The Saito–Kurokawa lift and Siegel’s theta series

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The aim of this paper is to give another short proof of the Saito–Kurokawa lift based on a converse theorem of Imai as was already done by Duke and Imamoglu. In contrast to their proof we avoid spectral analysis but use a real analytic Eisenstein series in a suitable Rankin–Selberg integral involving Siegel’s theta series.

Keywords: Saito–Kurokawa lift; Siegel modular forms; Siegel’s theta series.

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1. Introduction

The Saito–Kurokawa lift gives a correspondence between classical modular forms contained in the Kohnen space and the Maass Spezialschar of Siegel modular forms. By the work of Maass, Andrianov and Zagier this lift was originally produced as a composition of two liftings involving Jacobi forms, see [3]. Krieg [9] generalized this lifting to the case of Hermitian modular forms.

In [14] Sugano generalized this lifting to holomorphic cusp forms on $SO(2, m+2)$, also using Jacobi forms. Note that $m = 1$ refers to Siegel- and $m = 2$ to Hermitian modular forms.

A different strategy of proof in the case of Siegel modular forms was given by Duke and Imamoglu in [2] who used a converse theorem of Imai [5]. Imai’s converse theorem makes use of spectral theory of the Laplacian and in order to establish the lift, Duke and Imamoglu prove functional equations for certain spectral Koecher–Maass series.

The purpose of this paper is to give a shorter proof along the lines of Duke and Imamoglu but avoiding spectral theory. Instead a real analytic Eisenstein series and

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Siegel's theta series enter the picture. In a forthcoming paper, we shall apply our method also to the higher dimensional case.

2. Saito–Kurokawa Lift

Recall that for k an integer the Kohnen space $S_{k-\frac{1}{2}}^+ = S_{k-\frac{1}{2}}^+(\Gamma_0(4))$ is a subspace of the space $S_{k-\frac{1}{2}}(\Gamma_0(4))$ of holomorphic cusp forms for $\Gamma_0(4)$ of weight $k - \frac{1}{2}$ whose Fourier coefficients $a(n)$ vanish if $(-1)^{k+1}n \equiv 2, 3 \pmod{4}$, see [7].

A classical Siegel modular form of weight k and degree 2 is a function $F(Z)$ on the Siegel upper halfspace of degree 2

$$\mathbf{H} = \{Z = X + iY, X, Y \in M(2, \mathbb{R}) : Z^t = Z, Y \text{ positive definite}\}$$

which is holomorphic such that

$$F(MZ) = \det^k(CZ + D)F(Z),$$

for any $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in the Siegel modular group $\mathrm{Sp}(2, \mathbb{Z})$.

It is well known, that any Siegel modular form can be expanded in a Fourier series

$$F(Z) = \sum_{0 \leq T \in L} A(T)e(\mathrm{tr} TZ),$$

where L refers to the set of all half integral symmetric matrices of size two and where as usual $e(z) = e^{2\pi iz}$. Recall that a matrix $T = \begin{pmatrix} a & \alpha \\ \alpha & d \end{pmatrix}$ is called half integral, if $a, d, 2\alpha$ are integers.

We use the notations $T \geq 0$ and $T > 0$ for positive semidefinite or definite symmetric matrices, respectively. It is easy to verify the unimodularity property

$$A([U]T) = \det^k(U)A(T) \quad (2.1)$$

for $U \in \mathrm{GL}(2, \mathbb{Z})$ where U acts via $[U]T = UTU^t$.

F is called a cusp form if $A(T) = 0$ provided $\det T = 0$.

The classical result concerning the Saito–Kurokawa lift for even weight $k \geq 2$ establishes the following map $f(z) \rightarrow F(Z)$:

Theorem 2.1 (Saito–Kurokawa lift). *Let k denote an even positive integer and let $f \in S_{k-\frac{1}{2}}^+$ with Fourier series*

$$f(z) = \sum_{n \equiv 0, 3 \pmod{4}} a(n)e(nz).$$

For $0 < T \in L$ put $e(T) := \max\{q \in \mathbb{N} : T/q \in L\}$ and let

$$A(T) = \sum_{d|e(T)} d^{k-1} a((4 \det T)/d^2).$$

Then the function $F(Z) = \sum_{0 < T \in L} A(T)e(\mathrm{tr} TZ)$, called the Saito–Kurokawa lift of f , is a degree 2 Siegel cusp form of weight k .

3. Imai’s Converse Theorem

In her converse theorem Imai considers for real matrices $Y > 0$ the Mellin transform of $F(iY)$, where F is function on $\mathbf{H}$ given by a convergent Fourier series

$$F(Z) = \sum_{0 < T \in L} A(T) e(\mathrm{tr} TZ) \quad (3.1)$$

which is unimodular in the sense that the Fourier coefficients also satisfy (2.1). It is easy to see that this implies

$$F([U]Z) = F(Z) \quad (3.2)$$

for $U \in \mathrm{S}(2, \mathbb{Z})$

Therefore we shall call such a Fourier series a *unimodular Fourier series*.

