NONCOMMUTATIVE BTZ BLACK HOLE IN DIFFERENT COORDINATES*

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We consider noncommutative BTZ black hole solutions in two different coordinate systems, the polar and rectangular coordinates. The analysis is carried out by obtaining noncommutative solutions of $U(1, 1) \times U(1, 1)$ Chern-Simons theory on AdS_3 in the two coordinate systems via the Seiberg-Witten map. This is based on the noncommutative extension of the equivalence between the classical BTZ solution and the solution of ordinary $SU(1, 1) \times SU(1, 1)$ Chern-Simons theory on AdS_3 . The obtained solutions in these noncommutative coordinate systems become different in the first order of the noncommutativity parameter θ .

1. Introduction

The most used canonical commutation relation for noncommutative spacetime is modeled on the commutation relation of quantum mechanics:

$$[\hat{x}^\alpha, \hat{x}^\beta] = i\theta^{\alpha\beta}, \quad (1)$$

where $\theta^{\alpha\beta} = -\theta^{\beta\alpha}$ are constants. It has been also known that a theory on a deformed spacetime with the above canonical relation is equivalent to a theory on commutative spacetime in which any product of functions of noncommutative coordinates are replaced with a deformed $\star$ -product of the same functions of commutative coordinates, the so-called Moyal product¹ which is defined by

$$(f \star g)(x) \equiv \exp \left[\frac{i}{2} \theta^{\alpha\beta} \frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial y^\beta} \right] f(x)g(y) \Big|_{x=y}. \quad (2)$$

Using the Moyal product many works on noncommutative spacetime have been carried out and especially in² a map between a gauge theory on noncommutative spacetime and one on commutative spacetime, the so-called Seiberg-Witten map, was established.

So far, we are accustomed to take general covariance for granted. General covariance in “a noncommutative space^a” would mean the equivalence among different

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^a We only deal with space-space noncommutativity here, and we use (noncommutative) space and (noncommutative) spacetime interchangeably.

coordinate systems in this noncommutative space. However, different coordinate systems in “a noncommutative space” generally have different commutation relations which are not exactly equivalent to each other. Therefore, if we work with different coordinate systems in “a noncommutative space”, we may end up with different results. If this happens, it would contradict our usual notion of general covariance. Seiberg³ has already pointed out that general covariance would be broken in theories with emergent spacetime among which model theories on noncommutative spaces are included. As a step on this issue, here we investigate the noncommutative BTZ solutions in the polar and rectangular coordinate systems. We carry this by obtaining the solutions of $U(1, 1) \times U(1, 1)$ Chern-Simons theory on noncommutative AdS_3 in the two coordinate systems via the Seiberg-Witten map.

2. Noncommutative Chern-Simons Gravity

The action of the $(2 + 1)$ dimensional noncommutative $U(1, 1) \times U(1, 1)$ Chern-Simons theory with the negative cosmological constant $\Lambda = -1/l^2$ is given by up to boundary terms,^{4,5}

$$\begin{aligned}\hat{S}(\hat{\mathcal{A}}^+, \hat{\mathcal{A}}^-) &= \hat{S}_+(\hat{\mathcal{A}}^+) - \hat{S}_-(\hat{\mathcal{A}}^-), \\ \hat{S}_\pm(\hat{\mathcal{A}}^\pm) &= \beta \int \text{Tr}(\hat{\mathcal{A}}^\pm \star d\hat{\mathcal{A}}^\pm + \frac{2}{3}\hat{\mathcal{A}}^\pm \star \hat{\mathcal{A}}^\pm \star \hat{\mathcal{A}}^\pm),\end{aligned}\quad (3)$$

