AN ACCURATE SOLUTION TO THE LOTKA-VOLTERRA EQUATIONS BY MODIFIED HOMOTOPY PERTURBATION METHOD

M. S. H. CHOWDHURY

*Department of Science in Engineering,
 International Islamic University Malaysia,
 Jalan Gombak, 53100 Kuala Lumpur, Malaysia*

M. M. RAHMAN

*Department of Physics, Universiti Putra Malaysia,
 43400 UPM Serdang, Selangor Darul Ehsan, Malaysia
 mahmud@science.upm.edu.my*

In this paper, we suggest a method to solve the multispecies Lotka-Volterra equations. The suggested method, which we call modified homotopy perturbation method (HPM) which is very efficient in solving a variety of differential and algebraic equations. The HPM is modified in order to obtain the approximate solutions of Lotka-Volterra equation response in a sequence of time intervals. In particular, the example of two species is considered. The accuracy of this method is examined by comparison with the numerical solution of the Runge-Kutta-Verner method. The results prove that the modified HPM is a powerful tool for the solution of nonlinear equations.

Keywords: Homotopy-perturbation method; Lotka-Volterra systems; long term behavior; Adomian decomposition method.

PACS numbers: 02.60.Lj, 05.45.-a

1. Introduction

Homotopy perturbation method^{1–6} is an analytical method which can be applied to the solution of nonlinear deterministic and stochastic operator equations. HPM deforms a difficult problem into an infinite set of problems which are easier to solve without any need to transform nonlinear terms. The applications of HPM in nonlinear problems have been demonstrated by many researchers. In recent years, much attention has been devoted to the application of the HPM, to the solutions of various scientific models.^{7–10} HPM yields rapidly convergent series solutions.¹¹ Very recently, Chowdhury *et al.*,¹² Chowdhury and Hashim,¹³ Hashim and Chowdhury¹⁴ and Hashim *et al.*¹⁵ were the first to successfully apply the multistage homotopy-

perturbation method (MHPM) to the chaotic Lorenz system, Chen system and a class of systems of ODEs. This paper deals with the analysis of the Lotka-Volterra equations model the dynamical behaviour of an arbitrary number of competitors. Though originally formulated to describe the time history of a biological system, these equations find their application in a number of engineering fields such as simultaneous chemical and nonlinear control. In fact, the one-predator one-prey Lotka-Volterra model is one of the most popular ones to demonstrate a simple nonlinear control system. The accurate solutions of the Lotka-Volterra equations may become a difficult task either if the equations are stiff (even with a small number of species), or when the number of species is large. Very recently, Olek¹⁶ employed the Adomian decomposition method (ADM) and Chowdhury et al.¹⁷ applied HPM to solve the multispecies Lotka-Volterra equations. Inspired by the results,¹²⁻¹⁴ we are again interested to apply the MHPM for finding accurate approximate solutions to the Lotka-Volterra equations for long time span. Comparison with the Runge-Kutta method shall be made for two species Lotka-Volterra system to determine the performance of MHPM.

2. Analysis of Multispecies Lotka-Volterra Equations

Consider the general Lotka-Volterra model for an m species system given as

$$x'_i = x_i \left(b_i + \sum_{j=1}^m a_{ij} x_j \right), \quad i = 1, 2, \dots, m, \quad (1)$$

where the prime denotes differentiation with respect to t .

These equations may represent either predator-prey or competition cases.

3. Solution Approaches

Consider a general system of first-order ODEs:

$$\frac{u_1}{dt} + g_1(t, u_1, u_2, \dots, u_m) = f_1(t), \quad (2)$$

$$\frac{u_2}{dt} + g_2(t, u_1, u_2, \dots, u_m) = f_2(t), \quad (3)$$

$$\vdots$$

$$\frac{u_m}{dt} + g_m(t, u_1, u_2, \dots, u_m) = f_m(t), \quad (4)$$

subject to the initial conditions

$$u_1(t_0) = c_1, u_2(t_0) = c_2, \dots, u_m(t_0) = c_m. \quad (5)$$

First write system (2)–(4) in the operator form:

$$L(u_1) + N_1(u_1, u_2, \dots, u_m) - f_1 = 0, \quad (6)$$

$$L(u_2) + N_2(u_1, u_2, \dots, u_m) - f_2 = 0, \quad (7)$$

$$\vdots$$

$$L(u_m) + N_m(u_1, u_2, \dots, u_m) - f_m = 0, \quad (8)$$

subject to the initial conditions (5), where $L = d/dt$ is a linear operator and $N_1, N_2, \dots, N_m$ are the nonlinear operators. We shall next present the solution approaches for (6)–(8) based on the standard HPM and MHPM separately.