Also it is assumed, that the Fourier coefficients are of at most polynomial growth, i.e.

$$A(T) \ll (\det T)^a \quad (3.3)$$

for some $a > 0$.

Under these assumptions it is shown in [5] that the series $F(X + iY)$ converges absolutely and uniformly in any domain $Y \geq Y_0 > \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $F(Z)$ is bounded on this region.

Furthermore it is shown, cf. [5, Proposition 2.5 and Remark 2.6, p. 910f], that for $W > 0$ with $\det W = 1$ the partial Mellin transform

$$\tilde{F}(W, s) = \int_0^\infty F(iu^{\frac{1}{2}}W) u^{s-1} du \quad (3.4)$$

exists and is holomorphic in s for $\Re s$ sufficiently large.

There is the following well-known bijective correspondence between matrices $W > 0$ with $\det W = 1$ and the upper halfplane $\mathbb{H}$ given by a Cholesky decomposition of W :

Lemma 3.1. *Let $W > 0$ with $\det W = 1$. Then*

$$W = \begin{pmatrix} x & y \\ y & z \end{pmatrix} = \omega \omega^t \quad (3.5)$$

with uniquely determining

$$\omega = \begin{pmatrix} \sqrt{v} & u/\sqrt{v} \\ 0 & 1/\sqrt{v} \end{pmatrix} = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{v} & 0 \\ 0 & \frac{1}{\sqrt{v}} \end{pmatrix}$$

where $w = u + iv \in \mathbb{H}$ and $x = |w|^2/v$, $y = u/v$, $z = 1/v$.

Furthermore the correspondence $\phi : W \rightarrow w$ is compatible with the action of $\mathrm{SL}(2, \mathbb{R})$ in the sense that

$$\phi([U]W) = Uw$$

for $U \in \mathrm{SL}(2, \mathbb{R})$. Especially one finds that $\phi(W^{-1}) = -\frac{1}{w}$.

For a proof in a more general situation see e.g., [4]. □

Due to this correspondence the partial Mellin transform can be regarded as a function of w and by abuse of notation we shall write $\tilde{F}(w, s)$ instead of $\tilde{F}(W, s)$. The unimodularity property for F thus implies $\tilde{F}(Uw, s) = \tilde{F}(w, s)$ for $U \in \mathrm{SL}(2, \mathbb{Z})$ and therefore $\tilde{F}(w, s)$ can be viewed upon as an ordinary modular form for $\mathrm{SL}(2, \mathbb{Z})$.

We shall now state the converse theorem proved by Imai in [5] in a formulation that does not involve spectral properties of automorphic forms and which can be looked upon as a weaker form of Imai's original statement.

Theorem 3.2 ([5]). *Let F be defined by a unimodular convergent Fourier series as in (3.1) where the Fourier coefficients are of at most polynomial growth and $A(T) = 0$, if $\det T = 0$. If for each $w \in \mathbb{H}$ the partial Mellin transform $\tilde{F}(w, s)$ fulfills the following three conditions:*

- (i) *it is entire as a function of s*
- (ii) *it tends to zero as $\Im s \rightarrow \pm\infty$ uniformly in every vertical strip*
- (iii) *it satisfies the functional equation*

$$\tilde{F}(w, s) = (-1)^k \tilde{F}(w, k - s) \quad (3.6)$$

then $F(Z)$ is a Siegel cusp form of degree 2 and weight k .

Proof. Let W be the preimage of w under ϕ and $t > 0$. We use the Mellin inversion formula, which is valid for $\Re s > c$, for some suitable $c \in \mathbb{R}$ and obtain

$$\begin{aligned} F(it^{\frac{1}{2}}W) &= \frac{1}{2\pi i} \int_{\Re s=c} \tilde{F}(w, s) t^{-s} ds \\ &= \frac{(-1)^k}{2\pi i} \int_{\Re s=c} \tilde{F}(w, k-s) t^{-s} ds \\ &= \frac{(-1)^k}{2\pi i} \int_{\Re s=k-c} \tilde{F}(w, s) t^{-k+s} ds \\ &= \frac{(-1)^k t^{-k}}{2\pi i} \int_{\Re s=k-c} \tilde{F}(w, s) \left(\frac{1}{t}\right)^{-s} ds \\ &= (-1)^k t^{-k} F(it^{-\frac{1}{2}}W). \end{aligned}$$