where $\beta = l/16\pi G_N$ and G_N is the three dimensional Newton constant. Here $\hat{\mathcal{A}}^\pm = \hat{A}^{A\pm}\tau_A = \hat{A}^{a\pm}\tau_a + \hat{B}^\pm\tau_3$, with $A = 0, 1, 2, 3$, $a = 0, 1, 2$, $\hat{\mathcal{A}}^{a\pm} = \hat{A}^{a\pm}$, $\hat{\mathcal{A}}^{3\pm} = \hat{B}^\pm$, and the deformed wedge product $\star$ denotes that $A \star B \equiv A_\mu \star B_\nu dx^\mu \wedge dx^\nu$. The noncommutative $SU(1, 1) \times SU(1, 1)$ gauge fields $\hat{A}$ are expressed in terms of the triad $\hat{e}$ and the spin connection $\hat{\omega}$ as $\hat{A}^{a\pm} := \hat{\omega}^a \pm \hat{e}^a/l$. In terms of $\hat{e}$ and $\hat{\omega}$ the action becomes⁵

$$\begin{aligned}\hat{S} &= \frac{1}{8\pi G_N} \int \left(\hat{e}^a \star \hat{R}_a + \frac{1}{6l^2} \epsilon_{abc} \hat{e}^a \star \hat{e}^b \star \hat{e}^c \right) \\ &\quad - \frac{\beta}{2} \int \left(\hat{B}^+ \star d\hat{B}^+ + \frac{i}{3} \hat{B}^+ \star \hat{B}^+ \star \hat{B}^+ \right) + \frac{\beta}{2} \int \left(\hat{B}^- \star d\hat{B}^- + \frac{i}{3} \hat{B}^- \star \hat{B}^- \star \hat{B}^- \right) \\ &\quad + \frac{i\beta}{2} \int (\hat{B}^+ - \hat{B}^-) \star \left(\hat{\omega}^a \star \hat{\omega}_a + \frac{1}{l^2} \hat{e}^a \star \hat{e}_a \right) \\ &\quad + \frac{i\beta}{2l} \int (\hat{B}^+ + \hat{B}^-) \star \left(\hat{\omega}^a \star \hat{e}_a + \hat{e}^a \star \hat{\omega}_a \right),\end{aligned}\quad (4)$$

up to surface terms, where $\hat{R}^a = d\hat{\omega}^a + \frac{1}{2}\epsilon^{abc}\hat{\omega}_b \star \hat{\omega}_c$. The equation of motion can be written as follows.

$$\hat{\mathcal{F}}^\pm \equiv d\hat{\mathcal{A}}^\pm + \hat{\mathcal{A}}^\pm \star \hat{\mathcal{A}}^\pm = 0. \quad (5)$$

In the commutative limit this becomes,

$$F^\pm \equiv dA^\pm + A^\pm \wedge A^\pm = 0, \quad dB^\pm = 0, \quad (6)$$

and the first one can be rewritten as

$$R^a + \frac{1}{2l^2} \epsilon^{abc} e_b \wedge e_c = 0, \quad T^a \equiv de^a + \epsilon^{abc} \omega_b \wedge e_c = 0. \quad (7)$$

The solution of the decoupled EOM for $SU(1,1) \times SU(1,1)$ part was obtained in:⁶

$$\begin{aligned} e^0 &= m \left(\frac{r_+}{l} dt - r_- d\phi \right), \quad e^1 = \frac{l}{n} dm, \quad e^2 = n \left(r_+ d\phi - \frac{r_-}{l} dt \right), \\ \omega^0 &= -\frac{m}{l} \left(r_+ d\phi - \frac{r_-}{l} dt \right), \quad \omega^1 = 0, \quad \omega^2 = -\frac{n}{l} \left(\frac{r_+}{l} dt - r_- d\phi \right), \end{aligned} \quad (8)$$

where $m^2 = (r^2 - r_+^2)/(r_+^2 - r_-^2)$, $n^2 = (r^2 - r_-^2)/(r_+^2 - r_-^2)$, and r_+ , r_- are the outer and inner horizons respectively. There it was also shown to be equivalent to the ordinary BTZ black hole solution:⁷

$$ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 (d\phi + N^\phi dt)^2, \quad (9)$$

where $N^2 = (r^2 - r_+^2)(r^2 - r_-^2)/l^2 r^2$ and $N^\phi = -r_+ r_- / l r^2$.

