NON-COMMUTATIVE EINSTEIN EQUATIONS AND SEIBERG–WITTEN MAP

PAOLO ASCHIERI,^{1,2} ELISABETTA DI GREZIA,³ GIAMPIERO ESPOSITO³

¹*Dipartimento di Scienze e Tecnologie Avanzate, Università del Piemonte Orientale, Italy*

²*INFN, Sezione di Torino, Gruppo Collegato di Alessandria, Italy*

³*INFN, Sezione di Napoli, Complesso Universitario di Monte S. Angelo,
Via Cintia, Edificio 6, 80126 Napoli, Italy*

Received 30 May 2011

Revised 22 June 2010

The Seiberg–Witten map is a powerful tool in non-commutative field theory, and it has been recently obtained in the literature for gravity itself, to first order in non-commutativity. This paper, relying upon the pure-gravity form of the action functional considered in Ref. 2, studies the expansion to first order of the non-commutative Einstein equations, and whether the Seiberg–Witten map can lead to a solution of such equations when the underlying classical geometry is Schwarzschild. We find that, if one first obtains the non-commutative field equations by varying the action of Ref. 2 with respect to all non-commutative fields, and then tries to solve these equations by expressing the non-commutative fields in terms of the commutative ones via Seiberg–Witten map, no solution of these equations can be obtained when the commutative background is Schwarzschild.

Keywords: Non-commutative geometry; quantum gravity.

PACS numbers: 02.04.Gh, 04.60.-m

1. Introduction

Non-commutative gravity theories are receiving much attention in the literature because they are part of a promising research program aimed at developing an algebraic approach to the longstanding problem of quantum gravity.¹ In particular, we are here concerned with the developments described in Refs. 2 and 3.

On the one hand, the work in Ref. 2 has built a geometric theory of non-commutative gravity where the Lagrangian is a globally defined 4-form, invariant under diffeomorphisms as well as $*$ -diffeomorphisms and where in the commutative limit only the classical field degrees of freedom survive. For pure gravity, the work in Ref. 2 leads to the action functional

$$S = \int \text{Tr} \left(i \hat{R} \wedge_* \hat{V} \wedge_* \hat{V} \gamma_5 \right), \quad (1)$$

where $\hat{V}$ is the tetrad 1-form and $\hat{R}$ is the curvature 2-form

$$\hat{R} = d\hat{\Omega} - \hat{\Omega} \wedge_* \hat{\Omega}. \quad (2)$$

As in Ref. 2 we expand the tetrad on the basis of γ -matrices

$$\hat{V} = \hat{V}^a \gamma_a + \tilde{\hat{V}}^a \gamma_a \gamma_5. \quad (3)$$

On denoting by $\eta_{ab} = \text{diag}(1, -1, -1, -1)$ the Minkowski metric, and defining $\gamma_{ab} \equiv \gamma_a \gamma_b - I \eta_{ab}$, one similarly expands the spin-connection 1-form

$$\hat{\Omega} = \frac{1}{4} \hat{\omega}^{ab} \gamma_{ab} + i \hat{\omega} I + \tilde{\hat{\omega}} \gamma_5, \quad (4)$$

where $\hat{\omega} = \hat{\omega}_\mu dx^\mu$ and $\tilde{\hat{\omega}} = \tilde{\hat{\omega}}_\nu dx^\nu$ are 1-forms. From (3) and (4), one has

$$\hat{R} = \frac{1}{4} \hat{R}^{ab} \gamma_{ab} + i \hat{r} I + \tilde{\hat{r}} \gamma_5, \quad (5)$$

where $\hat{r}$ and $\tilde{\hat{r}}$ are 2-forms. We here correct some numerical factors in Eqs. (5.2) and (5.3) of Ref. 2 and write the following explicit formulae for the components in Eq. (5), i.e.

