IS QFT WITH EXTERNAL BACKGROUNDS A CONSISTENT MODEL?

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Received 27 June 2011

We discuss consistency of the concept of external background in QFT. Different restrictions on magnitude of magnetic and electric fields are analyzed. The back reaction due to strong electric field is calculated and restrictions on the magnitude and duration of such a field are obtained. The problem of consistency of Dirac equation with a superstrong Coulomb field is discussed.

Keywords: QFT; strong fields; particle creation; vacuum polarization.

PACS numbers: 11.10.-z, 03.70.+k

1. Introduction

QFT with an external background is an effective model when a part of a quantized field is strong enough to be treated as a given (external) classical one. Such a model is a certain approximation to the exact theory. In this approximation it is supposed that quantum processes do not affect significantly the external field which is equivalent to the supposition that the back reaction is small. Due to new challenges in astrophysics, discovery of quark-gluon plasma, heavy ions collisions, and creation of such materials as graphene, the problem of strong external background impact on the physical vacuum and the corresponding back reaction has already become of practical interest. Back-reaction has to be calculated, in particular, to verify consistency of QFT with external backgrounds. First of all one has to examine such

important and popular backgrounds as a constant electromagnetic field and the Coulomb field.

Both magnetic and electric fields polarize the vacuum, but only the electric field can produce the work acting on charged particles, even virtual ones, then destroying the physical vacuum. Qualitatively: if an electric field is strong enough, it can pass to a virtual electron-positron pair energy greater than its rest mass $2mc^2$ to transform it into real particles. The vacuum is practically stable if $E \ll E_c$, where E_c is determined from the following natural relations

$$2\lambda e E_c = 2mc^2 \xrightarrow{\lambda=\hbar/mc} E_c = \frac{m^2 c^3}{e\hbar} \approx 1.3 \times 10^{16} \text{V/cm} . \quad (1)$$

Creating pairs from the vacuum, the strong electric field loses its energy and thus destroys itself. That is why the principal part of the back reaction is due to the particle creation but not to the vacuum polarization.

The general theory and some nonperturbative calculations of particle creation on electric barriers, in time dependent (periodic) electric fields, in superstrong Coulomb field, and in gravitational fields are presented in the key books on this subject.¹⁻⁴

Total number of created particles by a constant electric field acting during the time T in the volume V has the form:^{5,6}

$$N = \frac{VTc}{4\pi^3\lambda^4} \left(\frac{E}{E_c} \right)^2 \exp \left(-\pi \frac{E_c}{E} \right) . \quad (2)$$

2. Heisenberg and Euler and Schwinger Calculations

Heisenberg and Euler⁷ have calculated the change of vacuum energy in QED with constant electric $E \ll E_c$ and arbitrary magnetic field $B \parallel E$. For instance, for $E = 0$, they had obtained a change $\Delta\mathcal{L}_M$ of the Maxwell Lagrangian $\mathcal{L}_M = (E^2 - B^2)/8\pi$. For $B \gg B_c = m^2 c^3 / e\hbar = 4.4 \times 10^{13} \text{G}$ is reads:

$$\Delta\mathcal{L}_M = - \left(\frac{\alpha}{3\pi} \ln \frac{B}{B_c} \right) \mathcal{L}_M , \quad \alpha = e^2 / \hbar c. \quad (3)$$

However, in such calculations the loop expansion makes sense only for the magnetic fields $B \ll B_{\max}$, where⁸

$$B_{\max} = B_c e^{\frac{3\pi}{\alpha}} \approx 10^{560} B_c. \quad (4)$$

Recently, Shabad and Usov,⁹ taking into account virtual particle interaction, have obtained another much stronger restriction,

$$B_{\max} = \frac{B_c}{4} \exp \left(\frac{\pi^{3/2}}{\sqrt{\alpha}} + 1, 2 \right) \approx 10^{28} B_c. \quad (5)$$