3. Noncommutative BTZ Solution in Polar Coordinates

Based on the above noncommutative extension of the equivalence between 3D gravity and $SU(1,1) \times SU(1,1)$ Chern-Simons theory, we now get the noncommutative BTZ solution in the polar coordinates following⁸ using the Seiberg-Witten map. The Seiberg-Witten map which dictates the following equivalence relation between ordinary and noncommutative gauge transformations,²

$$\hat{\mathcal{A}}_\mu(\mathcal{A}) + \hat{\delta}_\lambda \hat{\mathcal{A}}_\mu(\mathcal{A}) = \hat{\mathcal{A}}_\mu(\mathcal{A} + \delta_\lambda \mathcal{A}), \quad (10)$$

allows to express noncommutative gauge fields $\hat{\mathcal{A}}$ in terms of ordinary gauge fields $\mathcal{A}$ as

$$\hat{\mathcal{A}}_\mu(\mathcal{A}) = \mathcal{A}_\mu - \frac{i}{4} \theta^{\rho\sigma} \{ \mathcal{A}_\rho, \partial_\sigma \mathcal{A}_\mu + \mathcal{F}_{\sigma\mu} \} + \mathcal{O}(\theta^2), \quad (11)$$

where $\theta^{\rho\sigma}$ are noncommutativity parameters of the canonical commutation relation (1). Here, our chosen commutation relation for the polar coordinates is $[\hat{r}, \hat{\phi}] = i\theta \hat{r}^{-1}$, which is not in the canonical form. However, one can easily show that this is exactly equivalent to $[\hat{r}^2, \hat{\phi}] = 2i\theta$. Thus for computational convenience, we use the commutation relation $[\hat{R}, \hat{\phi}] = 2i\theta$ with $\hat{R} \equiv \hat{r}^2$. The Seiberg-Witten map (11) of the noncommutative gauge fields, $\hat{\mathcal{A}}^\pm := \hat{A}_\mu^{a\pm} \tau_a + \hat{B}_\mu^\pm \tau_3$, yields the following.

$$\hat{\mathcal{A}}_\mu^\pm = \left(A_\mu^{a\pm} - \frac{\theta}{2} B_\phi^\pm \partial_R A_\mu^{a\pm} \right) \tau_a + B_\mu^\pm \tau_3 + \mathcal{O}(\theta^2). \quad (12)$$

Setting the two $U(1)$ fluxes as $B_\mu^\pm = Bd\phi$ with constant B , we obtain the following expressions for the noncommutative triad and spin connection:

$$\begin{aligned}\hat{e}^0 &= \left(m - \frac{\theta B}{2}m'\right) \left(\frac{r_+}{l}dt - r_-d\phi\right) + \mathcal{O}(\theta^2), \\ \hat{e}^1 &= l \left[\frac{m'}{n} - \frac{\theta B}{2} \left(\frac{m'}{n}\right)' \right] dR + \mathcal{O}(\theta^2), \\ \hat{e}^2 &= \left(n - \frac{\theta B}{2}n'\right) \left(r_+d\phi - \frac{r_-}{l}dt\right) + \mathcal{O}(\theta^2), \\ \hat{\omega}^0 &= -\frac{1}{l} \left(m - \frac{\theta B}{2}m'\right) \left(r_+d\phi - \frac{r_-}{l}dt\right) + \mathcal{O}(\theta^2), \\ \hat{\omega}^1 &= \mathcal{O}(\theta^2), \\ \hat{\omega}^2 &= -\frac{1}{l} \left(n - \frac{\theta B}{2}n'\right) \left(\frac{r_+}{l}dt - r_-d\phi\right) + \mathcal{O}(\theta^2),\end{aligned}\tag{13}$$

where $'$ denotes the differentiation with respect to $R = r^2$. We now define the metric in the noncommutative case as

$$d\hat{s}^2 := \hat{g}_{\mu\nu}dx^\mu dx^\nu \equiv \eta_{ab}\hat{e}_\mu^a \star \hat{e}_\nu^b dx^\mu dx^\nu,\tag{14}$$

and with this definition we get a real metric ($\hat{e}_\mu \star \hat{e}_\nu = \hat{e}_\mu \hat{e}_\nu$). Rewriting R in terms of r , we now get