For the last equality we move the path of integration back to $\Re s = c$, which is possible because $\tilde{F}(w, s)$ is entire and tends to zero as $\Im s \rightarrow \pm\infty$ uniformly in every vertical strip. One should notice that

$$W^{-1} = \left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right] W.$$

Since $\tilde{F}(w, s)$ is a modular function a similar computation as above gives

$$F(it^{-\frac{1}{2}}W^{-1}) = (-1)^k t^k F(it^{\frac{1}{2}}W)$$

or equivalently for $Y > 0, \det Y > 0$

$$F(iY^{-1}) = (-1)^k (\det Y)^k F(iY).$$

The statement of the theorem now follows from the fact, that the symplectic group is known to be generated by the matrices

$$\begin{pmatrix} I_2 & X \\ 0 & I_2 \end{pmatrix}, \begin{pmatrix} U & 0 \\ 0 & U^{-1} \end{pmatrix}, \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix},$$

with X an integral matrix, I_2 the identity and $U \in \mathrm{GL}(2, \mathbb{Z})$. $\square$

As mentioned above this is not the original formulation by Imai, but the arguments of the proof can all be found in Imai’s paper [5, p. 930f].

4. Proof of the Saito–Kurokawa lift

The aim of this section is to prove that the Saito–Kurokawa lift satisfies the conditions of Imai’s converse theorem, esp. the functional equation (3.6) and thereby prove the theorem. For this we shall represent the partial Mellin transform as a Rankin–Selberg integral involving Siegel’s theta series.

4.1. Siegel’s Θ -series

We consider the lattice $\Lambda = \mathbb{Z}^3$ equipped with the symmetric bilinear form

$$2\Omega = \begin{pmatrix} 0 & 0 & 4 \\ 0 & -2 & 0 \\ 4 & 0 & 0 \end{pmatrix}.$$

Then $\Omega(\lambda) = \lambda^t \Omega \lambda$ is the corresponding quadratic form of signature $(1, 2)$. Let D be such that

$$\Omega = D^t \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} D.$$

Then $D^t D$ is called a majorant for Ω . Of course there is no unique choice for D . It is at once seen, that if $S \in \mathrm{SO}(1, 2)$ then D can be replaced by SD and all majorants in the above sense are of the form $D^t S^t S D$ for some fixed special majorant D . The group $\mathrm{SO}(\Omega)$ acts transitively on the space of majorants in an obvious way, providing it thus with the structure of a symmetric space.

For our purposes it will be convenient to fix

$$D = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

Let

$$\hat{\Lambda} = \{\mu \in \mathbb{Q}^3 : 2\Omega\mu \in \Lambda\} = \{\mu = (\mu_a, \mu_b, \mu_c) \in \mathbb{Q}^3 : 4\mu_a, 4\mu_c, 2\mu_b \in \mathbb{Z}\}$$

be the lattice dual to Λ and $\mathcal{D} = \hat{\Lambda}/\Lambda$ the discriminant group.

For $\kappa \in \mathcal{D}$, $z = x + iy \in \mathbb{H}$ and any majorant Ω^+ Siegel's theta series is defined by

$$\Theta(\kappa, z, \Omega^+) := y^{\frac{3}{4}} \sum_{\mu \equiv \kappa \pmod{\Lambda}} e(x\Omega(\mu) + iy\Omega^+(\mu)).$$

Introduce a unitary vector space of dimension $|\mathcal{D}|$ with a unitary basis $v_{\mathbf{r}}$ indexed by $\mathbf{r}$. Subject to this basis we build a vector-valued theta series by

$$\Theta(z, \Omega^+) = \sum_{\mathbf{r}} \Theta(\mathbf{r}, z, \Omega^+) v_{\mathbf{r}}.$$

Siegel's theta series can be constructed using the Weil representation associated to the lattice Λ as was done by Shintani in [13]. (See also [6].) From this it becomes evident, see [13, Proposition 1.6], that for $\gamma \in \mathrm{SL}(2, \mathbb{Z})$ there is a transformation formula

$$\Theta(\gamma z, \Omega^+) = j_{-\frac{1}{2}}(\gamma, z) \chi(\gamma) \Theta(z, \Omega^+)$$

with the automorphy factor $j_{-\frac{1}{2}}(\gamma, z) = \left(\frac{cz+d}{c\bar{z}+d}\right)^{-1/4}$ and a unitary matrix

$$\chi(\gamma) = (\chi_{\mathbf{r}, \mathbf{s}}(\gamma))_{\mathbf{r}, \mathbf{s} \in \mathcal{D}}.$$

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ one has

$$\chi_{\mathbf{r}, \mathbf{s}}(\gamma) = \begin{cases} e\left(\frac{1}{8}\right) |\det(2c\Omega)|^{-\frac{1}{2}} & \\ \sum_{g \in \Lambda/c\Lambda} e\left(\frac{a\Omega[g + \mathbf{r}] - 2s^t\Omega(g + \mathbf{r}) + d\Omega[\mathbf{s}]}{c}\right), & c \neq 0, \\ e(ab\Omega[\mathbf{r}]) & c = 0, \text{ } a\mathbf{r} = \mathbf{s}, \\ 0 & c = 0, \text{ } a\mathbf{r} \neq \mathbf{s} \end{cases}$$

as was originally shown by Siegel in [12, Hilfssatz 1].

