ESTIMATION OF FLUX BETWEEN INTERACTING NODES ON HUGE INTER-FIRM NETWORKS

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We analyze Japanese inter-firm network data showing scale-free properties as an example of a real complex network. The data contains information on money flow (annual transaction volume) between about 7000 pairs of firms. We focus on this money-flow data and investigate the correlation between various quantities such as sales or link numbers. We find that the flux from a buyer to a supplier is given by the product of the fractional powers of both sales with different exponents. This result indicates that the principle of detailed balance does not hold in the real transport of money; therefore, random walk type transport models such as PageRank are not suitable.

Keywords: Complex network; non-linear interaction; scale free.

1. Introduction

Studies on complex networks have ignited by the concept of preferential attachment proposed by Barabasi and Albert¹, and there has been a strong trend toward researching various properties of complex networks^{2,3}. Recently, transport models on complex networks have been attracting growing attention⁴. For example, random walks on complex networks are being intensively investigated following the success of PageRank, which is an index to evaluate the importance of Web pages by considering artificial random walk jumping over homepages. There are many studies

on the transportation on complex networks for evaluating nodes and explaining the characteristic value of nodes from network topologies using mathematics and physics, both fundamentally and technically⁵⁻⁹; however, there have been no empirical studies on the analysis of flow among nodes in a real complex network, which is scale free.

In view of the empirical studies on complex systems and scale-free networks in the real world, business firms are attracting the attention of statistical physicists because they show various types of scale-free properties confirmed by data in the form of financial reports¹⁰⁻¹³. The cumulative distribution function of sales, employees, and degrees of trading links are known to obey power laws with power exponents of 1.0, 1.3, and 1.3-1.4, respectively. In addition, the inter-firm network has been reported¹⁰ to be a small world network with a mean pair distance of about 5, nonlinear correlation has been confirmed between sales and link degrees^{11,13}, and a weak negative degree-degree correlation exists as a characteristic property of the inter-firm network^{11,12}.

In this study, we analyzed data of an inter-firm network compiled by Teikoku Databank^a (TDB) in 2011, which describes details of trading relations. This dataset contains partial information about annual transaction volumes, and we can directly observe the money flow from a buyer to a supplier of materials or services. In the next section, we explain the basic properties of the data we analyzed. In sections 3 and 4, we review the scaling properties of individual firms and network structures, respectively. In section 5, we estimate the functional form of the interaction between a pair of firms by analyzing the money-flow data and compare the results with that of the PageRank model. Finally, we conclude with a discussion in section 6.

2. Dataset

The data we analyzed were compiled by TDB in January 2011. It consists of finance time-series data (COSMOS2) and buyer-supplier relation data (CCR). There are 13 job categories in the data, and we analyzed all the data except those of the firms with jobs related to finance/insurance and governmental associations. These data were found to be unsuitable. For finance/insurance, the amount of total deposits is described in the column of sales; however, this value is quite different from the sales for ordinary firms. For governmental associations, all important values such as sales and capitals were missing.

COSMOS2 (time-series) data consist of 2,334,118 Japanese firms and their corresponding financial values (sales, capitals, profits, number of employees, job category, fiscal year, address) for 20-30 years. Some of the gross properties of this data are listed in Table 1. We can compare these numbers with the official numbers

^aTEIKOKU DATABANK, LTD. is one of the largest corporate credit research company in Japan and was established in 1900. They have built a corporate database to estimate credit risks since 1972.

Table 1. Summary of gross quantities in the data for most recent five years (except finance and government categories).

year	total firms	total sales (trillion yen)	total employees (person)	total capitals (trillion yen)
2005	1213776	1294	2478	124
2006	1208060	1338	2481	123
2007	1213495	1385	2508	127
2008	1237443	1408	2527	130
2009	1297249	1319	2563	135

Table 2. Financial quantities for most recent five years (except finance/insurance and government categories from the Financial Statements Statistics of Corporations by Industry, Annual, Ministry of Finance, Japan).

	total firms	total sales
2005	2718777	1508
2006	2735630	1566
2007	2759279	1580
2008	2774434	1508
2009	2771912	1368

announced by the Ministry of Finance, Japan in the report¹⁴ of Financial Statements Statistics of Corporations listed in Table 2. When comparing Table 2 with Table 1, the COSMOS2 data contain about 50% of the firms and 85% of the sales of the official data each year. The firms missing in this dataset are known to be mostly very small firms with only a few workers, and we believe that the data are sufficient for investigating the statistical properties of the interaction of business firms.