Schwinger had defined an effective Lagrangian $\mathcal{L}_{\text{eff}}$ of the electromagnetic field by the relation

$$c_v = \exp \left(\frac{i}{\hbar} \int dx \mathcal{L}_{\text{eff}} \right) = \exp \left[-\text{Tr} \log (S^c / S_0^c) \right], \quad (6)$$

where S^c and S_0^c are causal electron Green functions with and without external field respectively, and c_v is probability amplitude for a vacuum to remain the vacuum. For the constant electric field E he had obtained the following result¹⁰

$$\begin{aligned} P^v &= |c_v|^2 = \exp \left\{ -\frac{VTc}{\hbar} 2\text{Im}\mathcal{L}_{\text{eff}} \right\} \\ &= \exp \left\{ -\frac{VTc}{4\pi^3\lambda^4} \left(\frac{E}{E_c} \right)^2 \sum_{n=1}^{\infty} n^{-2} \exp \left(-\pi n \frac{E_c}{E} \right) \right\}. \end{aligned} \quad (7)$$

Using Schwinger effective Lagrangian for the case $E \gg E_c$ and $B = 0$, one can find the change of the vacuum energy due to the vacuum polarization

$$\text{Re}\mathcal{L}_{\text{eff}} = -\frac{\alpha E^2}{24\pi^2} \ln \frac{E}{E_c} = \Delta\mathcal{L}_M = \Delta\mathcal{E}_M < 0. \quad (8)$$

Thus, the total energy density of the electric field $\mathcal{E} = E^2/8\pi + \Delta\mathcal{E}_M$ vanishes at E_{max} ,

$$E_{\text{max}} = E_c \exp \left(\frac{3\pi}{\alpha} \right) \approx E_c \exp(10^3). \quad (9)$$

However, E_{max} cannot be considered a maximal strength of external constant electric field because the local quantity $\text{Re}\mathcal{L}_{\text{eff}}$ describes effects of vacuum polarization only, and the analysis of $\text{Im}\mathcal{L}_{\text{eff}}$ is not able to answer all the questions related to the back reaction of created particles to the initial electric background. Indeed, $\text{Im}\mathcal{L}_{\text{eff}}$ is also a local quantity which does not depends on the field duration T and, therefore, is not related to particle creation. The complete back reaction can be obtained by nonperturbative calculating mean value of energy-momentum tensor of the matter field.

3. T -Constant External Electric Field

In the works,^{11,12} it was calculated the mean value of the energy density of matter in one-loop approximation, taking an exact account of the interaction with T -const. external electric field.

The T -const. electric field turns on from zero to E at $-T/2 = t_1$ and turns off to zero at $T/2 = t_2$. It can be given by a vector potential of the form $\mathbf{A} = (0, 0, A_z)$, where $A_z(t) = -Et$, $t \in [t_1, t_2]$, being constant for $t \in (-\infty, t_1)$ and $t \in (t_2, +\infty)$.

For $T \gg T_{\text{stab}}$,

$$\begin{aligned} T_{\text{stab}} &= \sqrt{\frac{\hbar}{eEc}} [1 + E_c/E], \quad \frac{dT_{\text{stab}}}{dE} < 0, \\ T_{\text{stab}}|_{E=E_c} &= 2\frac{\lambda_C}{c} \approx 1.3 \times 10^{-21} \text{ s} \end{aligned} \quad (10)$$

all the finite effects of particle-creation reach their asymptotic and details of field switching on and off can be neglected.¹³