4.2. Space of majorants and hyperbolic space

The group $\mathrm{SL}(2, \mathbb{R})$ acts on the set of symmetric real matrices with given determinant via $[\sigma]T = \sigma T \sigma^t$ and under the identification

$$T = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \rightarrow \lambda_T = (a, b, c) \quad (4.1)$$

we have

$$4 \det T = \Omega(\lambda_T).$$

This gives rise to an isomorphism of groups

$$\begin{aligned}\mathrm{SL}(2, \mathbb{R}) / \{\pm I\} &\rightarrow \mathrm{SO}^+(\Omega) = D^{-1} \mathrm{SO}^+(1, 2) D \\ \sigma &\rightarrow D^{-1} S_\sigma D\end{aligned}$$

satisfying

$$D^{-1} S_\sigma D \lambda_T = \lambda_{[\sigma]T}. \quad (4.2)$$

Here, the “plus” shall denote the identity component of the orthogonal group.

Now to each $w = u + iv \in \mathbb{H}$ we associate as in Lemma 3.1 the matrix $\omega_w = \begin{pmatrix} \sqrt{v} & u/\sqrt{v} \\ 0 & 1/\sqrt{v} \end{pmatrix}$ which in turn corresponds to an element $D^{-1} S_w D \in \mathrm{SO}^+(\Omega)$, where we write for short S_w instead of S_{ω_w} .

At least we map w to a majorant

$$w \rightarrow \Omega_w^+ = D^t S_{-\frac{1}{w}}^{-t} S_{-\frac{1}{w}}^{-1} D \quad (4.3)$$

(where we write A^{-t} for $(A^{-1})^t$) and define

$$\Theta(\kappa, z, w) := \Theta(\kappa, z, \Omega_w^+).$$

For our purposes we rewrite the defining sum for the theta series (at least for $\kappa = 0$) in terms of half integral symmetric matrices.

For this recall, that the map in Eq.(4.1) gives a bijective correspondence between Λ and the set L of half integral symmetric matrices in such a way that $\Omega(\lambda_T) = 4 \det T$.

We show the following easy lemma.

Lemma 4.1. *Let $w = u + iv \in \mathbb{H}$, $\omega_w = \begin{pmatrix} \sqrt{v} & u/\sqrt{v} \\ 0 & 1/\sqrt{v} \end{pmatrix}$ and $W = \omega_w \omega_w^t$, so that $w = \phi(W)$ as in lemma 3.1. Then for any real symmetric matrix T we obtain*

$$\Omega_w^+(\lambda_T) = 2 \mathrm{tr}^2((TW)) - 4 \det(T).$$

Proof. Define $\tau(x_1, x_2, x_3) := \begin{pmatrix} x_1 & x_2/2 \\ x_2/2 & x_3 \end{pmatrix}$. So τ is the inverse map for $T \rightarrow \lambda_T$ and the compatibility condition (4.2) implies

$$D^{-1} S_{-\frac{1}{w}}^{-1} D = \tau^{-1} [\omega_{-\frac{1}{w}}^{-1}] \tau. \quad (4.4)$$

Further $\Omega_w^+(\lambda) = \lambda^t D^t S_{-\frac{1}{w}}^{-t} S_{-\frac{1}{w}}^{-1} D \lambda = \|S_{-\frac{1}{w}}^{-1} D \lambda\|^2$ (with the usual Euclidean vector norm $\|\cdot\|$) which by Eq. (4.4) yields

$$\Omega_w^+(\lambda_T) = \|D \tau^{-1} \left[\omega_{-\frac{1}{w}}^{-1} \right] \tau(\lambda_T)\|^2 = \|D \tau^{-1} \left[\omega_{-\frac{1}{w}}^{-1} \right] T\|^2.$$

A straightforward computation shows that for any matrix A

$$\|D \tau^{-1} A\|^2 = 2 \mathrm{tr}(A^t A).$$

For symmetric A we further obtain

$$\mathrm{tr}(A^t A) = \mathrm{tr}(A^2) = \mathrm{tr}^2(A) - 2 \det(A).$$

Another direct calculation gives

$$\mathrm{tr}([\omega_{-\frac{1}{w}}^{-1}]T) = \mathrm{tr}(TW)$$

and the proof is complete since $\det W = 1$. $\square$

As a consequence of this lemma we find

$$\Theta(0, z, w) = y^{\frac{3}{4}} \sum_{T \in L} e(4x \det T) e^{-2\pi y[T, w]}, \quad (4.5)$$

where we write for short $[T, w]$ for $2\mathrm{tr}^2(TW) - 4\det(TW)$.

