Erratum

A CAT(0)-valued pointwise ergodic theorem

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In my paper [1] there is a mistake in the proof of Lemma 2.3. The statement of the lemma is correct, and this erratum offers a corrected proof:

Let (X, d) be a separable complete CAT(0)-space, and let $P_2(X)$ be the space of Borel probability measures on X for which the distance function from any fixed point is square-integrable. For p equal to 1 or 2, let W_p denote the Wasserstein p -metric on $P_2(X)$, defined by

$$W_p(\mu, \nu) := \inf_{\lambda \text{ a coupling of } \mu, \nu} \left[\int_{X^2} d(x, y)^p \lambda(dx, dy) \right]^{1/p}.$$

Finally, for any $\mu \in P_2(X)$, let $b(\mu) \in X$ be its barycentre.

Lemma 2.3 of [1] asserts that

$$d(b(\mu), b(\nu)) \leq W_2(\mu, \nu) \quad \forall \mu, \nu \in P_2(X). \quad (1)$$

However, the proof in that paper considers only the product coupling of μ and ν , which may be far from achieving the infimum that defines W_2 .

In order to correct this, we will actually prove that

$$d(b(\mu), b(\nu)) \leq W_1(\mu, \nu) \quad \forall \mu, \nu \in P_2(X). \quad (2)$$

This is stronger than (1), simply because any coupling of μ and ν is a probability measure, and for a probability measure the L^2 -norm dominates the L^1 -norm.

The inequality (2) is just a small generalization of a known result: [2, Lemma 4.2], which amounts to (2) for measures supported on finitely many atoms. We quickly repeat the argument from [2] in the generality we need.

For any $x, y \in X$, we have

$$d(x, b(\nu))^2 + d(y, b(\mu))^2 \leq d(x, b(\mu))^2 + d(y, b(\nu))^2 + 2d(b(\mu), b(\nu))d(x, y)$$

(this is a standard inequality for any four points in a CAT(0)-space: see [2, Lemma 2.1]). Letting λ be any coupling of μ and ν , we may integrate this inequality with respect to $\lambda(dx, dy)$ to obtain

$$\begin{aligned} & \int d(x, b(\nu))^2 \mu(dx) + \int d(y, b(\mu))^2 \nu(dy) \\ & \leq \int d(x, b(\mu))^2 \mu(dx) + \int d(y, b(\nu))^2 \nu(dy) + 2d(b(\mu), b(\nu)) \int d(x, y) \lambda(dx, dy). \end{aligned} \quad (3)$$

On the other hand, another standard inequality for baycentres in CAT(0)-spaces (see [2, Lemma 4.1]) gives that

$$\int d(x, b(\nu))^2 \mu(dx) \geq \int d(x, b(\mu))^2 \mu(dx) + d(b(\mu), d(\nu))^2,$$

and similarly with the roles of μ and ν reversed. Inserting these two inequalities into (3), we obtain

$$\begin{aligned} & \int d(x, b(\mu))^2 \mu(dx) + \int d(y, b(\nu))^2 \nu(dy) + 2d(b(\mu), b(\nu))^2 \\ & \leq \int d(x, b(\mu))^2 \mu(dx) + \int d(y, b(\nu))^2 \nu(dy) + 2d(b(\mu), b(\nu)) \int d(x, y) \lambda(dx, dy). \end{aligned}$$

Upon simplifying, this yields

$$d(b(\mu), b(\nu)) \leq \int d(x, y) \lambda(dx, dy).$$

Since this holds for any coupling λ , it proves (2).

Acknowledgment

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References

1. T. Austin, A CAT(0)-valued pointwise ergodic theorem, *J. Topol. Anal.* **3** (2011) 145–152.
2. U. Lang, B. Pavlović and V. Schroeder, Extensions of Lipschitz maps into Hadamard spaces, *Geom. Funct. Anal.* **10** (2000) 1527–1553.