ERRATUM

PRIME DIVISORS ARE POISSON DISTRIBUTED

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Kevin Ford has pointed out that the proof of Theorem 3 of [1] contains several significant mistakes (perhaps I did not check it over carefully enough because it is the most easily proved result in the paper!). We give here a correct proof of a slightly modified version of that theorem.

Almost all integers up to x have $\sim \log \log x$ distinct prime factors. In our article [1], we examined the distribution of the sizes of those prime factors, and Theorem 3 looked at the integers with far fewer prime factors than is typical. If $S_k(x)$ denotes the set of integers $n \leq x$ which have exactly k distinct prime factors then it is well known that

$$|S_k(x)| = \frac{x}{\log x} \frac{(\log \log x)^{k-1}}{(k-1)!} \left\{ 1 + O\left(\frac{k}{\log \log x}\right) \right\} \quad (1)$$

for $k = o(\log \log x)$. We will denote by $p_1(n) < p_2(n) < \cdots < p_k(n)$ the distinct prime factors of $n \in S_k(x)$. Theorem 3 of [1] shows that the numbers

$$\{\log \log p_i(n) / \log \log n : 1 \leq i \leq k-1\}$$

are distributed on $(0, 1)$ like $k - 1$ random numbers, as we vary over $n \in S_k(x)$, for $k = o(\log \log x)$. More precisely we prove the following:

Theorem 3 ([1]). *Suppose that $2 \leq k = o(\log \log x)$. Let $\epsilon \in [1/\log \log x, 1/k)$ and $\alpha_0 = 0 < \alpha_1 < \alpha_2 < \dots < \alpha_{k-1} \leq \alpha_k = 1$ where $\alpha_{j+1} - \alpha_j \geq \epsilon$ for each j and $\alpha_k - \alpha_{k-1} \geq \epsilon + \log \log \log x / \log \log x$. Then there are*

$$(k-1)! \epsilon^{k-1} |S_k(x)| \left\{ 1 + O\left(\frac{1}{\epsilon \log \log x}\right) \right\}$$

integers $n \in S_k(x)$ for which $\log \log p_i(n) / \log \log x \in [\alpha_i, \alpha_i + \epsilon)$ for every i in the range $1 \leq i \leq k - 1$.

Proof. The number of $n \in S_k(x)$ for which $\log \log p_i(n) / \log \log x \in [\alpha_i, \alpha_i + \epsilon)$ for each $1 \leq i \leq k - 1$, equals

$$\sum_{p_1 \in I_1, p_2 \in I_2, \dots, p_{k-1} \in I_{k-1}} \sum_{a_1, a_2, \dots, a_{k-1} \geq 1} \pi^* \left(p_{k-1}; \frac{x}{p_1^{a_1} \dots p_{k-1}^{a_{k-1}}} \right) \quad (2)$$

where $I_j = (\exp((\log x)^{\alpha_j}), \exp((\log x)^{\alpha_j + \epsilon}))$, and $\pi^*(b; T)$ counts the number of prime powers $p^a \leq T$ with prime $p > b$. By the prime number theorem we have $\pi^*(b; T) = T / \log T (1 + O(1/\log T))$ if $b \leq T / \log T$, and we always have $\pi^*(b; T) \leq T$.

Since $(\log x)^{\alpha_{j+1}} \geq (\log x)^{\alpha_j + \epsilon} \geq e(\log x)^{\alpha_j}$, we deduce that

$$\log(p_1 \dots p_{k-1}) < \sum_{j=1}^{k-1} (\log x)^{\alpha_j + \epsilon} \leq (1 - 1/e)^{-1} (\log x)^{\alpha_{k-1} + \epsilon} < 2 \frac{\log x}{\log \log x}.$$

This implies that in each term of the outer sum in (2) we have $x/(p_1 \dots p_{k-1}) > x^{1-o(1)}$ and $p_{k-1} \leq x^{o(1)}$. In order to estimate the inner sum in (2) we treat the terms differently depending on whether $p_1^{a_1} \dots p_{k-1}^{a_{k-1}}$ is smaller or larger than x^δ , where $\delta = 100/\log \log x$. The terms with $p_1^{a_1} \dots p_{k-1}^{a_{k-1}} \leq x^\delta$ each contribute

$$\frac{x}{\log x} \frac{1}{p_1^{a_1} \dots p_{k-1}^{a_{k-1}}} \left\{ 1 + O\left(\frac{1}{\log \log x}\right) \right\}. \quad (3)$$

The terms with $p_1^{a_1} \dots p_{k-1}^{a_{k-1}} > x^\delta$ each contribute $\leq x^{1-\delta}$ and there are $\ll (\log x)^k/k!$ such terms (since we must have $p_1^{a_1} \dots p_{k-1}^{a_{k-1}} \leq x$). Thus their total contribution is $\ll x^{1-\delta/2}$. Therefore, summing up (3) over all values of the a_i we find that the inner sum in (2) equals

$$\frac{x}{\log x} \frac{1}{(p_1 - 1) \dots (p_{k-1} - 1)} \left\{ 1 + O\left(\frac{1}{\log \log x}\right) \right\}. \quad (4)$$

Therefore, the sum in (2) becomes, since the intervals I_j are disjoint,

$$\left\{ 1 + O\left(\frac{1}{\log \log x}\right) \right\} \frac{x}{\log x} \prod_{j=1}^{k-1} \sum_{p_j \in I_j} \frac{1}{p_j - 1}. \quad (5)$$

Now, $\sum_{a < p < b} 1/(p-1) = \log \log b - \log \log a + O(1/\log a)$, so that

$$\sum_{p \in I_j} \frac{1}{p-1} = \epsilon \log \log x + O\left(\frac{1}{(\log x)^{\alpha_j}}\right) = \epsilon \log \log x + O\left(\frac{1}{e^j}\right),$$

and hence (5) equals

$$\frac{x}{\log x} (\epsilon \log \log x)^{k-1} \left\{ 1 + O\left(\frac{1}{\epsilon \log \log x}\right) \right\}. \quad (6)$$

The result follows by comparing this to (1). $\square$

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Reference

- [1] A. Granville, Prime divisors are Poisson distributed, *Int. J. Number Theory* **3** (2007) 1–18.