3.1. Solution by HPM

According to HPM, we construct a homotopy for (6)–(8) which satisfies the following relations:

$$L(u_1) - L(v_1) + pL(v_1) + p[N_1(u_1, u_2, \dots, u_m) - f_1] = 0, \quad (9)$$

$$L(u_2) - L(v_2) + pL(v_2) + p[N_2(u_1, u_2, \dots, u_m) - f_2] = 0, \quad (10)$$

$$\vdots$$

$$L(u_m) - L(v_m) + pL(v_m) + p[N_m(u_1, u_2, \dots, u_m) - f_m] = 0, \quad (11)$$

where $p \in [0, 1]$ is an embedding parameter and $v_1, v_2, \dots, v_m$ are initial approximations satisfying the given conditions. It is obvious that when the perturbation parameter $p = 0$, Eqs. (9)–(11) become a linear system and when $p = 1$ we get the original nonlinear system.

Let us take the initial approximations as follows:

$$u_1(t) = u_{1,0}(t) + pu_{1,1}(t) + p^2u_{1,2}(t) + p^3u_{1,3}(t) + \dots, \quad (12)$$

$$u_2(t) = u_{2,0}(t) + pu_{2,1}(t) + p^2u_{2,2}(t) + p^3u_{2,3}(t) + \dots, \quad (13)$$

$$\vdots$$

$$u_m(t) = u_{m,0}(t) + pu_{m,1}(t) + p^2u_{m,2}(t) + p^3u_{m,3}(t) + \dots, \quad (14)$$

and

$$u_{1,0}(t) = v_1(t) = u_1(t_0) = c_1, \quad (15)$$

$$u_{2,0}(t) = v_2(t) = u_2(t_0) = c_2, \quad (16)$$

$$\vdots$$

$$u_{m,0}(t) = v_m(t) = u_m(t_0) = c_m, \quad (17)$$

where $u_{i,j}$, ($i = 1, 2, \dots, m$; $j = 1, 2, \dots$) are functions yet to be determined. Substituting (12)–(17) into (9)–(11) and arranging the coefficients of the same powers

of p , we get

$$L(u_{1,1}) + L(v_1) + N_1(u_{1,0}, u_{2,0}, \dots, u_{m,0}) - f_1 = 0, \quad u_{1,1}(t_0) = 0, \quad (18)$$

$$L(u_{2,1}) + L(v_2) + N_2(u_{1,0}, u_{2,0}, \dots, u_{m,0}) - f_2 = 0, \quad u_{2,1}(t_0) = 0, \quad (19)$$

$$\vdots$$

$$L(u_{m,1}) + L(v_m) + N_m(u_{1,0}, u_{2,0}, \dots, u_{m,0}) - f_m = 0, \quad u_{m,1}(t_0) = 0, \quad (20)$$

$$L(u_{1,2}) + N_1(u_{1,1}, u_{2,1}, \dots, u_{m,1}) = 0, \quad u_{1,2}(t_0) = 0, \quad (21)$$

$$L(u_{2,2}) + N_2(u_{1,1}, u_{2,1}, \dots, u_{m,1}) = 0, \quad u_{2,2}(t_0) = 0, \quad (22)$$

$$\vdots$$

$$L(u_{m,2}) + N_m(u_{1,1}, u_{2,1}, \dots, u_{m,1}) = 0, \quad u_{m,2}(t_0) = 0, \quad (23)$$

etc. We solve the above systems of equations for the unknowns $u_{i,j}$ ($i = 1, 2, \dots, m; j = 1, 2, \dots$) by applying the inverse operator

$$L^{-1}(\cdot) = \int_{t_0}^t (\cdot) dt. \quad (24)$$

Therefore, according to HPM the n -term approximations for the solutions of (6)–(8) can be expressed as

$$\phi_{1,n}(t) = u_1(t) = \lim_{p \rightarrow 1} u_1(t) = \sum_{k=0}^{n-1} u_{1,k}(t), \quad (25)$$

$$\phi_{2,n}(t) = u_2(t) = \lim_{p \rightarrow 1} u_2(t) = \sum_{k=0}^{n-1} u_{2,k}(t), \quad (26)$$

$$\vdots$$

$$\phi_{m,n}(t) = u_m(t) = \lim_{p \rightarrow 1} u_m(t) = \sum_{k=0}^{n-1} u_{m,k}(t). \quad (27)$$

