

Keywords: Killing–Yano symmetry; Torsion; black hole; supergravity.

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1. Introduction

Killing–Yano tensors were introduced first by Yano¹ and were later generalized to conformal Killing–Yano tensors by Tachibana² and Kashiwada.³ Due to the presence of such tensors, the Kerr’s black hole spacetime possesses many remarkable properties in mathematical physics. Furthermore, it is known that the existence of the Killing–Yano symmetry extends to many higher dimensional vacuum solutions of Einstein’s equations with cosmological constant describing rotating black holes with spherical horizon topology.^{4–6}

Unfortunately, the standard Killing–Yano symmetry disqualifies supergravity black hole solutions in the presence of fluxes. Therefore, a notion of generalized Killing–Yano symmetry was introduced⁷ by considering a skew-symmetric torsion. Given a metric g and a skew-symmetric torsion T on a D -dim manifold, a *rank- p conformal Killing–Yano (CKY) tensor with torsion k* is defined as a p -form satisfying

$$\nabla_X^T k = \frac{1}{p+1} X \lrcorner d^T k - \frac{1}{D-p+1} X^b \wedge \delta^T k \quad (1)$$

for any vector field X . In particular, a rank-2 CKY tensor with torsion h satisfying $d^T h = 0$ is called a *principal Killing–Yano (PKY) with torsion*. Such generalized Killing–Yano symmetry naturally occurs in some higher dimensional charged black hole spacetimes of supergravity theories, where the torsion is identified with the fluxes of the matter fields.

When the torsion is absent, metrics with the Killing–Yano symmetry were classified in four and higher dimensions, and it was shown that a vacuum solution admitting the PKY tensor without torsion is uniquely given by the black hole metric found by Chen, Lü and Pope.⁶ Thus, it is natural task to attempt to classify spacetimes admitting the Killing–Yano symmetry when the torsion is present. In particular, we concentrate on metrics admitting the PKY tensor with torsion.

2. Local Metrics Admitting a PKY Tensor with Torsion

The classification problem is reduced to some algebraic equations and a system of nonlinear partial differential equations.⁸ It is demonstrated from the algebraic equations that there are three possible distinct types of metrics, which we call types A, B and C. For each type, we are able to find large families of solutions of the partial differential equations in all dimensions, which cover a wide variety of solutions of various supergravity theories.

2.1. Type A

Type A metrics can be regarded as a natural generalization of black hole spacetimes and provide a unified description of several known solutions in supergravity theories. The metric in $2n + \varepsilon$ dimensions where $\varepsilon = 0$ for even dimensions or $\varepsilon = 1$ for odd dimensions is given in the form

$$ds^2 = \sum_{\mu=1}^n \frac{U_\mu}{X_\mu} dx_\mu^2 + \sum_{\mu=1}^n \frac{f_\mu^2 X_\mu}{U_\mu} \left(\sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k - A_{(1)} \right)^2 + \varepsilon \left(\frac{k_1}{\sigma} \left(\sum_{k=0}^n A^{(k)} d\psi_k - A_{(1)} \right) + k_2 \sigma (d\psi_n - A_{(1)}) \right)^2, \quad (2)$$

where k_1 and k_2 are constant, $A_\mu^{(k)}$ and $A^{(k)}$ are elementary symmetric polynomials in x_μ^2 of k -th order, the other functions and one-form are

$$U_\mu = \prod_{\nu \neq \mu} (x_\mu^2 - x_\nu^2), \sigma = \prod_{\rho=1}^n x_\rho, \quad (3)$$

$$A_{(1)} = \frac{1}{\Phi} \sum_{\mu=1}^n \frac{N_\mu}{U_\mu} \sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k, \Phi = 1 + \sum_{\mu=1}^n \frac{N_\mu}{U_\mu}.$$

This metric contains $3n$ unknown functions $f_\mu(x_\mu)$, $N_\mu(x_\mu)$ and $X_\mu(x_\mu)$ which depend only on one variable x_μ , and covers the four⁹ and higher^{10,11} dimensional Kerr–Sen black hole metrics in Abelian heterotic supergravity when provided that the functions f_μ , N_μ and X_μ take the form

$$f_\mu = 1, \quad N_\mu = \sum_{k=0}^{n-1} b_k x_\mu^{2k} + a m_\mu x_\mu^{1-\varepsilon}, \quad (4)$$

$$X_\mu = \sum_{k=0}^{n-1} c_k x_\mu^{2k} + m_\mu x_\mu^{1-\varepsilon} + \varepsilon \frac{(-1)^n c}{x_\mu^2},$$

where m_μ ($\mu = 1, \dots, n$), a , b_k , c_k ($k = 0, \dots, n-1$) and c are free parameters. Then the torsion T is identified with the 3-form field strength H .

In odd dimensions, we additionally obtain another family of metrics which is given by

$$ds^2 = \sum_{\mu=1}^n \frac{U_\mu}{X_\mu} dx_\mu^2 + \sum_{\mu=1}^n \frac{X_\mu}{U_\mu} \left(\sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k \right)^2 + k^2 \sigma^2 (d\psi_n - B_{(1)})^2 \quad (5)$$

where

$$B_{(1)} = \sum_{\mu=1}^n \frac{N_\mu}{U_\mu} \sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k. \quad (6)$$

This metric, this time, contains $2n$ unknown functions $N_\mu(x_\mu)$ and $X_\mu(x_\mu)$ which depend only on one variable x_μ . In five dimensions, this metric covers the charged rotating black hole metric of five-dimensional gauged minimal supergravity, discovered by Chong, Cvetič, Lü and Pope,¹² when provided that the functions N_μ and X_μ ($\mu = 1, 2$) take the form

$$\begin{aligned} N_\mu &= b_1 x_\mu^2 + b_0 + \frac{q_\mu}{x_\mu^2}, \\ X_\mu &= c_2 x_\mu^4 + c_1 x_\mu^2 + m_\mu + \frac{k^2 q_\mu^2}{x_\mu^2}, \end{aligned} \quad (7)$$

where c_1 , b_0 , b_1 , m_μ , k , q_1 and q_2 are free parameters. The torsion is identified with the Hodge dual of the Maxwell field: $T = *F/\sqrt{3}$.

2.2. Type $B\mathcal{E}C$

Types B and C are exceptional metrics appearing only when the torsion is present. The physical significance of the metrics is not well understood at the moment. It would be interesting to examine the physical meaning of types B and C metrics in the future.

3. Recent Developments

Geometry with the generalized Killing–Yano symmetry in the presence of torsion is also related to the Kählerian geometry studied by Apostolov, Calderbank and Gauduchon¹³ in even dimensions and the Sasakian geometry discussed by Gauntlett, Martelli, Sparks and Waldram^{14,15} in odd dimensions. In fact, it is shown that the Killing–Yano tensors with torsion arise on these Kähler and Sasaki manifolds.^{8,16} As these geometries are deeply related to supersymmetric compactifications and AdS/CFT correspondence in string theories, this work might provide an *ansatz* for finding other new exact solutions of various supergravity theories.

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