$$\begin{aligned} \hat{R}^{ab} = & d\hat{\omega}^{ab} - \frac{1}{2} \hat{\omega}_c^a \wedge_* \hat{\omega}^{cb} + \frac{1}{2} \hat{\omega}_c^b \wedge_* \hat{\omega}^{ca} - i(\hat{\omega}^{ab} \wedge_* \hat{\omega} + \hat{\omega} \wedge_* \hat{\omega}^{ab}) \\ & - \frac{i}{2} \varepsilon^{ab}_{cd} (\hat{\omega}^{cd} \wedge_* \tilde{\hat{\omega}} + \tilde{\hat{\omega}} \wedge_* \hat{\omega}^{cd}), \end{aligned} \quad (6)$$

$$\hat{r} = d\hat{\omega} - i \left(\frac{1}{8} \hat{\omega}^{ab} \wedge_* \hat{\omega}_{ab} + \hat{\omega} \wedge_* \hat{\omega} - \tilde{\hat{\omega}} \wedge_* \tilde{\hat{\omega}} \right), \quad (7)$$

$$\tilde{\hat{r}} = d\tilde{\hat{\omega}} - i \left(\hat{\omega} \wedge_* \tilde{\hat{\omega}} + \tilde{\hat{\omega}} \wedge_* \hat{\omega} \right) + \frac{i}{16} \varepsilon_{abcd} \hat{\omega}^{ab} \wedge_* \hat{\omega}^{cd}. \quad (8)$$

We explicitly write in components the equations obtained by varying the action (1) with respect to the tetrad components $\hat{V}^a$ and $\tilde{\hat{V}}^a$, they respectively read as

$$\begin{aligned} & - \left(\hat{V}^d \wedge_* \hat{R}^{ab} + \hat{R}^{ab} \wedge_* \hat{V}^d \right) \varepsilon_{abcd} + i(\eta_{bc} \eta_{ad} - \eta_{ac} \eta_{bd}) \left(\hat{R}^{ab} \wedge_* \tilde{\hat{V}}^d - \tilde{\hat{V}}^d \wedge_* \hat{R}^{ab} \right) \\ & + 4i\eta_{dc} \left(\tilde{\hat{r}} \wedge_* \hat{V}^d - \hat{V}^d \wedge_* \tilde{\hat{r}} \right) + 4\eta_{dc} \left(\tilde{\hat{V}}^d \wedge_* \hat{r} + \hat{r} \wedge_* \tilde{\hat{V}}^d \right) = 0, \end{aligned} \quad (9)$$

$$\begin{aligned} & - \left(\tilde{\hat{V}}^d \wedge_* \hat{R}^{ab} + \hat{R}^{ab} \wedge_* \tilde{\hat{V}}^d \right) \varepsilon_{abcd} + i(\eta_{bc} \eta_{ad} - \eta_{ac} \eta_{bd}) \left(\hat{R}^{ab} \wedge_* \hat{V}^d - \hat{V}^d \wedge_* \hat{R}^{ab} \right) \\ & + 4i\eta_{dc} \left(\tilde{\hat{r}} \wedge_* \tilde{\hat{V}}^d - \tilde{\hat{V}}^d \wedge_* \tilde{\hat{r}} \right) + 4\eta_{dc} \left(\hat{V}^d \wedge_* \hat{r} + \hat{r} \wedge_* \hat{V}^d \right) = 0. \end{aligned} \quad (10)$$

We notice that Eq. (10) can be obtained by the replacements $\hat{V}^a \rightarrow \tilde{\hat{V}}^a$ and $\tilde{\hat{V}}^b \rightarrow \hat{V}^b$ in Eq. (9).

The torsion 2-form is defined by

$$\hat{T} \equiv d\hat{V} - \hat{\Omega} \wedge_* \hat{V} - \hat{V} \wedge_* \hat{\Omega}, \quad (11)$$

and expanding $\hat{T}$ on the basis of γ -matrices as in Eq. (2) we have $\hat{T} = \hat{T}^a \gamma_a + \tilde{\hat{T}}^a \gamma_a \gamma_5$. Variation of the action with respect to the spin-connection gives the remaining field

equation $\{\hat{T} \wedge_* \hat{V} - \hat{V} \wedge_* \hat{T}, \gamma_5\} = 0$, where we use curly brackets for anticommutators. Since $\hat{T} \wedge_* \hat{V}$ is even in the γ -matrices we eventually obtains

$$\hat{T} \wedge_* \hat{V} - \hat{V} \wedge_* \hat{T} = 0, \quad (12)$$

which is satisfied by a vanishing torsion.