3.2. Solution by MHPM

The approximate solutions (25)–(27) are generally, as shall be shown in the numerical experiments of this paper, not valid for large t . A simple way of ensuring validity of the approximations for large t is to treat (18)–(23) as an algorithm for approximating the solutions of (2)–(5) in a sequence of intervals choosing the initial approximations as

$$u_{1,0}(t) = v_1(t) = u_1(t^*) = c_1^*, \quad (28)$$

$$u_{2,0}(t) = v_2(t) = u_2(t^*) = c_2^*, \quad (29)$$

$$\vdots$$

$$u_{m,0}(t) = v_m(t) = u_m(t^*) = c_m^*, \quad (30)$$

where t^* is the left-end point of each subinterval.

Now we solve (18)–(23) for the unknowns $u_{i,j}$ ($i = 1, 2, \dots, m; j = 1, 2, \dots$) by applying the inverse linear operator

$$L^{-1}(\cdot) = \int_{t^*}^t (\cdot) dt. \quad (31)$$

In order to carry out the iterations in every subinterval of equal length Δt , $[0, t_1)$, $[t_1, t_2)$, $[t_2, t_3)$... $[t_{j-1}, t)$, we need to know the values of the following,

$$u_{1,0}^*(t) = u_1(t^*), \quad u_{2,0}^*(t) = u_2(t^*), \quad \dots, \quad u_{m,0}^*(t) = u_m(t^*). \quad (32)$$

But, in general, we do not have these information at our clearance except at the initial point $t^* = t_0$. A simple way for obtaining the necessary values could be by means of the previous n -term approximations $\phi_{1,n}, \phi_{2,n}, \dots, \phi_{m,n}$ of the preceding subinterval given by Eqs. (25)–(27), *i.e.*

$$u_{1,0}^* \simeq \phi_{1,n}(t^*), \quad u_{2,0}^* \simeq \phi_{2,n}(t^*), \quad \dots, \quad u_{m,0}^* \simeq \phi_{m,n}(t^*). \quad (33)$$

4. Numerical Application

In this section, we shall study the HPM and the MHPM to solve the two-species Lotka-Volterra equation:

$$x'_1 = x_1(b_1 + a_{11}x_1 + a_{12}x_2), \quad (34)$$

$$x'_2 = x_2(b_2 + a_{21}x_1 + a_{22}x_2), \quad (35)$$

where $a_{11}, a_{12}, a_{21}, a_{22}, b_1$ and b_2 are constants and subject to the initial conditions

$$x_1(t^*) = c_1, \quad x_2(t^*) = c_2. \quad (36)$$

According to the HPM, we can construct a homotopy of system Eq. (34)–(35) as follows:

$$v'_1 - x'_0 + p(x'_0 - b_1v_1 - a_{11}v_1^2 - a_{12}v_1v_2) = 0, \quad (37)$$

$$v'_2 - y'_0 + p(y'_0 - b_2v_2 - a_{22}v_2^2 - a_{21}v_1v_2) = 0. \quad (38)$$

The initial approximations are

$$v_{1,0}(t) = x_0(t) = x_1(t^*) = c_1, \quad (39)$$

$$v_{2,0}(t) = y_0(t) = x_2(t^*) = c_2, \quad (40)$$

and

$$v_1(t) = v_{1,0}(t) + pv_{1,1}(t) + p^2v_{1,2}(t) + p^3v_{1,3}(t) + \dots, \quad (41)$$

$$v_2(t) = v_{2,0}(t) + pv_{2,1}(t) + p^2v_{2,2}(t) + p^3v_{2,3}(t) + \dots, \quad (42)$$