$$d\hat{s}^2 = -f^2 dt^2 + \hat{N}^{-2} dr^2 + 2r^2 N^\phi dt d\phi + \left(r^2 + \frac{\theta B}{2}\right) d\phi^2 + \mathcal{O}(\theta^2),\tag{15}$$

where $N^\phi = -\frac{r_+r_-}{l^2r^2}$, $f^2 = \frac{(r^2 - r_+^2 - r_-^2)}{l^2} - \frac{\theta B}{2l^2}$, $\hat{N}^2 = \frac{1}{l^2r^2}[(r^2 - r_+^2)(r^2 - r_-^2) - \frac{\theta B}{2}(2r^2 - r_+^2 - r_-^2)]$. In this solution, the apparent horizon (denoted as $\hat{r}$) which is determined by

$$\hat{g}^{rr} = \hat{g}_{rr}^{-1} = \hat{N}^2 = 0,\tag{16}$$

and the Killing horizon (denoted as $\tilde{r}$) which is determined by

$$\hat{g}_{tt} - \hat{g}_{t\phi}^2/\hat{g}_{\phi\phi} = 0,\tag{17}$$

are given as follows:

$$\hat{r}_\pm^2 = r_\pm^2 + \frac{\theta B}{2} + \mathcal{O}(\theta^2),\tag{18}$$

$$\tilde{r}_\pm^2 = r_\pm^2 \pm \frac{\theta B}{2} \left(\frac{r_+^2 + r_-^2}{r_+^2 - r_-^2} \right) + \mathcal{O}(\theta^2).\tag{19}$$

In the classical case, the apparent and Killing horizons coincide for stationary black holes. Note that here the apparent and Killing horizons do not coincide. Only in the non-rotating limit in which the classical inner horizon vanishes, $r_- = 0$, we see that the two outer horizons coincide as in the classical case.

4. Noncommutative BTZ Solution in Rectangular Coordinates

We now consider the noncommutative BTZ solution in the rectangular coordinates following.⁹ In order to evaluate the Seiberg-Witten map in the rectangular coordinates we first have to express the classical solution of the $U(1, 1) \times U(1, 1)$ gauge fields in terms of the rectangular coordinates. We again set the two $U(1)$ fluxes as $B_\mu^\pm = B d\phi = B(xdy - ydx)/r^2$ with constant B , and the classical $SU(1, 1) \times SU(1, 1)$ solution in the rectangular coordinates is given by

$$\begin{aligned} A^{0\pm} &= \pm \frac{m(r_+ \pm r_-)}{l^2} \left[dt \pm \frac{l}{r^2} (ydx - xdy) \right], \\ A^{1\pm} &= \pm \frac{1}{\sqrt{(r^2 - r_+^2)(r^2 - r_-^2)}} (xdx + ydy), \\ A^{2\pm} &= -\frac{n(r_+ \pm r_-)}{l^2} \left[dt \pm \frac{l}{r^2} (ydx - xdy) \right]. \end{aligned} \quad (20)$$

Now performing the Seiberg-Witten map as in the previous section, and using the relations $\hat{e}/l = \hat{\mathcal{A}}^+ + \hat{\mathcal{A}}^-$ and $\hat{\omega} = \hat{\mathcal{A}}^+ - \hat{\mathcal{A}}^-$, we obtain the noncommutative triad and spin connection in the rectangular coordinates up to first order in θ as follows.