The work in Ref. 3 has studied the Seiberg–Witten map⁴ which relates non-commutative degrees of freedom for spin-connection, tetrad and gauge parameter to their commutative counterparts. The relation is such that non-commutative gauge transformations (with non-commutative gauge parameter $\hat{\Lambda}$) correspond to commutative gauge transformations (with commutative gauge parameter Λ). For example for the tetrad we have

$$\hat{V}_\mu + \delta_{\hat{\Lambda}} \hat{V}_\mu = \hat{V}_\mu (V + \delta_\Lambda V, \omega + \delta_\Lambda \omega), \quad (13)$$

where $\hat{V} = \hat{V}_\mu dx^\mu$ is the non-commutative tetrad 1-form, while $V = V^a \gamma_a = V_\mu^a dx^\mu \gamma_a$ and $\omega = \frac{1}{4} \omega^{ab} \gamma_{ab} = \frac{1}{4} \omega_\mu^{ab} dx^\mu \gamma_{ab}$ are the usual commutative tetrad and spin-connection. The non-commutativity we consider is given by the Moyal–Weyl $\star$ -product associated with a constant antisymmetric matrix $\theta^{\rho\sigma}$ in the (not necessarily Cartesian) coordinates x^μ ; we have $x^\rho \star x^\sigma - x^\sigma \star x^\rho = i\theta^{\rho\sigma}$. Equation (13) can be solved order by order in perturbation theory. The solution of Eq. (13) to first order in the non-commutativity $\theta^{\rho\sigma}$ has the general structure

$$\hat{V}_\mu = \hat{V}_\mu^{(0)a} \gamma_a + \hat{V}_{\mu 5}^{(1)a} \gamma_5 \gamma_a + O(\theta^2). \quad (14)$$

The zeroth- and first-order terms in $\theta^{\rho\sigma}$ in Eq. (14) are found to be³

$$\hat{V}_\mu^{(0)a} = V_\mu^a, \quad (15)$$

$$\hat{V}_{\mu 5}^{(1)a} = \frac{1}{4} \theta^{\lambda\sigma} \omega_\lambda^{eb} \left(\partial_\sigma V_\mu^c - \frac{1}{2} \omega_\sigma^{cd} V_{\mu d} \right) \varepsilon_{ebc}^a. \quad (16)$$

Similarly, on writing for the spin-connection

$$\hat{\omega}_\mu = \frac{1}{2} \hat{\omega}_\mu^{(0)ab} \sigma_{ab} + \hat{a}_\mu^{(1)} + i \hat{b}_{\mu 5}^{(1)} \gamma_5, \quad (17)$$

one finds,³ to first order in $\theta^{\rho\sigma}$,

$$\hat{\omega}_\mu^{(0)ab} = \omega_\mu^{ab}, \quad (18)$$

$$\hat{a}_\mu^{(1)} = -\frac{1}{16} \theta^{\lambda\sigma} \omega_\lambda^{ab} \left(\partial_\sigma \omega_\mu^{cd} + R_{\sigma\mu}^{cd} \right) \eta_{ac} \eta_{bd}, \quad (19)$$

$$\hat{b}_{\mu 5}^{(1)} = -\frac{1}{32} \theta^{\lambda\sigma} \omega_\lambda^{ab} \left(\partial_\sigma \omega_\mu^{cd} + R_{\sigma\mu}^{cd} \right) \varepsilon_{abcd}. \quad (20)$$

In this note we consider the non-commutative fields appearing in the action (1) and in the corresponding field equations (9) and (10) as dependent on the commutative ones via Seiberg–Witten map. Hence we expand equations (9) and (10) in terms of

the commutative tetrad and spin-connection. This is done to first order in $\theta^{\mu\nu}$ by inserting the first-order Seiberg–Witten map (15), (16) and (18)–(20). We have done so when the underlying classical geometry is the spherically symmetric solution of the classical Einstein equations in vacuum, i.e. Schwarzschild.

