ANGULAR EIGENVALUES OF HIGHER-DIMENSIONAL KERR-(A)dS BLACK HOLES WITH TWO ROTATIONS

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In this paper, following the work of Chen, Lü and Pope, we present the general metric for Kerr-(A)dS black holes with two rotations. The corresponding Klein-Gordon equation is separated explicitly, from which we develop perturbative expansions for the angular eigenvalues in powers of the rotation parameters with $D \geq 6$.

Keywords: Kerr-(A)dS black holes; spheroidal harmonics; angular eigenvalues.

PACS numbers: 02.30.Mv, 04.50.-h, 04.70.-s, 11.25.-w

1. Introduction

After the advent of the brane world scenario¹ and the AdS/CFT correspondence², there has been an increase in interest in the study of higher-dimensional black holes. The Kerr metric was first generalized to higher dimensions in the seminal paper by Myers and Perry³. One of the unexpected results is that for some rotating black holes event horizons exist for arbitrarily large values of the rotation parameters. The stability of such black holes are certainly in question⁴. These are asymptotically flat

black holes. The first asymptotically non-flat higher-dimensional Kerr metric was given by Hawking *et al.*⁵ for five dimensions. Subsequent generalizations to arbitrary dimensions were done by Gibbons *et al.*⁶, finally resulting in the most general Kerr-(A)dS-NUT metric by Chen, Lü and Pope⁷.

The study of the wave equations in these black hole spacetimes was initiated by Frolov and Stojkovic⁸ by analyzing the Klein-Gordon equation in five dimensions. The analysis relied crucially on the method of the separation of variables. The problem of separability of these wave equations in higher dimensions is a difficult one. Even for the Klein-Gordon case, early attempts were only aimed at special cases⁹. Finally, using the Chen-Lü-Pope metric, Frolov, Krtous and Kubiznak¹⁰ were able to separate the geodesic equation and the Klein-Gordon equation in the most general setting.

However, the research on the scalar wave equation so far is focused mostly on the case with one rotation, the so-called simply-rotating case. Notably, the stability of the scalar perturbation in six and higher dimensions was considered in the ultra-rotating cases^{11,12,13}. Apparently no instability was found. Then, due to the interests in AdS spacetimes, these considerations were extended to Kerr-AdS black holes¹⁴. Here the expected superradiant instability did indeed show up. In addition the Hawking radiation in these spacetimes^{15,16} are calculated with possible application to the production and decay of LHC black holes¹⁷.

In this paper we would like to explore the cases with two rotations. In the next section we present the general metric of the Kerr-(A)dS black hole with two rotations. The corresponding Klein-Gordon equation will be separated into one radial equation and two angular ones. In Section 3, we shall develop perturbative expansions of the angular eigenvalues in powers of the rotation parameters. Conclusions and discussions are given in Section 4.

2. The metric and the separated equations of the Klein-Gordon equation with two rotations

In this section we shall first present the Kerr-(A)dS metric with two rotations. We start with the metric for $D = 2n$, that is, for even dimensions. However, the result we obtain in the end should also be valid for odd dimensional cases. For $D = 2n$, there are at most $n - 1$ rotation directions so we have a_i , $i = 1, 2, \dots, n - 1$. This metric, satisfying $R_{\mu\nu} = -3g^2g_{\mu\nu}$, can be expressed as follows⁷.

$$ds^2 = \frac{U}{X} dr^2 + \sum_{\alpha=1}^{n-1} \frac{U_{\alpha}}{X_{\alpha}} dy_{\alpha}^2 - \frac{X}{U} \left[W d\tilde{t} - \sum_{i=1}^{n-1} \gamma_i d\tilde{\phi}_i \right]^2 - \sum_{\alpha=1}^{n-1} \frac{X_{\alpha}}{U_{\alpha}} \left[\frac{(1 + g^2 r^2) W}{1 - g^2 y_{\alpha}^2} d\tilde{t} - \sum_{i=1}^{n-1} \frac{(r^2 + a_i^2) \gamma_i}{a_i^2 - y_{\alpha}^2} d\tilde{\phi}_i \right]^2, \quad (1)$$

where

$$U = \prod_{\alpha=1}^{n-1} (r^2 + y_\alpha^2), \quad U_\alpha = -(r^2 + y_\alpha^2) \prod_{\beta=1, \beta \neq \alpha}^{n-1} (y_\beta^2 - y_\alpha^2), \quad (2)$$