Furthermore for $\gamma \in \mathrm{SL}(2, \mathbb{R})$ we obtain by help of Lemma 3.1

$$[[\gamma^t]T, w] = [T, \gamma w]$$

and since $\mathrm{SL}(2, \mathbb{Z})$ acts on $L_n = \{T \in L : 4\det T = n\}$ this implies

$$\Theta(0, z, \gamma w) = \Theta(0, z, w). \quad (4.6)$$

4.3. Kohnen's plus space and the theta multiplier system

4.3.1. Vector valued new forms

New forms from the Kohnen space can also be looked upon as vector valued forms for $\mathrm{SL}(2, \mathbb{Z})$ as was done by Eichler and Zagier in [3, Theorem 5.4]. See also the paper of Borcherds [1, Example 2.4]

Namely, let $k \in 2\mathbb{Z}$ and $f = \sum_{n \equiv 0, 3 \pmod{4}} a(n)e(nz) \in S_{k-\frac{1}{2}}^+$. Then define

$$f_i(z) = \sum_{n \equiv i \pmod{2}} a(n)e(nz/4), \quad i = 0, 1$$

and the vector valued function

$$\mathbf{f}(z) = y^{\frac{k}{2}-\frac{1}{4}} \begin{pmatrix} f_0(z) \\ f_1(z) \end{pmatrix}.$$

Under the action of the modular group this function transforms according to

$$\mathbf{f}(\gamma z) = e\left(\frac{1}{8}\right) j_{k-\frac{1}{2}}(\gamma, z) \mathcal{U}(\gamma) \mathbf{f}(z), \quad \gamma \in \mathrm{SL}(2, \mathbb{Z})$$

with a unitary multiplier system $\mathcal{U}$ of weight $k - \frac{1}{2}$ on $\mathrm{SL}(2, \mathbb{Z})$. For further details see also [7, 8].

The multiplier $\mathcal{U}$ is determined by its values on the generators $V = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ of $\mathrm{SL}(2, \mathbb{Z})$

$$\mathcal{U}(V) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and

$$\mathcal{U}(T) = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}.$$

4.3.2. Subrepresentation of the theta representation

We can realize $\mathcal{U}$ as a subrepresentation of χ associated to the (standard-)lattice signature $(1, 2)$, level 4 and discriminant group $\mathbb{Z}/2\mathbb{Z}$.

For this define

$$\begin{aligned}\Theta_0(z, w) &:= \sum_{r_a, r_c \in \{0, \frac{1}{2}\}} \Theta((r_a, 0, r_c), z, w), \\ \Theta_1(z, w) &:= \sum_{r_a, r_c \in \{0, \frac{1}{2}\}} \Theta\left(\left(r_a, \frac{1}{2}, r_c\right), z, w\right)\end{aligned}$$

and the vector-valued function

$$\Theta_{\text{red}}(z, w) = \begin{pmatrix} \Theta_0(z, w) \\ \Theta_1(z, w) \end{pmatrix}.$$

Rewrite the defining equation for $\Theta_j(z, w)$

$$\begin{aligned}\Theta_j(z, w) &= \sum_{r_a, r_c \in \{0, \frac{1}{2}\}} \Theta((r_a, j/2, r_c), z, w) \\ &= y^{\frac{3}{4}} \sum_{a \in \mathbb{Z}} \sum_{c \in \mathbb{Z}} \sum_{b \equiv j(2)} e\left(x\Omega\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right) + iy\Omega_w^+\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)\right).\end{aligned}$$

Thus it is evident, that $\Theta_j(z, w)$ are related to the quadratic form $q(a, b, c) = ac - b^2$, which has signature $(1, 2)$, level 4 and discriminant group $\mathbb{Z}/2\mathbb{Z}$.

Furthermore observe, that $\Omega(\lambda) \equiv 0, 3 \pmod{4}$ for $\lambda \in \Lambda = \mathbb{Z}^3$. So, as a function in z , $\Theta(0, z, w)$ satisfies the Kohnen plus-condition for even k and from Eq. (4.5) we obtain a Fourier expansion of the form

$$\Theta(0, z, w) = y^{\frac{3}{4}} \sum_{n \equiv 0, 3 \pmod{4}} e(nx) \theta_n(y, w) \quad (4.7)$$

with

$$\theta_n(y, w) = \sum_{T \in L_n} e^{-2\pi y[T, w]},$$

where $L_n = \{T \in L : 4 \det L = n\}$.