The other dataset, the CCR (trade) data, consists of information on firms' trade relationships and was compiled in January 2011. The data contain 3,841,496 partnership relations among 612,133 firms and the corresponding information of each trade (e.g., name of the buyer/supplier, category of goods, etc.). The data contain 7,851 annual transaction volumes (money flow to the supplier). By removing the firms with job categories of finance/insurance and government, we eventually had 3,678,870 business links among 604,663 firms with 7,779 links characterized quantitatively by the amount of annual money flow. More details on the CCR data are summarized in Table 3.

In the Japanese economy, firms in construction, manufacturing, and whole-sale/retail trade account for 65% of all firms and 70%-80% trades of all observed trades. We can say that the Japanese inter-firm trades mainly consist of firms in these three categories.

Table 3. Job categories of buyers/suppliers in CCR data.

	total firms	total trades (as a buyer)	total trades (as a supplier)
agriculture	2852	9960	7902
forestry	299	642	832
fisheries	726	2688	1609
mining	1142	5015	7680
construction	144463	733788	525097
manufacturing	115739	972082	1050261
wholesale/retail trade	192452	1235640	1573191
finance/insurance	3281	40628	22857
real estate	17962	70612	38233
transport/information/communications	28769	186822	158633
electricity/gas/heat/water	383	12560	3891
services	99876	470438	448021
government	4189	100621	3271
total	612133	3841496	3841496

3. Basic Properties

Here, we review the basic properties of individual business firms by analyzing the CCR data. As mentioned above, we analyzed all of the data except those of the finance/insurance and government categories. We have already pointed out that the distributions of sales and employees obey power laws; in particular, for sales, the exponent is close to -1, which is called Zipf's law. We confirmed these results in the CCR data, as shown in Figs. 1 and 2. Each distribution was approximated by a power law with exponents of 1.05 and 1.3, respectively. This is consistent with the former results based on different data^{10,12,13}. In addition, the distribution of firms' capitals is plotted in Fig. 3. This distribution has several characteristic steps and hence cannot be approximated by a power law. This deviation from a power law may have been caused by minimum capital regulation or a discrete change in tax rate as a function of the capital. The distribution of the growth rate of firms is known to obey a tent-like distribution with the peak value at unity, where the

Fig. 1. Cumulative distribution of sales in log-log scale

Fig. 2. Cumulative distribution of the number of employees in log-log scale

Fig. 3. Cumulative distribution of capitals in log-log scale

Fig. 4. Probability density of growth rate from 2009 to 2010 in log-log scale

growth rate is defined by the value of a quantity such as sales in the specific year divided by the same quantity one year before¹³. The probability density of growth rates for sales, incomes, and number of employees are plotted in a semi-log plot in Fig. 4. In this plot, the Gaussian distribution is represented by a parabolic curve, and we confirmed that the growth rate probability density is higher at the center and that there are long tails in both sides for all cases, as already pointed out.

Next, we confirm the correlation between sales and the number of employees^{15,16}. In Fig. 5, these two values are plotted in log-log scale. The points are scattered around the nonlinear relation $S \propto n^{1.3}$ for n larger than 1.5×10^3 , and the correlation coefficient is 0.50 for all the plots. The fact that the amount of sales is given by a nonlinear function of the number of employees implies that larger firms are more efficient in gathering money than smaller firms.

4. Properties of the inter-firm network

In general, a business transaction is made between a pair of firms, where one is a buyer and the other is a supplier, with the money flow from buyer to supplier. The CCR data contain lists of business partners for each firm with their roles, and

Fig. 5. Relation between sales and the number of employees

we can draw a direct line for each pair of business partners showing the money flow, as shown in Fig. 6. The opposite direction of the link is the direction of the goods/service flow.

We can define the adjacency matrix corresponding to the money flow network as follows.

$$A_{ij} = \begin{cases} 1 & \text{if money flow } i \rightarrow j \text{ exists} \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

The adjacency matrix has $604,663 \times 604,663$ elements with 3,678,870 non-zero elements, and all other matrix elements (about 3.6×10^{11}) are 0; namely, the connection is very sparse with the average number of links per node being 6.1. In Fig. 8, the cumulative distributions of link numbers for in-links and out-links, which are also called in-degrees and out-degrees, are plotted in log-log scale. Both distributions are quite similar, and they are approximated by power laws with exponents of about 1.4; this is consistent with the results already reported from the analysis of a different database in a different year^{10–13}. As a consequence of this scale-free property, a majority of firms are transacting with only a few firms; on the other hand, there are big firms with thousands of business partners.