$$X = (1 + g^2 r^2) \prod_{k=1}^{n-1} (r^2 + a_k^2) - 2Mr, \quad X_\alpha = -(1 - g^2 y_\alpha^2) \prod_{k=1}^{n-1} (a_k^2 - y_\alpha^2), \quad (3)$$

$$W = \prod_{\alpha=1}^{n-1} (1 - g^2 y_\alpha^2), \quad \gamma_i = \prod_{\alpha=1}^{n-1} (a_i^2 - y_\alpha^2), \quad (4)$$

$$t = \tilde{t} \prod_{i=1}^{n-1} (1 - g^2 a_i^2), \quad \phi_i = a_i (1 - g^2 a_i^2) \tilde{\phi}_i \prod_{k=1, k \neq i}^{n-1} (a_i^2 - a_k^2), \quad (5)$$

with $1 \leq \alpha, i \leq n-1$. ϕ_i is the azimuthal angle for each a_i . With the direction cosines μ_i , $i = 1, \dots, n$, the metric for a unit $D-2$ sphere is just

$$d\Omega^2 = \sum_{i=1}^n d\mu_i^2 + \sum_{i=1}^{n-1} \mu_i^2 d\phi_i^2, \quad (6)$$

subject to the constraint $\sum_{i=1}^n \mu_i^2 = 1$. This constraint can be solved in terms of the unconstrained latitude variables y_α 's,

$$\mu_i^2 = \frac{\prod_{\alpha=1}^{n-1} (a_i^2 - y_\alpha^2)}{a_i^2 \prod_{k=1, k \neq i}^{n-1} (a_i^2 - a_k^2)}, \quad \mu_n^2 = \frac{\prod_{\alpha=1}^{n-1} y_\alpha^2}{\prod_{i=1}^{n-1} a_i^2}. \quad (7)$$

The metric for the unit sphere can then be written as

$$d\Omega^2 = \sum_{\alpha=1}^{n-1} g_\alpha dy_\alpha^2 + \sum_{i=1}^{n-1} \mu_i^2 d\phi_i^2, \quad (8)$$

with

$$g_\alpha = \frac{\prod_{\beta=1, \beta \neq \alpha}^{n-1} (y_\beta^2 - y_\alpha^2)}{\prod_{k=1}^{n-1} (a_k^2 - y_\alpha^2)}. \quad (9)$$

Now, to obtain a general metric with two rotations, we take the limit $a_3, a_4, \dots, a_n \rightarrow 0$ while assuming that $a_1 > a_2 > a_3 > \dots > a_{n-1}$. From the definition in Eq. (7) we see that y_i is of the same order of magnitude as a_i . With this in mind, we take the limit to arrive at the metric with two rotations for general dimensions

$D \geq 6$,

$$\begin{aligned}
 ds^2 = & -\frac{\Delta_r}{(r^2 + y_1^2)(r^2 + y_2^2)} \left[\frac{(1 - g^2 y_1^2)(1 - g^2 y_2^2)}{(1 - g^2 a_1^2)(1 - g^2 a_2^2)} dt - \frac{(a_1^2 - y_1^2)(a_1^2 - y_2^2)}{(1 - g^2 a_1^2)(a_1^2 - a_2^2)} \frac{d\phi_1}{a_1} \right. \\
 & \quad \left. - \frac{(a_2^2 - y_1^2)(a_2^2 - y_2^2)}{(1 - g^2 a_2^2)(a_2^2 - a_1^2)} \frac{d\phi_2}{a_2} \right]^2 \\
 & + \frac{\Delta_{y_1}}{(r^2 + y_1^2)(y_2^2 - y_1^2)} \left[\frac{(1 + g^2 r^2)(1 - g^2 y_2^2)}{(1 - g^2 a_1^2)(1 - g^2 a_2^2)} dt - \frac{(r^2 + a_1^2)(a_1^2 - y_2^2)}{(1 - g^2 a_1^2)(a_1^2 - a_2^2)} \frac{d\phi_1}{a_1} \right. \\
 & \quad \left. - \frac{(r^2 + a_2^2)(a_2^2 - y_2^2)}{(1 - g^2 a_2^2)(a_2^2 - a_1^2)} \frac{d\phi_2}{a_2} \right]^2 \\
 & + \frac{\Delta_{y_2}}{(r^2 + y_2^2)(y_1^2 - y_2^2)} \left[\frac{(1 + g^2 r^2)(1 - g^2 y_1^2)}{(1 - g^2 a_1^2)(1 - g^2 a_2^2)} dt - \frac{(r^2 + a_1^2)(a_1^2 - y_1^2)}{(1 - g^2 a_1^2)(a_1^2 - a_2^2)} \frac{d\phi_1}{a_1} \right. \\
 & \quad \left. - \frac{(r^2 + a_2^2)(a_2^2 - y_1^2)}{(1 - g^2 a_2^2)(a_2^2 - a_1^2)} \frac{d\phi_2}{a_2} \right]^2 \\
 & + \frac{(r^2 + y_1^2)(r^2 + y_2^2)}{\Delta_r} dr^2 + \frac{(r^2 + y_1^2)(y_2^2 - y_1^2)}{\Delta_{y_1}} dy_1^2 + \frac{(r^2 + y_2^2)(y_1^2 - y_2^2)}{\Delta_{y_2}} dy_2^2 \\
 & + r^2 \left(\frac{y_1^2 y_2^2}{a_1^2 a_2^2} \right) d\Omega_{D-6}^2, \tag{10}
 \end{aligned}$$