2. Expansion to First Order of the Non-Commutative Einstein Equations

By virtue of the wedge- $*$ product of forms defined in Ref. 2, for any 1-form α^1, β^1 and any 2-form γ^2 , one can write (provided that $\partial_\mu x^\rho = \delta_\mu^\rho$)

$$\alpha^1 \wedge_* \beta^1 = \alpha^1 \wedge \beta^1 + \frac{i}{2} \theta^{\rho\sigma} \left(\partial_\rho \alpha_\mu^1 \right) \left(\partial_\sigma \beta_\nu^1 \right) dx^\mu \wedge dx^\nu + O(\theta^2), \quad (21)$$

$$\alpha^1 \wedge_* \gamma^2 = \alpha^1 \wedge \gamma^2 + \frac{i}{2} \theta^{\rho\sigma} \left(\partial_\rho \alpha_\mu^1 \right) \left(\partial_\sigma \gamma_{\nu\lambda}^2 \right) dx^\mu \wedge dx^\nu \wedge dx^\lambda + O(\theta^2). \quad (22)$$

Equation (22) can be applied repeatedly to the θ -expansion of Eqs. (9) and (10). For this purpose, we need from Eq. (22) the identities

$$\alpha^1 \wedge_* \gamma^2 + \gamma^2 \wedge_* \alpha^1 = 2\alpha^1 \wedge \gamma^2 + O(\theta^2), \quad (23)$$

$$\alpha^1 \wedge_* \gamma^2 - \gamma^2 \wedge_* \alpha^1 = i\theta^{\rho\sigma} \left(\partial_\rho \alpha_\mu^1 \right) \left(\partial_\sigma \gamma_{\nu\lambda}^2 \right) dx^\mu \wedge dx^\nu \wedge dx^\lambda + O(\theta^2). \quad (24)$$

Moreover, from the work in Ref. 2 we know that, in non-commutative field theory, charge-conjugation conditions imply that

$$\tilde{V}^a(\theta) = -\tilde{V}^a(-\theta), \quad \hat{\omega}(\theta) = -\hat{\omega}(-\theta), \quad \tilde{\hat{\omega}}(\theta) = -\tilde{\hat{\omega}}(-\theta), \quad (25)$$

and hence all non-commutative fields that are not present in the commutative case are at least proportional to θ (and hence must vanish in the commutative limit),

$$\tilde{V}_\mu^a = O(\theta), \quad \hat{\omega} = O(\theta), \quad \tilde{\hat{\omega}} = O(\theta), \quad \hat{r} = O(\theta), \quad \tilde{\hat{r}} = O(\theta). \quad (26)$$

By virtue of (23), (24) and (26), Eq. (9) reduces to

$$\varepsilon_{abcd} V^d \wedge \left(R^{(0)ab} + R^{(1)ab} \right) + O(\theta^2) = 0, \quad (27)$$

where $R^{(n)ab}$ denotes the n -th order part of the curvature 2-form in powers of $\theta^{\rho\sigma}$, while Eq. (10) becomes

$$\left[-\varepsilon_{abcd} \tilde{V}_\mu^d R_{\nu\lambda}^{(0)ab} + \theta^{\rho\sigma} \left(\partial_\rho V_\mu^d \right) \left(\partial_\sigma R_{dc\nu\lambda}^{(0)} \right) + 4V_{c\mu} r_{\nu\lambda}^{(1)} \right] dx^\mu \wedge dx^\nu \wedge dx^\lambda + O(\theta^2) = 0, \quad (28)$$

having defined

$$r_{\nu\lambda}^{(1)} \equiv \theta^{\rho\sigma} r_{\nu\lambda\rho\sigma}^{(1)}. \quad (29)$$

We note that, in Eq. (27), since V^d is the classical tetrad, the term $\varepsilon_{abcd} V^d \wedge R^{(0)ab}$ vanishes for any solution of the vacuum Einstein equations.