It is apparent that

$$\begin{aligned}\Theta_j(z, w) &= y^{\frac{3}{4}} \sum_{\lambda \in \mathbb{Z}^3, \Omega(\lambda) \equiv j(2)} e(x\Omega(\lambda/2) + iy\Omega^+(\lambda/2)) \\ &= y^{\frac{3}{4}} \sum_{\lambda \in \mathbb{Z}^3, \Omega(\lambda) \equiv j(2)} e\left(\frac{x}{4}\Omega(\lambda) + i\frac{y}{4}\Omega^+(\lambda)\right).\end{aligned}$$

So we see that up to the factor $2\sqrt{2}$ the vector valued theta series $\Theta_{\text{red}}(z, w)$ is constructed in the same way from $\Theta(0, z, w)$ as the vector valued forms of the Kohnen space. Indeed, $\Theta_{\text{red}}(z, w)$ also behaves similarly under the action of the

modular group which would follow from the transformation formulas given by Shimura in [13] applied to the theta series associated to the quadratic form q mentioned above.

Alternatively one may check by a direct calculation, using the above formula for χ that

$$\Theta_{\text{red}}\left(\frac{-1}{z}, w\right) = e\left(\frac{1}{8}\right) j_{-\frac{1}{2}}(V, z) \mathcal{U}(V) \Theta_{\text{red}}(z, w)$$

and

$$\Theta_{\text{red}}(z + 1, w) = \mathcal{U}(T) \Theta_{\text{red}}(z, w).$$

4.4. Rankin–Selberg integral

We are now in a position to write $\tilde{F}(w, s)$ as a Rankin–Selberg integral. For this we use the real analytic Eisenstein series for $\text{SL}(2, \mathbb{Z})$ of weight $l \in \mathbb{Z}$ which is given by

$$E_l(z, s) = \sum_{T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{\infty} \backslash \text{SL}(2, \mathbb{Z})} y_T^s e^{-il \arg(cz+d)}$$

As a function of s the Eisenstein series $E_l(z, s)$ has an analytic continuation to the whole complex plane. If $l \neq 0$ then $\zeta(2s)E_l(z, s)$ is an entire function in s and it satisfies the functional equation

$$E_l(z, s) = \phi_l(s) E_l(z, 1 - s) \quad (4.8)$$

with

$$\phi_l(s) = \frac{i^{-l} 2^{2-2s} \pi \Gamma(2s-1) \zeta(2s-1)}{\zeta(2s) \Gamma(s-\frac{l}{2}) \Gamma(s+\frac{l}{2})}.$$

This is well known. For a proof see e.g., [10, Lemma 3.2(ii)].

Now put for $w \in \mathbb{H}$, $f \in S_{k-\frac{1}{2}}^+$ and $\mathbf{f}$ as introduced above

$$\mathcal{S}_f(w, s) := \int_{\Gamma_0(1) \backslash \mathbb{H}} \langle \mathbf{f}(z), \Theta_{\text{red}}(z, w) \rangle E_{-k}(z, s) \frac{dx dy}{y^2}, \quad (4.9)$$

with $\langle \mathbf{g}, \mathbf{h} \rangle = \sum_{i=1}^2 g_i \overline{h_i}$. The integral exists for any s which is not a pole of the Eisenstein series, since f is a cusp form. (See [11] for growth estimates for the theta series.) From the above proven transformation properties it is readily seen, that the integrand is $\text{SL}(2, \mathbb{Z})$ -invariant.

4.5. Completing the proof

We are going to prove our main result:

Theorem 4.2. *Let F be as in Theorem 2.1, and $W > 0$ with $\det W = 1$, $w = \phi(W)$. Then F is a unimodular convergent Fourier series as in (3.1) where the*

Fourier coefficients are of at most polynomial growth and $A(T) = 0$, if $\det T = 0$. Furthermore the partial Mellin transform $\tilde{F}(w, s)$ has the following properties:

- (i) For all $s \in \mathbb{C}$ with $\Re s$ sufficiently large, we have as an identity of meromorphic functions

$$\tilde{F}(w, s) = \frac{1}{2} \pi^{-\frac{1}{2}-s} \Gamma\left(s + \frac{1}{2}\right) \zeta(2s - k + 1) \mathcal{S}_f\left(w, s - \frac{k-1}{2}\right).$$

- (ii) The Mellin transform $\tilde{F}(w, s)$ extends to an entire function in s and it satisfies for all $W > 0$ the functional equation

$$\tilde{F}(w, k - s) = \tilde{F}(w, s).$$

- (iii) $\tilde{F}(w, s)$ tends to zero as $\Im s \rightarrow \pm\infty$ uniformly in every vertical strip.