As a basic property of the network, we calculate the correlation coefficient defined with the following formula.

$$r = \frac{4M \sum_{i,j}^N A_{ij} k_i k_j - \left[\sum_{i,j}^N A_{ij} (k_i + k_j) \right]^2}{2M \sum_{i,j}^N A_{ij} (k_i^2 + k_j^2) - \left[\sum_{i,j}^N A_{ij} (k_i + k_j) \right]^2} \quad (2)$$

where M is the total number of links, N is the total number of nodes, k_i denotes the degree (either in-degree or out-degree) of node i , and A_{ij} is an adjacency matrix of this network. The correlation coefficients for out-degree:out-degree, in-degree:in-degree, out-degree:in-degree, in-degree:out-degree and total-degree:total-degree (total-degree is the sum of in-degree and out-degree) were -0.047, -0.052, -0.065, -0.034, and -0.072, respectively. Weak negative correlations were confirmed in this inter-firm network as reported earlier^{11,12}.

In order to clarify this negative correlation of links, we focused on the nodes with given link numbers and observed the average link numbers of its neighbors. In Fig. 9,

Fig. 6. Schematic view of a money flow

Fig. 7. Example of the inter-firm network (part of business relations in Aomori Prefecture)

Fig. 8. Cumulative distribution of out-degree and in-degree in log-log scale

Fig. 9. Average degree of nearest neighbours vs degree

these conditional averages of the neighbors' degrees are plotted in log-log scale. As shown in this figure, larger degree nodes tend to be connected with smaller degree nodes. Link numbers are known to be strongly correlated with sales¹¹. Figs. 10 and 11 show the scatter plots for sales and degrees in log-log scale. The correlation coefficients of sales:in-degree and sales:out-degree in log-log scale were 0.37 and 0.54, respectively. For larger degrees, we can find a nonlinear power law relation $s_i \propto k_i^{1.3}$ like the case of employees for both in-degrees and out-degrees. Clearly, the network structure of firms plays an important role in determining the sales.

5. Interaction between nodes

As already mentioned, the CCR data contain detailed information on the money flow between pairs of business partners: there were data on 7,779 annual transactions involving 7,300 firms. Table 4 lists the money flow data in each job category. We analyzed this money flow data and observed the interaction among firms. As shown

Fig. 10. Relation between sales and out-degree

Fig. 11. Relation between sales and in-degree

in the table, the firms in the manufacturing and wholesale/retail categories were dominant in this data. As shown in Table 3, these job categories were also dominant in terms of overall number of firms and overall sales.

5.1. *PageRank model*

One of the most popular theoretical models of transport defined on directed complex networks is the PageRank model¹⁷. PageRank at the i -th node x_i is defined by the steady solution of the following equation.

$$x_i(t+1) = r \sum_{j=1}^N A_{ji} \frac{x_j}{k_j^{out}} + 1 - r \tag{3}$$

where A_{ij} is the adjacency matrix, k_i^{out} is the out-degree of the i -th node, and r is a parameter taking a value between 0 and 1 that stands for the ratio of random jump to any node. The value of PageRank in the limit of $r \rightarrow 0$ is the existence probability of a random walker on the network. PageRank was also studied analytically, and by applying the mean field approximation, it was proved to behave as

$$x_i \propto k_i^{in} \tag{4}$$

Namely, x_i is simply proportional to the value of in-degree, which has been confirmed numerically for the business firm network¹⁸. This result is unfavorable for PageRank to become a model of money flow on the network, because if we regard the value of PageRank to be proportional to the sales, then, the solution does not fulfill the empirical relation of Fig. 11, $S_i \propto (k_i^{in})^{1.3}$. It is likely that real transport between firms is not approximated by the simple random walk.