3. Non-Commutative Einstein Equations Under the First-Order Seiberg–Witten Map

Now we consider the coordinates $x^1 = t, x^2 = r, x^3 = \vartheta, x^4 = \varphi$ and the tetrad $V^a = V_\mu^a dx^\mu$ given by

$$V^{(t)} = \sqrt{1 - \frac{2M}{r}} dt, \quad V^{(r)} = \frac{dr}{\sqrt{1 - \frac{2M}{r}}}, \quad V^{(\vartheta)} = r d\vartheta, \quad V^{(\varphi)} = r \sin \vartheta d\varphi. \quad (30)$$

This tetrad with the associated spin-connection [see formula (34) below] describe a Schwarzschild geometry solution of the classical Einstein equations in first-order formalism. For this tetrad we have $R_{\mu\nu}^{(1)ab} = 0$, so that Eq. (27) becomes an identity. In order to try to solve the first-order non-commutative Eqs. (28) we consider for the first-order form of the non-commutative tetrad the expression given by Eq. (16), which is the first-order Seiberg–Witten map for the tetrad. This ansatz is supported by the following argument:

To first order in θ the non-commutative Einstein action (1) is the same as the commutative one if the non-commutative fields are re-expressed in terms of the commutative ones by using the Seiberg–Witten map (indeed, the non-commutative fields satisfy the charge conjugation conditions of Ref. 2). Therefore it is expected that the non-commutative Einstein equations are automatically satisfied if the non-commutative fields are expressed via Seiberg–Witten map in terms of the commutative ones.

We then substitute Eq. (16) into Eq. (28), we relabel the indices summed over, and assuming (for ease of calculations) that only the non-commutativity component θ^{23} (that we rename θ) is non-vanishing, i.e.

$$\theta^{\gamma\sigma} \frac{\partial}{\partial x^\gamma} \wedge \frac{\partial}{\partial x^\sigma} = \theta \left(\frac{\partial}{\partial r} \otimes \frac{\partial}{\partial \vartheta} - \frac{\partial}{\partial \vartheta} \otimes \frac{\partial}{\partial r} \right), \quad (31)$$

we re-express Eq. (28) in the form (with our notation $dx^1 = dt, dx^2 = dr, dx^3 = d\vartheta, dx^4 = d\varphi$)

$$\begin{aligned} & \left\{ \varepsilon_{abcd} \varepsilon_{epq} {}^d R_{\nu\lambda}^{(0)ab} \left[\left(\partial_3 V_\mu^q \right) \omega_2^{ep} - \left(\partial_2 V_\mu^q \right) \omega_3^{ep} + \frac{1}{2} V_{\mu f} \left(\omega_2^{qf} \omega_3^{ep} - \omega_3^{qf} \omega_2^{ep} \right) \right] \right. \\ & + 4 \left[\left(\partial_2 V_\mu^q \right) \left(\partial_3 R_{qc\ \nu\lambda}^{(0)} \right) - \left(\partial_3 V_\mu^q \right) \left(\partial_2 R_{qc\ \nu\lambda}^{(0)} \right) \right] \\ & \left. + 16 V_{c\mu} \left(r_{\nu\lambda 23}^{(1)} - r_{\nu\lambda 32}^{(1)} \right) \right\} dx^\mu \wedge dx^\nu \wedge dx^\lambda = 0. \end{aligned} \quad (32)$$

At this stage, the identity

$$\varepsilon^{abc p} \varepsilon_{def p} = -\det \begin{pmatrix} \delta_d^a & \delta_d^b & \delta_d^c \\ \delta_e^a & \delta_e^b & \delta_e^c \\ \delta_f^a & \delta_f^b & \delta_f^c \end{pmatrix} \quad (33)$$

can be exploited, jointly with the standard evaluation of classical curvature 2-form and classical spin-connection 1-form, the latter being given by

$$\omega_{\mu}^{ab} = \frac{1}{2} V^{a\nu} \left(V_{\nu,\mu}^b - V_{\mu,\nu}^b \right) - \frac{1}{2} V^{b\nu} \left(V_{\nu,\mu}^a - V_{\mu,\nu}^a \right) + \frac{1}{2} V^{a\nu} V^{b\sigma} \left(V_{\nu,\sigma}^c - V_{\sigma,\nu}^c \right) V_{c\mu}. \quad (34)$$