and

$$\Delta_r = (1 + g^2 r^2)(r^2 + a_1^2)(r^2 + a_2^2) - 2Mr^{7-D}, \tag{11}$$

$$\Delta_{y_1} = (1 - g^2 y_1^2)(a_1^2 - y_1^2)(a_2^2 - y_1^2), \tag{12}$$

$$\Delta_{y_2} = (1 - g^2 y_2^2)(a_1^2 - y_2^2)(a_2^2 - y_2^2). \tag{13}$$

Here we have kept the variables y_1 and y_2 instead of writing them in terms of angle variables. This is because the relationship between them, as shown in Eq. (7), is a bit complicated to write out explicitly. If we solve y_1 and y_2 in terms of μ_1 and μ_2 , we have

$$\begin{aligned}
 y_{1,2}^2 = & \frac{1}{2} \left[a_1^2(1 - \mu_1^2) + a_2^2(1 - \mu_2^2) \right. \\
 & \quad \left. \pm \sqrt{4a_1^2 a_2^2 (\mu_1^2 + \mu_2^2 - 1) + (a_1^2(1 - \mu_1^2) + a_2^2(1 - \mu_2^2))^2} \right]. \tag{14}
 \end{aligned}$$

However, it is important to note that y_1 and y_2 must be constrained to be

$$a_2 \leq y_1 \leq a_1 \quad ; \quad 0 \leq y_2 \leq a_2 \tag{15}$$

in order to have Eq. (7) well-defined.

The separation of the Klein-Gordon equation in the general Kerr-(A)dS metric has been achieved by Frolov *et al.*¹⁰. Here we shall present explicitly the separated equation for the case with two rotations. Writing the Klein-Gordon field as

$$\Phi = R_r(r) R_{y_1}(y_1) R_{y_2}(y_2) e^{i\psi_0 \Psi_0} e^{i\psi_1 \Psi_1} e^{i\psi_2 \Psi_2} Y(\Omega), \tag{16}$$

where

$$\Psi_0 = -\omega + g^2(m_1 a_1 + m_2 a_2), \quad (17)$$

$$\Psi_1 = -\omega(a_1^2 + a_2^2) + m_1 a_1(1 + g^2 a_2^2) + m_2 a_2(1 + g^2 a_1^2), \quad (18)$$

$$\Psi_2 = -\omega a_1^2 a_2^2 + m_1 a_1 a_2^2 + m_2 a_1^2 a_2, \quad (19)$$

the Klein-Gordon equation $\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi) = 0$ can be simplified to the radial equation

$$\begin{aligned} & \frac{1}{r^{D-6}} \frac{d}{dr} \left(r^{D-6} \Delta_r \frac{dR_r}{dr} \right) + \left[\frac{(r^2 + a_1^2)^2 (r^2 + a_2^2)^2 \omega^2}{\Delta_r} \right. \\ & - \frac{2\omega(1 + g^2 r^2)(r^2 + a_1^2)^2 (r^2 + a_2^2)^2}{\Delta_r} \left(\frac{m_1 a_1}{r^2 + a_1^2} + \frac{m_2 a_2}{r^2 + a_2^2} \right) \\ & + \frac{(1 + g^2 r^2)^2 (r^2 + a_1^2)^2 (r^2 + a_2^2)^2}{\Delta_r} \left(\frac{m_1 a_1}{r^2 + a_1^2} + \frac{m_2 a_2}{r^2 + a_2^2} \right)^2 \\ & \left. - \frac{a_1^2 a_2^2 j(j + D - 7)}{r^2} - b_1 r^2 - b_2 \right] R_r = 0, \end{aligned} \quad (20)$$