Vacuum energy change in T -const electric field reads

$$\langle \hat{H}_{QED} \rangle = wV, \quad w = \frac{1}{2} \langle 0, \text{in} | [\hat{\psi}(x)^\dagger, \hat{H}_D \hat{\psi}(x)] | 0, \text{in} \rangle \Big|_{x^0=T/2-0}, \quad (11)$$

where $\hat{H}_D$ is the one-particle Dirac Hamiltonian and $\hat{\psi}(x)$ obeys the Dirac equation with T -const electric field. In these calculations, we have a complete account of the vacuum polarization as well as the pair-creation during the entire time T . The energy density w is given by

$$w = \frac{1}{4} \left(\lim_{t' \rightarrow t-0} + \lim_{t' \rightarrow t+0} \right) \text{tr} (\partial_{t'} - \partial_t) S_{\text{in}}^c(x, x')|_{\mathbf{x}=\mathbf{x}', t=t_2-0}. \quad (12)$$

Here $S_{\text{in}}^c(x, x')$ is the so-called *in-in* Green function,⁴

$$\begin{aligned} S_{\text{in}}^c(x, x') &= i \langle 0, \text{in} | T \hat{\psi}(x) \hat{\bar{\psi}}(x') | 0, \text{in} \rangle = S^c(x, x') + S^p(x, x'), \\ S^c(x, x') &= i \langle 0, \text{out} | T \hat{\psi}(x) \hat{\bar{\psi}}(x') | 0, \text{in} \rangle c_v^{-1}, \end{aligned} \quad (13)$$

$|0, \text{in}\rangle$ -free particle vacuum at $t = -T/2$, or at $A_z = ET/2$, and $|0, \text{out}\rangle$ -free particle vacuum at $t = T/2$, or at $A_z = -ET/2$, $S^c(x, x')$ is the causal (Feynman) Green function.

The separation of S_{in} into the c - and p -parts is responsible for the separation of w into the two respective summands, $w = w^c + w^p$, where w^c is a local quantity and has a finite limit at $T \rightarrow \infty$, i.e., it has a finite limit in the constant electric field, such that w^c can be calculated with S^c taken in the constant electric field.

One can see that w^c is expressed in terms of the real-valued part of Schwinger effective Lagrangian (at $B = 0$). This contribution is due to vacuum polarization. In a superstrong electric field, it has the form

$$w^c = E \frac{\partial \text{Re} \Delta \mathcal{L}_M}{\partial E} - \text{Re} \Delta \mathcal{L}_M \approx - \left(\frac{\alpha}{3\pi} \ln \frac{E}{E_c} \right) \mathcal{L}_M. \quad (14)$$

The contribution w^p due to particle-creation, it is nonlocal and is calculated via $S^p(x, x')$. In the external field under consideration, quantum numbers of particles are $n = (\mathbf{p}, r)$, $r = \pm 1$, and

$$S^p(x, x') = i \sum_n -\psi_n(x) [G(+|^-)G(-|^-)^{-1}]_{nn}^\dagger \bar{\psi}_n(x'), \quad (15)$$

where $\{\pm \psi_n(x)\}$ are *in*-solutions, their asymptotics at $t \leq t_1$ are free stationary states of the Dirac Hamiltonian with $A_z = ET/2$; $\{\pm \psi_n(x)\}$ are *out*-solutions, their asymptotics at $t \geq t_2$ are free stationary states of the Dirac Hamiltonian with $A_z = -ET/2$. The matrices $G(\pm|\pm)$ are defined as

$$\pm \psi(x) = +\psi(x)G(+|\pm) + -\psi(x)G(-|\pm). \quad (16)$$

In particular, differential mean numbers of electrons created are $\aleph_n = |G(-|+)|_{nn}^2$.

For calculating w^p one needs $S^p(x, x')$ at $x \approx x'$, which is

$$S^p = -i \int d\mathbf{p} \sum_{r=\pm 1} \aleph_{\mathbf{p},r} \left[{}^+\psi_{\mathbf{p},r}(x) {}^+\bar{\psi}_{\mathbf{p},r}(x') - {}^-\psi_{\mathbf{p},r}(x) {}^-\bar{\psi}_{\mathbf{p},r}(x') \right]. \quad (17)$$

In such a way, we obtain

$$w^p = \frac{1}{4\pi^3} \int d\mathbf{p} \sum_{r=\pm 1} \aleph_{\mathbf{p},r} \varepsilon_{\mathbf{p},r}, \quad (18)$$

where $\varepsilon_{\mathbf{p},r} = \sqrt{m^2 + \mathbf{p}_\perp^2 + (eET/2 + p_3)^2}$ are energies of out-particles.