Proof. By our assumption F is a function on $\mathbf{H}$ given by a convergent series

$$F(Z) = \sum_{0 < T \in L} A(T) e(\mathrm{tr} TZ),$$

where $A(T) = \sum_{d|e(T)} d^{k-1} a((4 \det T)/d^2)$ with $e(T) = \max\{q \in \mathbb{N} : T/q \in L\}$.

So trivially condition (2.1) is fulfilled. The estimate $A(T) \ll \det^a T$ follows from well known estimates for the Fourier coefficients of modular functions $a(n) \ll n^r$ for some suitable $r > 0$. Then we know from [5, p. 910f], that the partial Mellin transform exists for $\Re s$ sufficiently large.

(i) We begin with the evaluation of the Rankin–Selberg integral. For this we have the Fourier expansion (see Eq. (4.7))

$$\Theta_j(z, w) = y^{\frac{3}{4}} \sum_{n \equiv j \pmod{2}} e\left(n \frac{x}{4}\right) \theta_n\left(\frac{y}{4}, w\right)$$

with

$$\theta_n\left(\frac{y}{4}, w\right) = 2 \sum_{0 < T \in L_n} e^{-\frac{\pi}{2} y [T, w]}$$

if $n > 0$.

Now the inner product $\langle \mathbf{f}(z), \Theta_{\mathrm{red}}(z, w) \rangle$ takes the form

$$\begin{aligned} & \langle \mathbf{f}(z), \Theta_{\mathrm{red}}(z, w) \rangle \\ &= y^{\frac{k}{2}-\frac{1}{4}} \left(f_0(z) \overline{\Theta_0(z, w)} + f_1(z) \overline{\Theta_1(z, w)} \right) \\ &= y^{\frac{k}{2}+\frac{1}{2}} \sum_{i=0,1} \sum_{n \equiv i \pmod{2}} \sum_{m \equiv i \pmod{2}} a(n) e^{-\frac{\pi}{2} ny} \theta_m\left(\frac{y}{4}, w\right) e\left((n-m)\frac{x}{4}\right). \end{aligned}$$

Unfolding the fundamental domain gives

$$\begin{aligned}\mathcal{S}_f\left(w, s - \frac{k-1}{2}\right) &= \int_0^\infty \int_0^1 \langle \mathbf{f}(z), \Theta_{\text{red}}(z, w) \rangle y^{s-(k-1)/2-2} dx dy \\ &= 2 \int_0^\infty \sum_{n>0} a(n) \sum_{0<T \in L_n} e^{-\frac{\pi}{4} y \text{tr}^2(TW)} y^{s-1} dy \\ &= 2^{2s+1} \pi^{-s} \Gamma(s) \sum_{n>0} a(n) \sum_{0<T \in L_n} \text{tr}^{-2s}(TW).\end{aligned}$$

We write $T = \sqrt{\det TS}$ and arrive at

$$\mathcal{S}_f\left(w, s - \frac{k-1}{2}\right) = 2^{2s+1} \pi^{-s} \Gamma(s) \sum_{n>0} \frac{a(n)}{n^s} \sum_{0<\sqrt{n}S \in L_n} \text{tr}^{-2s}(SW). \quad (4.10)$$

We compare this with the Dirichlet series we obtain when calculating the Mellin transform of F .

$$\begin{aligned}\int_0^\infty F(iu^{\frac{1}{2}}W) u^{s-1} du &= \sum_{0<T \in L} A(T) \int_0^\infty e^{-2\pi \text{tr}(\sqrt{u}TW)} u^{s-1} du \\ &= \sum_{0<T \in L} \frac{A(T)}{(2\pi \text{tr}(TW))^{2s}} \int_0^\infty e^{-v} v^{2s-1} dv \\ &= 2\Gamma(2s)(2\pi)^{-2s} \sum_{n>0} \frac{1}{n^s} \sum_{0<\sqrt{n}S \in L_n} \frac{A(\sqrt{n}S)}{\text{tr}^{2s}(SW)} \\ &= 2\Gamma(2s)(2\pi)^{-2s} \sum_{n>0} \frac{\sum_{d|n} d^{k-1} a(4n/d^2)}{n^s} \\ &\quad \times \sum_{0<\sqrt{n}S/d \in L_{n/d^2}} \text{tr}^{-2s}(SW).\end{aligned}$$

For the last equality observe that $d|e(T)$ and $T \in L_n$ if and only if $T/d \in L_{n/d^2}$. If we use the identity for Dirichlet series:

$$\zeta(2s - k + 1) \sum_{n>0} \frac{c(n)}{n^s} = \sum_{n>0} \frac{\sum_{d^2|n} d^{k-1} c(n/d^2)}{n^s}$$

and the well-known identity for the Gamma-function

$$\frac{\Gamma(2s)}{\Gamma(s)} = \frac{2^{2s-1}}{\sqrt{\pi}} \Gamma\left(s + \frac{1}{2}\right), \quad (4.11)$$

the statement of (i) follows from (4.10) for all s with $\Re s$ sufficiently large.