and the angular equations, for $i = 1, 2$,

$$\begin{aligned} & \left(\frac{a_i}{y_i} \right)^{D-6} \frac{d}{dy_i} \left[\left(\frac{y_i}{a_i} \right)^{D-6} (1 - g^2 y_i^2)(a_1^2 - y_i^2)(a_2^2 - y_i^2) \frac{dR_{y_i}}{dy_i} \right] \\ & + \left\{ - \frac{(a_1^2 - y_i^2)(a_2^2 - y_i^2)\omega^2}{1 - g^2 y_i^2} + 2\omega[m_1 a_1(a_2^2 - y_i^2) + m_2 a_2(a_1^2 - y_i^2)] \right. \\ & - 2m_1 a_1 m_2 a_2 (1 - g^2 y_i^2) - \frac{m_1^2 a_1^2 (1 - g^2 y_i^2)(a_2^2 - y_i^2)}{a_1^2 - y_i^2} \\ & \left. - \frac{m_2^2 a_2^2 (1 - g^2 y_i^2)(a_1^2 - y_i^2)}{a_2^2 - y_i^2} - \frac{a_1^2 a_2^2 j(j + D - 7)}{y_i^2} - b_1 y_i^2 + b_2 \right\} R_{y_i} = 0. \end{aligned} \quad (21)$$

In the following we shall solve the coupled angular equations above for the case with two rotations to obtain the angular eigenvalues b_1 and b_2 . Assuming that the rotation parameters a_1 and a_2 are small, we shall express these eigenvalues perturbatively as power series of a_1 and a_2 . However, the case with $D = 5$ is exceptional in which there is only one angular equation which has been dealt with recently¹⁸. Therefore, we shall consider in the next section the case with $D \geq 6$.

3. Angular equation and eigenvalue expansion for $D \geq 6$

With two rotations there are two angular equations involving the latitude coordinates y_1 and y_2 as in Eq. (21) for dimensions $D \geq 6$. In this section we develop expansions in powers of the rotation parameters a_1 and a_2 for the angular eigenvalues. Since the latitude coordinates are restricted to $a_2 \leq y_1 \leq a_1$ and $0 \leq y_2 \leq a_2$,

it is convenient to change variables to x_1 and x_2 with

$$y_1^2 = \frac{1}{2} (a_1^2 + a_2^2) - \frac{1}{2} (a_1^2 - a_2^2) x_1 \quad ; \quad y_2^2 = \frac{1}{2} a_2^2 (1 - x_2), \quad (22)$$

where $-1 \leq x_1, x_2 \leq 1$.

To further simplify the consideration we shall specialize to the case with $D = 6$, $j = 0$, and $m_1 = m_2 = 1$. The analysis for other values of these parameters can be carried out in a similar fashion. In this special case the angular equations in Eq. (21) are $\mathcal{O}_i R_i = B_i R_i / 4$ with $i = 1, 2$. Here, the operators are

$$\begin{aligned} \mathcal{O}_1 = & -\frac{(1+x_1)}{2} [(1-x_1) + \epsilon^2(1+x_1)] \{2 - g^2 a_1^2 [(1-x_1) + \epsilon^2(1+x_1)]\} \frac{d^2}{dx_1^2} \\ & + \left\{ \left[\frac{1}{2}(1+5x_1) - \frac{1}{4}g^2 a_1^2(1-x_1)(3+7x_1) \right] \right. \\ & \quad - \frac{\epsilon^2}{2} \left(\frac{1+x_1}{1-x_1} \right) [(1-5x_1) + 7g^2 a_1^2 x_1(1-x_1)] \\ & \quad \left. + \frac{g^2 a_1^2 \epsilon^4 (1+x_1)^2 (3-7x_1)}{4(1-x_1)} \right\} \frac{d}{dx_1} \\ & + \left\{ \frac{a_1^2 \omega^2 (1+x_1)(1-\epsilon^2)^2}{4[2 - g^2 a_1^2 ((1-x_1) + \epsilon^2(1+x_1))]} \right. \\ & \quad + \frac{[(1-x_1)^2 + \epsilon^2(1+x_1)^2][2 - g^2 a_1^2 ((1-x_1) + \epsilon^2(1+x_1))]}{4(1-x_1)^2(1+x_1)} \\ & \quad \left. - \frac{\epsilon^2(1+x_1)B_1}{4(1-x_1)} + \frac{B_2}{2(1-x_1)} \right\}, \end{aligned} \quad (23)$$