Hence we find in a Schwarzschild background, bearing also in mind Eqs. (7), the Seiberg–Witten map for the spin-connection Eq. (19), and the definition (29), that the left-hand side of Eq. (32) takes the form

$$K_{c123} dx^1 \wedge dx^2 \wedge dx^3 + K_{c124} dx^1 \wedge dx^2 \wedge dx^4 + K_{c134} dx^1 \wedge dx^3 \wedge dx^4 + K_{c234} dx^2 \wedge dx^3 \wedge dx^4,$$

where each $K_{c\mu\nu\lambda}$ can be written as the sum of 6 terms. We obtain, after a lengthy calculation, the simple formulae

$$K_{c123} = \frac{4M(5M - 2r)}{r^4 \sqrt{1 - \frac{2M}{r}}} \delta_{c1}, \quad (35)$$

$$K_{c124} = K_{c134} = 0 \quad \forall c = 1, 2, 3, 4, \quad (36)$$

$$K_{c234} = 4M \frac{\sin \vartheta}{r^2} \delta_{c4}. \quad (37)$$

Since K_{1123} and K_{4234} are non-vanishing, the field configurations given in Eqs. (16), (19) and obtained by applying the Seiberg–Witten map to the classical tetrad of Eq. (30) and spin-connection of Eq. (34), are not solutions of the non-commutative Einstein equations.

In order to search for solutions to non-commutative Einstein equations that in the commutative limit become Schwarzschild we have therefore to revert to Eq. (28), where no use of the Seiberg–Witten map is made, and look for solutions of Eq. (28) and also of the torsion constraints in Eq. (12).

4. Conclusions

In conclusion, the performed calculation shows that there is a mismatch between:

(I) using the Seiberg–Witten map in the non-commutative action (1) in order to express all non-commutative fields in terms of the commutative tetrad and spin connection, and then solving the action (that in general will be a higher derivative action) by varying with respect to only the classical fields.

(II) obtaining the non-commutative field equations by varying the action (1) with respect to all non-commutative fields, and then trying to solve these equations by expressing the non-commutative fields in terms of the commutative ones via Seiberg–Witten map.

This mismatch can be due to the fact that in case (II), in order to obtain the equations of motion we have to vary also with respect to the extra fields $\tilde{\tilde{V}}$, $\tilde{\tilde{\omega}}$ and $\hat{\omega}$. Exactly these corresponding extra equations of motion are not satisfied by

considering the field configurations $\tilde{V}$, $\tilde{\omega}$ and $\hat{\omega}$ obtained by the Seiberg–Witten map with V^a and ω^{ab} the classical black hole tetrad and spin-connection.

We also notice that we have chosen the non-commutativity directions $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \vartheta}$ not to be Killing vector fields for our classical black hole solution. This was done on purpose because, similarly to Refs. 5–7, it is possible to show that, when non-commutativity is (in part) obtained by using Killing vector fields of a given classical solution to Einstein equations, then this classical solution is also a solution of the non-commutative field equations, where all the extra non-commutative fields are taken to vanish.

Acknowledgments

G. Esposito is grateful to the Dipartimento di Scienze Fisiche of Federico II University, Naples, for hospitality and support; he dedicates to Maria Gabriella his contribution to this work. We are grateful to P. Vitale for scientific conversations.

References

1. J. M. Gracia-Bondia, *Lect. Notes Phys.* **807**, 3 (2010).
2. P. Aschieri and L. Castellani, *JHEP* **0906**, 086 (2009).
3. Y. G. Miao and S. J. Zhang, *Phys. Rev. D* **82**, 084017 (2010).
4. N. Seiberg and E. Witten, *JHEP* **9909**, 032 (1999).
5. P. Aschieri and L. Castellani, *J. Geom. Phys.* **60**, 375 (2010).
6. T. Ohl and A. Schenkel, *JHEP* **0910**, 052 (2009).
7. P. Schupp and S. Solodukhin, Exact black hole solutions in noncommutative gravity, arXiv:0906.2724v1.