where $v_{i,j}$, $i = 1, 2$, $j = 1, 2, 3, \dots$ are functions yet to be determined. Substituting (39) and (41) into (37) and collecting terms the same powers of p , we have

$$v'_{1,1} - b_1 v_{1,0} - a_{11} v_{1,0}^2 - a_{12} v_{1,0} v_{2,0} = 0, \quad (43)$$

$$v'_{1,2} - b_1 v_{1,1} - 2a_{11} v_{1,0} v_{1,1} - a_{12} v_{1,1} v_{2,0} - a_{12} v_{1,0} v_{2,1} = 0, \quad (44)$$

$$v'_{1,3} - b_1 v_{1,2} - 2a_{11} v_{1,0} v_{1,2} - a_{11} v_{1,1}^2 - a_{12} v_{1,2} v_{2,0} - a_{12} v_{1,1} v_{2,1} - a_{12} v_{2,2} v_{1,0} = 0, \quad (45)$$

$$v'_{2,1} - b_2 v_{2,0} - a_{21} v_{1,0} v_{2,0} - a_{22} v_{2,0}^2 = 0, \quad (46)$$

$$v'_{2,2} - b_2 v_{2,1} - a_{21} v_{1,1} v_{2,0} - a_{21} v_{2,1} v_{1,0} - 2a_{22} v_{2,0} v_{2,1} = 0, \quad (47)$$

$$v'_{2,3} - b_2 v_{2,2} - a_{21} v_{1,2} v_{2,0} - a_{21} v_{2,1} v_{1,1} - a_{21} v_{2,2} v_{1,0} - 2a_{22} v_{2,0} v_{2,2} - a_{22} v_{2,1}^2 = 0. \quad (48)$$

In order to obtain the six unknowns $v_{i,j}(t)$ where $i = 1, 2$ and $j = 1, 2, 3$, we consider the initial conditions $v_{i,j}(0) = 0$, where $i = 1, 2$ and $j = 1, 2, 3$.

Solving differential equations (43)–(48) we get,

$$v_{1,1}(t) = (4b_1 + 16a_{11} + 40a_{12})t, \quad (49)$$

$$v_{2,1}(t) = (10b_2 + 100a_{22} + 40a_{21})t, \quad (50)$$

and the rest of the components $v_{1,2}(t)$, $v_{1,3}(t)$, $v_{2,2}(t)$, $v_{2,3}(t)$ can be obtained by using the Maple Package.

From (34)–(35), if four-term approximations are sufficient, we obtain form:

$$x_1(t) = \lim_{p \rightarrow 1} v_1(t) = \sum_{k=0}^4 v_{1,k}(t), \quad (51)$$

$$x_2(t) = \lim_{p \rightarrow 1} v_2(t) = \sum_{k=0}^4 v_{2,k}(t). \quad (52)$$

To carry out the iterations on every subinterval of equal length Δt , we need to know the values of the following initial conditions,

$$c_1 = x_1(t^*), \quad c_2 = x_2(t^*). \quad (53)$$

In general, we do not have these information at our clearance except at the initial point $t^* = t_0 = 0$ but we can obtain these values following the MHPM as given in section (3.2). We note that the 4-term approximations of x_1 , and x_2 are denoted as $x_1(t) \simeq \phi_4(t) = \sum_{i=0}^3 v_{1,i}$ and $x_2(t) \simeq \psi_4(t) = \sum_{i=0}^3 v_{2,i}$.

5. Results and Discussions

The MHPM algorithm is coded in the computer algebra package Maple and we employ the Maple's built-in fourth-order Runge-Kutta procedure (**rk4**). The Maple environment variable **Digits** controlling the number of significant digits is set to 16 in all the calculations done in this paper. For the comparison, we set $b_1 = 0.1$, $a_{11} = -0.0014$, $a_{12} = -0.0012$, $b_2 = 0.08$, $a_{21} = -0.0009$, $a_{22} = -0.001$, $x_1(0) =$