$$\begin{aligned} \mathcal{O}_2 = & -\frac{1}{4}(1-x_2^2) [2 - \epsilon^2(1-x_2)] [2 - g^2 a_1^2 \epsilon^2(1-x_2)] \frac{d^2}{dx_2^2} \\ & - \left\{ \frac{1}{2}(1-3x_2) + \frac{1}{4}\epsilon^2(1+g^2 a_1^2)(1-x_2)(1+5x_2) \right. \\ & \quad \left. - \frac{1}{8}g^2 a_1^2 \epsilon^4 (1-x_2)^2 (3+7x_2) \right\} \frac{d}{dx_2} \\ & + \left\{ \frac{\omega^2 a_1^2 \epsilon^2 (1+x_2)(2 - \epsilon^2(1-x_2))}{8(2 - g^2 a_1^2 \epsilon^2(1-x_2))} \right. \\ & \quad + \frac{[2 - g^2 a_1^2 \epsilon^2(1-x_2)][4 - \epsilon^2(3 - 6x_2 - x_2^2) + \epsilon^4(1-x_2)^2]}{8(1+x_2)(2 - \epsilon^2(1-x_1))} \\ & \quad \left. + \frac{\epsilon^2(1-x_2)B_1}{8} \right\}, \end{aligned} \quad (24)$$

where we have defined $\epsilon = a_2/a_1$. Since we have assumed that $a_1 \geq a_2$, $\epsilon \leq 1$. Moreover, we have defined the constants

$$B_1 \equiv b_1 + 2\omega(a_1 + a_2) - 2g^2 a_1 a_2, \quad (25)$$

$$B_2 \equiv \frac{1}{a_1^2} [b_2 + 2\omega(a_1 a_2^2 + a_1^2 a_2) - 2a_1 a_2]. \quad (26)$$

With these definitions of B_1 and B_2 , one has only even powers of ϵ and a_1 .

We shall develop perturbative expansions for B_1 and B_2 . Since we have assumed that $a_1 \geq a_2$, we first expand the operators with respect to ϵ . Schematically we write

$$\begin{aligned} & (\mathcal{O}_{i0} + \mathcal{O}_{i2} + \mathcal{O}_{i4} + \mathcal{O}_{i6} + \cdots) (R_{i0} + R_{i2} + R_{i4} + R_{i6} + \cdots) \\ &= \frac{1}{4} (B_{i0} + B_{i2} + B_{i4} + B_{i6} + \cdots) (R_{i0} + R_{i2} + R_{i4} + R_{i6} + \cdots). \end{aligned} \quad (27)$$

To the zeroth order of ϵ , the eigenvalue equations are $\mathcal{O}_{i0}R_{i0} = B_{i0}R_{i0}/4$ where

$$\begin{aligned} \mathcal{O}_{10} = & -\frac{(1-x_1^2)}{2} [2 - g^2 a_1^2 (1-x_1)] \frac{d^2}{dx_1^2} \\ & + \left[\frac{1}{2} (1+5x_1) - \frac{1}{4} g^2 a_1^2 (1-x_1) (3+7x_1) \right] \frac{d}{dx_1} \\ & + \left\{ \frac{a_1^2 \omega^2 (1+x_1)}{4[2 - g^2 a_1^2 (1-x_1)]} + \frac{[2 - g^2 a_1^2 (1-x_1)]}{4(1+x_1)} + \frac{B_{20}}{2(1-x_1)} \right\}, \end{aligned} \quad (28)$$

$$\mathcal{O}_{20} = -(1-x_2^2) \frac{d^2}{dx_2^2} - \frac{1}{2} (1-3x_2) \frac{d}{dx_2} + \frac{1}{2(1+x_2)}. \quad (29)$$