Table 4. Job categories and number of observable money flow data.

	total firms	total trades (as a buyer)	total trades (as a supplier)
agriculture	8	4	8
forestry	0	0	0
fisheries	5	6	1
mining	5	5	3
construction	451	566	207
manufacturing	2645	2740	2665
wholesale/retail trade	3136	3537	4009
finance/insurance	32	28	9
real estate	44	46	31
transport/information/communications	380	353	483
electricity/gas/heat/water	17	19	13
services	549	517	417
government	28	30	5
total	7300	7851	7851

5.2. Estimating interaction

In order to clarify the underlying interaction among nodes, we observed the correlation between the strength of money flow for a link and characterizing quantities such as sales or link numbers at the nodes. We assumed that the money flow or flux is described by the following functional form with any function of f and g :

$$flux(A \rightarrow B) \propto f(X_A)g(X_B) \quad (5)$$

where A and B denote the buyer and supplier, respectively; the representative variable X actually means sales S; capital is C; the number of employees is E; the in-degree is I; and the out-degree is O. Let us estimate the functional form of f and g of X_A and X_B by observing the dependence on one variable while the other variable is fixed.

All figures listed below show the values of flux as the vertical axis and the values of a representative variable as the horizontal axis in log-log scale, while the values of fixed variables are categorized into three classes (small, medium and large). For example, in Fig. [12], the values of flux are plotted as a function of buyer's sales, while the supplier's sales are categorized. Here, the symbol $S_B 10^4$ [MYen] represents the selected data points where the supplier's sales S_B are approximately 10^4 million yen. The dotted straight lines in these figures show theoretical guide lines representing power law functions that fit well with the plots.

In the case of sales, we can find rather clear power law relations in the smaller-scale range of sales for both buyers and suppliers, as seen in Figs. 11 and [12]. As for the number of employees, we can also find rough power law relations, as seen in Figs. 13 and 14, although these relations look less clear for categories with small numbers of employees. For the capital shown in Figs. 15 and [16], no remarkable relation can be observed. From Figs. 17 to 20, we can find power law relations for categories with large degrees; however, the relations are not clear for categories with small degrees. Summarizing these results, we propose that the best combination of

Fig. 12. Relation between money flow and buyer's sales in log-log scale with fixed supplier's sales

Fig. 13. Relation between money flow and supplier's sales in log-log scale with fixed buyer's sales

Fig. 14. Relation between money flow and buyer's employees in log-log scale with fixed supplier's employees

Fig. 15. Relation between money flow and supplier's employees in log-log scale with fixed buyer's employees

Fig. 16. Relation between money flow and buyer's capitals in log-log scale with fixed supplier's capitals

Fig. 17. Relation between money flow and supplier's capitals in log-log scale with fixed buyer's capitals

Fig. 18. Relation between money flow and buyer's out-degree in log-log scale with fixed supplier's out-degree

Fig. 19. Relation between money flow and supplier's out-degree in log-log scale with fixed buyer's out-degree

Fig. 20. Relation between money flow and buyer's in-degree in log-log scale with fixed supplier's in-degree

Fig. 21. Relation between money flow and supplier's in-degree in log-log scale with fixed buyer's in-degree

representative variables is sales at the nodes, and the flux of money from a buyer of sales S_A to a seller of sales S_B is given as

$$flux(A \rightarrow B) \propto S_A^\alpha S_B^\beta \quad (6)$$

where the values of exponents are $\alpha = 0.85$ and $\beta = 0.65$, as estimated by fitting.

The functional form of flux given in the form of Eq.(6) is familiar in the fields of international economy and population sociology by the name of generalized gravity model; it is named so because it is analogousness with Newton's law of gravity^{8,9}. In these models, the amount of trade between two countries of GDPs S_A and S_B or traffic between two cities of population S_A and S_B is assumed to follow Eq.(6). Therefore, our results are not remarkable in terms of economy and sociology. Also, it is easy to show that in the special case of $\alpha \rightarrow 1$, $\beta \rightarrow 0$, the flux is equivalent to the case of a random walk on the network, namely, the case of PageRank. In this case, the amount of flow from a buyer node is proportional to the sales S_A , and all flows from this node are of the same amount, which is realized in the case of pure diffusion.

6. Conclusion

We analyzed the strength of interaction among nodes in the inter-firm network by observing large amounts of data of business firms. The relation we found is described by Eq.(6), which shows that the flux or annual money flow from a buyer firm to a supplier firm is proportional to the products of fractional powers of both sales with different exponents. The general form of flux given by Eq.(6) includes PageRank with the special case of $\alpha \rightarrow 1$, $\beta \rightarrow 0$. The values of the exponents that we obtained from the data ($\alpha = 0.85$, $\beta = 0.65$) are far from these values, indicating that the transport of money in the business firm network is far from the thermal diffusive transport in equilibrium.

In the near future, our task is to clarify the properties of network systems in which transport is governed by a form of Eq.(6). As this system is not in thermal equilibrium, we may expect various types of new behaviors.

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