Differential mean numbers $\aleph_{\mathbf{p},r}$ have the form

$$\aleph_{\mathbf{p},r} = \exp \left(-\pi \frac{m^2 + \mathbf{p}_\perp^2}{eE} \right) \quad (19)$$

for $|p_3| \leq \sqrt{eE} \left(\sqrt{eET}/2 - K \right)$, where K is an arbitrary large number, $\sqrt{eET} \gg K \gg 1 + m^2/eE$, and $\aleph_{\mathbf{p},r}$ are fast-decreasing out of this interval.

For T -leading terms, we obtain

$$w^p = eET \aleph, \quad \aleph = \frac{e^2 E^2 T}{4\pi^3} \exp \left(-\pi \frac{m^2}{eE} \right). \quad (20)$$

Particle-creation is the main reason for the change of matter energy. The smallness of the back-reaction, which implies $w^p \ll E^2/8\pi$, imposes a restriction on dimensionless parameter $(eEc/\hbar) T^2$,

$$(ceE/\hbar) T^2 \ll \frac{\pi^2}{2\alpha} \exp \left(\pi \frac{E_c}{E} \right), \quad (21)$$

or on both the magnitude E and duration T of electric field¹¹

$$eE \ll \frac{\pi^2 \hbar}{2\alpha c T^2} \exp \left(\pi \frac{E_c}{E} \right). \quad (22)$$

If the initial state is in thermal equilibrium at high temperatures $\theta \gg ecET$, we have a weaker restriction:

$$eE \ll \frac{\pi}{c} \sqrt{\frac{3\hbar\theta}{\alpha T^3}} \exp \left(\frac{\pi}{2} \frac{E_c}{E} \right). \quad (23)$$

4. Are There Some Restrictions on the Coulomb Field of Point-Like Charges?

For an electron, Coulomb field of a nucleus Ze can be considered as external since $m_p/m_e \sim 2000$ (for the uranium $E = 900E_c$, at $R = 1.2A^{1/3}$ fm.) Usually, electron spectrum in Coulomb field is identified with the Dirac operator

$$\hat{H}_D = -i\hbar c \alpha \nabla + \beta mc^2 - \frac{Ze^2}{r}, \quad (24)$$

spectrum in the point-like Coulomb field given in 1940 by Sommerfeld. Here there exists so-called “ $Z = 137$ catastrophe”. If $Za > j + 1/2$, the spectrum becomes

imaginary. For instance, for the lowest level

$$E = mc^2 \sqrt{1 - (Z\alpha)^2}. \quad (25)$$

One of the first who studied the problem Gordon (1928) claimed “no regular solutions for ground state are found beyond $Z_c = 137$.” This statement is repeated in all articles and books on this subject (see, e.g., Refs. 3 and 14). The common explanation: the catastrophe is due point-like nucleus in calculating the spectrum. Doing cutoff at $R \sim 1.2 \times 10^{-12}$ cm, the Dirac equation has physically reasonable solutions for $Z < Z_{sc} = 173$. But for $Z \geq Z_{sc}$ another difficulty arises,¹⁵ some discrete levels appear below the upper bound of the lower continuum. The explanation: the problem cannot longer be considered a one-particle one because there exists spontaneous pair production.