Since $\mathcal{O}_{20}$ does not involve a_1 , the corresponding eigenvalue equation can be solved readily. To fix notation and to anticipate the analysis in the following we define $\mathcal{O}_{200} \equiv \mathcal{O}_{20}$. The first zero in the subscript indicates that the operator involves zeroth power of ϵ while the second zero indicates that it involves zeroth power of a_1 . The eigenfunctions and eigenvalues of the equation $\mathcal{O}_{200}R_{200} = B_{200}R_{200}/4$ are

$$R_{200}(n_2) = \sqrt{\frac{(4n_2+3)(2n_2+1)}{2^{3/2}(n_2+1)}} (1+x_2)^{1/2} P_{n_2}^{(-\frac{1}{2}, 1)}(x_2), \quad (30)$$

$$B_{200}(n_2) = 2(n_2+1)(2n_2+1), \quad (31)$$

where $n_2 = 0, 1, 2, \dots$ and $P_n^{\alpha, \beta}(x)$ is the Jacobi polynomial. One can check that the orthonormal condition is

$$\frac{1}{4} \int_{-1}^1 \frac{dx_2}{\sqrt{1-x_2}} R_{200}(n_2) R_{200}(n'_2) = \delta_{n_2 n'_2}. \quad (32)$$

To be concrete in the following discussion we take $n_2 = 0$. Hence, we have

$$R_{200} = R_{200}(0) = 2^{-3/4} 3^{1/2} \sqrt{1+x_2}, \quad (33)$$

$$B_{200} = B_{200}(0) = 2. \quad (34)$$

Next, we work on the $\mathcal{O}_{10}$ equation. Following what we have been doing we expand the eigenvalue equation in powers of a_1 with the operators

$$\mathcal{O}_{100} = -(1-x_1^2)\frac{d^2}{dx_1^2} + \frac{1}{2}(1+5x_1)\frac{d}{dx_1} + \frac{3+x_1}{2(1-x_1^2)}, \quad (35)$$

$$\begin{aligned} \mathcal{O}_{102} = & \frac{1}{2}g^2a_1^2(1-x_1)(1-x_1^2)\frac{d^2}{dx_1^2} - \frac{1}{4}g^2a_1^2(1-x_1)(3+7x_1)\frac{d}{dx_1} \\ & + \frac{[\omega^2a_1^2(1+x_1)^2 - 2g^2a_1^2(1-x_1)]}{8(1+x_1)}, \end{aligned} \quad (36)$$

$$\mathcal{O}_{104} = \frac{1}{16}g^2\omega^2a_1^4(1-x_1^2), \quad (37)$$

$$\mathcal{O}_{106} = \frac{1}{32}g^4\omega^2a_1^6(1-x_1)(1-x_1^2). \quad (38)$$

Similarly we solve the zeroth order eigenvalue equation, $\mathcal{O}_{100}R_{100} = B_{100}R_{100}/4$, to give

$$R_{100}(n_1) = \sqrt{\frac{(4n_1+7)(2n_1+5)}{2^{7/2}(n_1+1)}} \sqrt{1-x_1^2} P_{n_1}^{(\frac{3}{2},1)}(x_1), \quad (39)$$

$$B_{100}(n_1) = 2(n_1+1)(2n_1+5), \quad (40)$$

with $n_1 = 0, 1, 2, \dots$ and the orthonormal condition

$$\frac{1}{4} \int_{-1}^1 dx_1 \sqrt{1-x_1} R_{100}(n_1) R_{100}(n'_1) = \delta_{n_1 n'_1}. \quad (41)$$

Again for simplicity we choose $n_1 = 0$ and we have

$$R_{100} = R_{100}(0) = 2^{-7/4} 5^{1/2} 7^{1/2} \sqrt{1-x_1^2}, \quad (42)$$

$$B_{100} = B_{100}(0) = 10. \quad (43)$$

To obtain B_{102} to B_{106} , we follow a perturbative procedure¹⁸. The result is

$$B_{102} = \frac{2}{9} (2\omega^2a_1^2 - 17g^2a_1^2), \quad (44)$$

$$B_{104} = -\frac{20}{8019} (\omega^4a_1^4 - 53g^2\omega^2a_1^4 + 196g^4a_1^4), \quad (45)$$

$$B_{106} = \frac{20}{8444007} (2\omega^6a_1^6 - 645g^2\omega^4a_1^6 + 28959g^4\omega^2a_1^6 - 105644g^6a_1^6). \quad (46)$$