$$\begin{aligned} \hat{e}^0 &= \frac{r_+ [r^2 - r_+^2 - \theta B/4]}{l \sqrt{(r^2 - r_+^2)(r_+^2 - r_-^2)}} dt + \frac{r_-}{r^2} \sqrt{\frac{r^2 - r_+^2}{r_+^2 - r_-^2}} \left[1 + \frac{\theta B}{4r^2} \left(\frac{r^2 - 2r_+^2}{r^2 - r_+^2} \right) \right] (ydx - xdy), \\ \hat{e}^1 &= -\frac{l(r^2 + r_-^2)}{(r^2 - r_+^2)(r^2 - r_-^2)} \left[1 - \frac{\theta B}{4r^2} \frac{r_+^4 (r^2 - 2r_-^2) - r_-^4 (r^2 - 2r_+^2)}{(r_+^2 - r_-^2)(r^2 + r_-^2) \sqrt{(r^2 - r_+^2)(r^2 - r_-^2)}} \right] (xdx + ydy), \\ \hat{e}^2 &= \frac{r_- [r^2 - r_-^2 - \theta B/4]}{l \sqrt{(r^2 - r_-^2)(r_+^2 - r_-^2)}} dt - \frac{r_+}{r^2} \sqrt{\frac{r^2 - r_-^2}{r_+^2 - r_-^2}} \left[1 + \frac{\theta B}{4r^2} \left(\frac{r^2 - 2r_-^2}{r^2 - r_-^2} \right) \right] (ydx - xdy), \quad (21) \\ \hat{\omega}^0 &= \frac{r_- [r^2 - r_+^2 - \theta B/4]}{l^2 \sqrt{(r^2 - r_+^2)(r_+^2 - r_-^2)}} dt + \frac{r_+}{lr^2} \sqrt{\frac{r^2 - r_+^2}{r_+^2 - r_-^2}} \left[1 + \frac{\theta B}{4r^2} \left(\frac{r^2 - 2r_+^2}{r^2 - r_+^2} \right) \right] (ydx - xdy), \\ \hat{\omega}^1 &= 0, \\ \hat{\omega}^2 &= -\frac{r_+ [r^2 - r_-^2 - \theta B/4]}{l^2 \sqrt{(r^2 - r_+^2)(r_+^2 - r_-^2)}} dt - \frac{r_-}{lr^2} \sqrt{\frac{r^2 - r_-^2}{r_+^2 - r_-^2}} \left[1 + \frac{\theta B}{4r^2} \left(\frac{r^2 - 2r_-^2}{r^2 - r_-^2} \right) \right] (ydx - xdy). \end{aligned}$$

Here we may define the metric by the same relation (14) as in the polar coordinates case. However, this definition yields complex valued metric components. Noting that the length element $d\hat{s}^2$ in (14) has symmetric summation after which its value becomes real, we now define the metric by $\hat{G}_{\mu\nu} \equiv (\hat{g}_{\mu\nu} + \hat{g}_{\nu\mu})/2$ as in Ref. 10.

Reexpress the rectangular coordinates back into the polar coordinates, we then get

$$\begin{aligned} d\hat{s}^2 &:= \hat{G}_{\mu\nu} dx^\mu dx^\nu \\ &= -\mathcal{F}^2 dt^2 + \hat{\mathcal{N}}^{-2} dr^2 + 2r^2 N^\phi \left(1 + \frac{\theta B}{2r^2}\right) dt d\phi + \left(r^2 + \frac{\theta B}{2}\right) d\phi^2, \end{aligned} \quad (22)$$

where

$$\begin{aligned} \mathcal{F}^2 &= \frac{(r^2 - r_+^2 - r_-^2)}{l^2} - \frac{\theta B}{2l^2} = f^2, \\ \hat{\mathcal{N}}^2 &= \frac{1}{l^2 r^2} \left[(r^2 - r_+^2)(r^2 - r_-^2) - \frac{\theta B}{2r^2} (r_+^2(r^2 - r_-^2) + r_-^2(r^2 - r_+^2)) \right]. \end{aligned}$$

This solution now yields the apparent and Killing horizons which are determined by the same relations (16) and (17), respectively, as in the previous section as follows.

$$\hat{r}_\pm^2 = r_\pm^2 + \frac{\theta B}{2} + \mathcal{O}(\theta^2), \quad (23)$$

$$\tilde{r}_\pm^2 = r_\pm^2 + \frac{\theta B}{2} + \mathcal{O}(\theta^2). \quad (24)$$

In this rectangular coordinates case, unlike the polar coordinates case, the apparent and Killing horizons do coincide. Although the inner and outer horizons are shifted from the classical value by the same amount $\theta B/2$ due to noncommutative effect of flux, the feature that the apparent and Killing horizons coincide matches with the classical result.

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