Returning again to the Dirac equations with the Coulomb field of point-like nucleus, Voronov, Gitman, Tyutin¹⁶ have demonstrated that:

- i. For any Z , the Dirac Hamiltonian $\hat{H}_D(Z)$ can be defined as a self-adjoint operator.
- ii. For $Z \geq Z_c^1 = (\sqrt{3}/2)\alpha^{-1} = 119$, there exist families of self-adjoint $\hat{H}_D(Z, \nu)$ parametrized by a parameter ν .
- iii. For any $Z \geq Z_c^1$, and any self-adjoint $\hat{H}_D(Z, \nu)$, spectra and inversion formulas are found. There is no any Z catastrophe. No any discrete levels appear below the upper bound of the lower continuum for any Z and any $\hat{H}_D(Z, \nu)$.
- iv. For any $Z \geq Z_c^1$ there exists ν_0 , such that

$$E_0(Z, \nu_0) = -mc^2, \quad -mc^2 < E_n(Z, \nu_0) < mc^2, \quad \forall n > 0, \\ -mc^2 < E_n(Z, \nu) < mc^2, \quad \forall n \geq 0, \quad \forall \nu \neq \nu_0.$$

Thus, formally, there are no problems with Dirac equation with Coulomb field of point-like nucleus for any Z . Generalizing the Zeldovich-Popov interpretation for point-like nucleus, one can suppose that particle creation must start for any $Z \geq 119$ in case if the nature is described by the Dirac Hamiltonians $\hat{H}_D(Z, \nu_0)$. In this respect an interesting coincidence has to be noted: only nuclei with $Z \leq 118$ have been synthesized until now.

Acknowledgments

The work of S.P.G. is supported by the program Bolsista CAPES/Brasil. S.P.G. thanks University of São Paulo for hospitality. D.M.G. acknowledges the permanent support of FAPESP and CNPq.

References

1. A. A. Grib, S. G. Mamaev and V. M. Mostepanenko, *Quantum Effects in Intensive External Fields* (Atomizdat, Moscow, 1980, in Russian); *Vacuum Quantum Effects in Strong Fields* (Atomizdat, Moscow, 1988, in Russian; English translation: Friedmann Laboratory Publishing, St. Petersburg, 1994).

2. N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982).
3. W. Greiner, B. Müller and J. Rafelski, *Quantum Electrodynamics of Strong Fields* (Springer-Verlag, Berlin, 1985).
4. E. S. Fradkin, D. M. Gitman and S. M. Shvartsman, *Quantum Electrodynamics with Unstable Vacuum* (Springer-Verlag, Berlin, 1991).
5. A. I. Nikishov, *Sov. Phys. JETP* **30**, 660 (1970).
6. V. G. Bagrov, D. M. Gitman and Sh. M. Shvartsman, *Sov. Phys. JETP* **41**, 191 (1975).
7. W. Heisenberg and H. Euler, *Z. Phys.* **98**, 714 (1936); English translation available at [arXiv:physics/0605038](https://arxiv.org/abs/physics/0605038).
8. V. I. Ritus, *Sov. Phys. JETP* **42**, 774 (1975); *ibid* **46**, 423 (1977).
9. A. E. Shabad and V. V. Usov, *Phys. Rev. Lett.* **96**, 180401 (2006).
10. J. Schwinger, *Phys. Rev.* **82**, 664 (1951).
11. S. P. Gavrilov and D. M. Gitman, *Phys. Rev. Lett.* **101**, 130403 (2008)
12. S. P. Gavrilov and D. M. Gitman, *Phys. Rev. D* **78**, 045017 (2008).
13. S. P. Gavrilov and D. M. Gitman, *Phys. Rev. D* **53**, 7162 (1996).
14. A. I. Akhiezer and V. B. Berestetskii, *Quantum Electrodynamics* (Interscience Publishers, New York, 1965)
15. Ya. B. Zeldovich and V. S. Popov, *Uspekhi Fiz. Nauk* **105**, 403 (1971); W. Piper and W. Greiner, *Z. Phys.* **218**, 327 (1969).
16. B. L. Voronov, D. M. Gitman and I. V. Tyutin, *Theor. Math. Phys.* **150**(1), 34 (2007).