If one iterates to higher orders, one can in principle obtain the eigenvalues in powers of a_1 and $a_2 = a_1\epsilon$. Here we just list the result. For $D = 6$, $j = 0$, $m_1 =$

$m_2 = 1$ and $n_1 = n_2 = 0$,

$$\begin{aligned}
 B_1 = & \left[10 + \frac{2}{9} (2\omega^2 - 17g^2) a_1^2 - \frac{20}{8019} (\omega^4 - 53g^2\omega^2 + 196g^4) a_1^4 \right. \\
 & \left. + \frac{20}{8444007} (2\omega^6 - 645g^2\omega^4 + 28959g^4\omega^2 - 105644g^6) a_1^6 \right] \\
 & + \epsilon^2 \left[\frac{2}{9} (2\omega^2 - 17g^2) a_1^2 + \frac{32}{8019} (\omega^4 - 53g^2\omega^2 + 196g^4) a_1^4 \right. \\
 & \left. - \frac{32}{2814669} (\omega^6 - 147g^2\omega^4 + 5178g^4\omega^2 - 18424g^6) a_1^6 \right] \\
 & + \epsilon^4 \left[-\frac{20}{8019} (\omega^4 - 53g^2\omega^2 + 196g^4) a_1^4 \right. \\
 & \left. - \frac{32}{2814669} (\omega^6 - 147g^2\omega^4 + 5178g^4\omega^2 - 18424g^6) a_1^6 \right] \\
 & + \epsilon^6 \left[\frac{20}{8444007} (2\omega^6 - 645g^2\omega^4 + 28959g^4\omega^2 - 105644g^6) a_1^6 \right] + \dots, \quad (47)
 \end{aligned}$$

$$\begin{aligned}
 B_2 = & 2 + \epsilon^2 \left[2 + \frac{2}{9} (4\omega^2 - 7g^2) a_1^2 - \frac{4}{8019} (\omega^4 - 53g^2\omega^2 + 196g^4) a_1^4 \right. \\
 & \left. + \frac{4}{8444007} (2\omega^6 - 645g^2\omega^4 + 28959g^4\omega^2 - 105644g^6) a_1^6 \right] \\
 & + \epsilon^4 \left[-\frac{4}{8019} (\omega^4 - 53g^2\omega^2 + 196g^4) a_1^4 \right. \\
 & \left. - \frac{32}{8444007} (4\omega^2 - 25g^2) (\omega^4 - 53g^2\omega^2 + 196g^4) a_1^6 \right] \\
 & + \epsilon^6 \left[\frac{4}{8444007} (2\omega^6 - 645g^2\omega^4 + 28959g^4\omega^2 - 105644g^6) a_1^6 \right] + \dots. \quad (48)
 \end{aligned}$$

4. Conclusions

In this paper we have started to study higher-dimensional Kerr-(A)dS black holes with two rotations. First, we have presented here the general metric for these doubly rotating black holes. We have also separated the Klein-Gordon equation, writing out the corresponding radial and angular equations explicitly. With these separated equations we could start to ask questions about stability, the various spectra of Hawking radiation and the phenomenon of superradiance.

To get some quantitative understanding of the angular equations, we have developed perturbative expansions in powers of the rotation parameters a_1 and a_2 for the angular eigenvalues. These expansions could be used in a variety of considerations. For example, in the evaluation of the frequencies of the low-lying quasinormal modes using a semi-analytic method like the WKB approximation²¹, it is necessary to keep the frequency ω arbitrary. Perturbative expansions of the angular eigenvalues that we have developed would be useful. In fact, we have already considered the

quasinormal modes of these doubly rotating black holes using a numerical approach recently²².

On the other hand, to discuss the stability of the ultra-spinning black holes, we need to consider the angular eigenvalues in the large rotation limit²³. For our case there are two rotation parameters. With one small and one large, the situation will be very much like the simply rotating case^{11,12} and no instability is expected. Therefore it is more interesting to consider the other case with both rotation parameters large. This work will be pursued subsequently.

Acknowledgments

HTC was supported in part by the National Science Council of the Republic of China under the Grant NSC 99-2112-M-032-003-MY3, and the National Science Centre for Theoretical Sciences. The work of JD was supported by the Japan Society for the Promotion of Science (JSPS), under fellowship no. P09749.

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