4 and $x_2(0) = 10$. The time range studied in this work is $[0, 100]$. In general, there is not known an exact solution for the two species Lotka-Volterra system. Thus, the accuracy of the present method will be determined by its comparison to the numerical solution by Runge-kutta method. In this analysis, we attempt to demonstrate the accuracy of the HPM and MHPM for the solutions of two species lotka-volterra systems. In this work we fix the number of terms used to be four and we select the time step size $\Delta t = 0.001$ in MHPM solutions and choose time step $\Delta t = 0.001$ for the RK4 method. The four-term approximate solutions to the two-species problem (34)–(35) obtained by HPM are

$$\begin{aligned}x_1(t) &\simeq 4 + \frac{206}{625}t + \frac{8643}{781250}t^2 + \frac{85978}{732421875}t^3, \\x_2(t) &\simeq 10 + \frac{83}{125}t + \frac{1347}{78125}t^2 + \frac{183589}{2343750000}t^3.\end{aligned}\tag{54}$$

In Figs. 1 and 2, we plot the 4-term HPM solutions and 4-term MHPM solutions (on $\Delta t = 0.001$) against the RK4 solutions on $\Delta t = 0.001$. Both the 4-term

Fig. 1. Graphical comparisons between HPM, MHPM ($\Delta t = 0.001$) and RK4 ($\Delta t = 0.001$) solutions for x_1 .

Fig. 2. Graphical comparisons between HPM, MHPM ($\Delta t = 0.001$) and RK4 ($\Delta t = 0.001$) solutions for x_2 .

MHPM solutions on $\Delta t = 0.001$ and RK4 solutions on $\Delta t = 0.001$ seem to overlap on the scale used in Figs. 1 and 2 but the numerical results from the standard HPM starts to stay away for both $x_1(t)$ and $x_2(t)$.

6. Conclusions

In this work, the MHPM was applied to the solutions of two species well known Lotka-Volterra system. The MHPM is only a simple modification of the standard HPM. This approximate analytic solution is an infinite power series. For practical computations, a finite number of terms in the series is used in a time step procedure. Comparisons between the HPM and MHPM solutions and the fourth-order Runge-Kutta (RK4) numerical solutions were made. For the two species system studied we found that the 4-term MHPM solutions achieved excellent accuracy compared with the RK4 solutions on a smiler time step. We note that the MHPM solutions were computed via a simple algorithm with less amount of computations and without any need for perturbation techniques, special transformations, linearization or discretization.

Acknowledgment

The author would like to acknowledge the financial supports received from the Ministry of Higher Education via the FRGS grant no. FRGS0409-106 and International Islamic University Malaysia through the research Endowment fund type B no. EDW B 0906-315.

References

1. J. H. He, *Comput. Methods Appl. Mech. Engrg.* **167**, 57 (1998).
2. J. H. He, *Intern. J. Non-Linear Mech.* **35**, 37 (2000).
3. J. H. He, *Chaos Solitons Fractals* **26**, 695, 700 (2005).
4. J. H. He, *Phys. Lett. A* **350**, 87 (2006).
5. J. H. He *Non-Perturbative Methods for Strongly Nonlinear Problems*, 1st edn. (Die Deutsche bibliothek, Germany, 2006).
6. J. H. He, *Int. J. Modern Phys. B* **20**(10), 1141 (2006).
7. M. S. H. Chowdhury and I. Hashim, *Nonlin. Anal. Real World Appl.* **10**, 104 (2009).
8. M. S. H. Chowdhury and I. Hashim, *Chaos Solitons Fractals* **39**, 1928 (2009).
9. M. S. H. Chowdhury and I. Hashim, *Phys. Lett. A* **365**, 439 (2007).
10. M. S. H. Chowdhury and I. Hashim, *Phys. Lett. A* **368**, 305 (2007).
11. M. El-Shahed, *Int. J. Nonlinear Sci. Numer. Simul.* **6**, 163 (2005).
12. M. S. H. Chowdhury, I. Hashim and S. Momani, *Chaos Solitons Fractals* **40**, 1929 (2009).
13. M. S. H. Chowdhury and I. Hashim, *Nonlin. Anal. Real World Appl.* **10**, 381 (2009).
14. I. Hashim and M. S. H. Chowdhury, *Phys. Lett. A* **372**, 470 (2008).
15. I. Hashim, M. S. H. Chowdhury and S. Mawa, *Chaos Solitons Fractals* **36**, 823 (2008).
16. S. Olek, *SIAM Review* **36**(3), 480 (1994).
17. M. S. H. Chowdhury, I. Hashim and O. Abdulaziz, *Phys. Lett. A* **368**, 251 (2007).