(ii) Since f is a cusp form the Rankin Selberg integral gives a meromorphic continuation for $\tilde{F}(w, s)$ as function of s to the whole complex plane. (See [11] for growth estimates for the theta series.)

It is entire since $\zeta(2s - k + 1)E_{-k}(z, s - k/2 + 1/2)$ is an entire function for even positive weight and the poles of $\Gamma(s + 1/2)$ at $s = -1/2 - n$ for $n \in \mathbb{N}$ are

annihilated by the zeroes of the zeta function. Here it is important to notice, that k is even.

For our purposes the most important fact is, that it inherits the functional equation of the Eisenstein series. Besides the functional equation for the Eisenstein series we use the well-known functional equation for the ζ -function

$$\zeta(s) = \pi^{s-\frac{1}{2}} \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \zeta(1-s) \quad (4.12)$$

and besides Eq. (4.11) we recall the familiar identity for the Gamma-function

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}. \quad (4.13)$$

We start with the functional equation for the Eisenstein series

$$E_{-k}\left(z, s - \frac{k-1}{2}\right) = \frac{i^k 2^{k+1-2s} \pi \Gamma(2s-k) \zeta(2s-k)}{\zeta(2s-k+1) \Gamma\left(s + \frac{1}{2}\right) \Gamma\left(s - k + \frac{1}{2}\right)} E_{-k}\left(z, \frac{1}{2} + \frac{k}{2} - s\right)$$

to obtain

$$\tilde{F}(w, s) = \frac{i^k 2^{k-2s} \pi^{\frac{1}{2}-s} \Gamma(2s-k) \zeta(2s-k)}{\Gamma\left(s - k + \frac{1}{2}\right)} \mathcal{S}_f(w, -s + (k+1)/2).$$

Now we apply Eq. (4.12), which gives

$$\begin{aligned} \tilde{F}(w, s) &= \frac{i^k 2^{k-2s} \pi^{s-k} \Gamma(2s-k) \Gamma\left(\frac{k+1}{2} - s\right) \zeta(1+k-2s)}{\Gamma\left(s - k + \frac{1}{2}\right) \Gamma\left(s - \frac{k}{2}\right)} \\ &\quad \times \mathcal{S}_f(w, -s + (k+1)/2). \end{aligned}$$

Next apply Eq. (4.11)

$$\begin{aligned} \tilde{F}(w, s) &= \frac{i^k 2^{-1} \pi^{-\frac{1}{2}+s-k} \Gamma\left(\frac{1}{2} - \frac{k}{2} + s\right) \Gamma\left(\frac{1}{2} + \frac{k}{2} - s\right) \zeta(1+k-2s)}{\Gamma\left(s - k + \frac{1}{2}\right)} \\ &\quad \times \mathcal{S}_f(w, -s + (k+1)/2). \end{aligned}$$

Now from Eq. (4.13) follows

$$\Gamma\left(\frac{1}{2} - \frac{k}{2} + s\right) \Gamma\left(\frac{1}{2} + \frac{k}{2} - s\right) = \frac{\pi}{\sin\left(\pi s + (1-k)\frac{\pi}{2}\right)}$$

and

$$\frac{1}{\Gamma\left(\frac{1}{2} + s - k\right)} = \frac{1}{\pi} \Gamma\left(\frac{1}{2} - s + k\right) \sin\left(\pi s + (1 - 2k)\frac{\pi}{2}\right).$$

Since k is an even integer

$$\frac{\sin\left(\pi s + (1 - 2k)\frac{\pi}{2}\right)}{\sin\left(\pi s + (1 - k)\frac{\pi}{2}\right)} = (-1)^{\frac{k}{2}}$$

and we obtain

$$\tilde{F}(w, s) = 2^{-1} \pi^{-\frac{1}{2} + s - k} \Gamma\left(\frac{1}{2} + k - s\right) \zeta(1 + k - 2s) \mathcal{S}_f(w, -s + (k + 1)/2),$$

which by part (i) of the theorem is the same as $\tilde{F}(w, k - s)$.

This proves (ii).

(iii) By Stirling's formula follows, that $s\tilde{F}(w, s)$ stays bounded for $\Re s = c$ for some suitable c . Because of the functional equation in (ii) the same is true for $(k - s)\tilde{F}(w, k - s)$. This means that $s\tilde{F}(w, s)$ stays bounded for $\Re s = c$ and also for $\Re s = k - c$. By the Phragmen–Lindelöf theorem it follows that we have boundedness of $s\tilde{F}(w, s)$ in the whole strip $k - c < \Re s < c$ and therefore (iii) follows. $\square$

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