Multi-Fidelity Modeling-Based Structural Reliability Analysis with the Boundary Element Method

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In this work, a method for the application of multi-fidelity modeling to the reliability analysis of 2D elastostatic structures using the boundary element method (BEM) is proposed. Reliability analyses were carried out on a rectangular plate with a center circular hole subjected to uniaxial tension using Monte Carlo simulations (MCS), the first-order reliability method (FORM), and the second-order reliability method (SORM). Two BEM models were investigated, a low-fidelity model (LFM) of 20 elements and a high-fidelity model (HFM) of 100 elements. The response of these models at several design points was used to create multi-fidelity models (MFMs) utilizing second-order polynomial response surfaces and their reliability, alongside that of the LFM and the HFM, was evaluated. Results show that the MFMs that directly called the LFM were significantly superior in terms of accuracy to the LFM, achieving very similar levels of accuracy to the HFM, while also being of similar computational cost to the LFM. These direct MFMs were found to provide good substitutes for the HFM for MCS, FORM, and SORM.

Keywords: Multi-fidelity modeling; boundary element method (BEM); reliability analysis; Monte Carlo simulation (MCS); first-order reliability method (FORM); second-order reliability method (SORM).

1. Introduction

All engineering parameters have uncertainties in their values, it is therefore important that these are considered during the design process. Failure to consider uncertainties can lead to the design failing or becoming overengineered, requiring large safety factors. As stated by Huang *et al.*,¹ adequate performance cannot be

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guaranteed when there is uncertainty. Therefore, a probability of success that the design meets certain criteria needs to be defined, this is referred to as *reliability*. There are several methods that can be used to evaluate reliability, such as Monte Carlo simulations (MCS), the first-order reliability method (FORM), and the second-order reliability method (SORM). Details of these methods can be seen in Refs. 2 and 3. In this work the reliability of a 2D elastostatic boundary element method (BEM) model is evaluated for various levels of uncertainty in design parameters such as the applied stress and the dimensions of the model.

The BEM is a numerical computational method for structural analysis that has found increasing use in the engineering industry over the past few decades since its introduction in the late 1960s. It has proven itself to be an effective tool for structural reliability analysis, with previous applications to 2D elastostatic structures,^{1,4} fatigue crack growth,^{5,6} and plate bending.⁷ It has developed as an effective alternative to the finite element method (FEM) and has several features that make it an attractive choice. One feature is that the formulations used for the BEM result in a significantly smaller number of equations being obtained than with the FEM, the BEM is also able to obtain a similar level of accuracy to the FEM while using a much coarser mesh.⁸ These advantages allow the BEM to be especially useful for structural reliability analysis. A comprehensive overview of the application of the BEM to structural analysis can be found in Ref. 8.

There have been several publications relating to structural reliability analysis with the BEM. One notable example is Huang and Aliabadi¹ where MCS and FORM were used to assess the reliability of a 2D elastostatics problem involving a rectangular plate with a center circular hole subjected to uniaxial tension. The implicit differentiation method (IDM) was employed to calculate the first-order derivatives for FORM and reliability was evaluated for various levels of uncertainty in material strength and center hole radius. Results showed that MCS and FORM obtained similar results. Huang and Aliabadi⁶ also addressed a more complex problem involving the reliability analysis of 2D cracked structures using the dual boundary element method (DBEM).⁸ The sensitivity of the stress intensity factors with respect to several random variables, including crack length and fracture toughness, was calculated using the IDM and compared to those obtained from the finite difference method (FDM) with both the DBEM and the FEM and also with MCS. Results showed good agreement between these methods. Lionel and Venturini⁵ built upon this work and assessed the reliability of a problem involving the modeling of crack growth in a 2D perforated panel using MCS and FORM. The failure condition of the structure was based on the number of cycles required for the structure to fail calculated from fatigue crack growth analyses performed using the DBEM. Reliability analyses were carried out using two methods. The first, the direct method, involved the direct coupling of FORM with the DBEM model and approximating the derivatives for FORM using the FDM. The second, the indirect method, involved approximating the response of the DBEM model using a polynomial response

surface. Results showed that the indirect coupling proved to be less accurate and more computationally expensive than the direct method.

Several recent publications^{4,7} have involved the use of a stochastic spline fictitious boundary element method (SFBEM). The SFBEM is a modified indirect BEM and its use makes it unnecessary to create an explicit expression of the limit state function (LSF), improving the efficiency of the reliability analyses. One example of its use is Su *et al.*⁴ where it was used in conjunction with the advanced first-order second moment (AFOSM) method, a derivative of FORM, to conduct reliability analyses of 2D elastostatic structures. Results showed that the AFOSM with the SFBEM can obtain good agreement with MCS. Su and Xu⁷ later applied the SFBEM to a more complex problem involving the structural reliability analysis of Reissner plate bending problems using the AFOSM. The proposed method was validated through comparison with a stochastic FEM method and MCS. Results showed close agreement between the three methods, with the SFBEM proving to be the most efficient, achieving computation times 80% lower than the stochastic FEM method. As well as the BEM, reliability analyses have also been conducted using the FEM^{9,10} and experimental data.¹¹

Although there have been several publications regarding the use of the BEM for structural reliability analysis,^{1,4-7} there have been none involving the use of multi-fidelity modeling with the BEM, whether for structural reliability analysis or otherwise. The application of multi-fidelity modeling to the topic of structural reliability analysis has also not been investigated thoroughly by the research community, although there have been several instances where surrogate models have been employed successfully.^{5,10,12} The development of reliability analysis formulations incorporating the use of multi-fidelity modeling with the BEM, with its many attractions, is therefore an area of great interest and has the potential to provide significant improvements in efficiency to the employed reliability analysis procedures, especially since assessing the reliability of a structure can be very computationally expensive, particularly for complex problems. With regards to MCS, for relatively simple problems with only two random variables the number of simulations required could be 5×10^5 (Ref. 1), however, for more complex problems, for example with 10 random variables, it could be much higher at 1×10^8 .⁹ By substituting a multi-fidelity model (MFM) for the high-fidelity model (HFM) used with MCS, similar accuracy to the HFM could be achieved at a fraction of the cost. Similar improvements could also be seen with FORM and SORM.

In the context of this work, multi-fidelity modeling, related to surrogate modeling or metamodeling, refers to the process of approximating the response of a HFM over a particular domain. The approximation is created by evaluating the response of the HFM and a low-fidelity model (LFM) at several points, called design points, within this domain. Types of metamodels that have been employed in the past include polynomials,^{9,10,13,14} Kriging,¹⁵⁻¹⁷ and radial basis neural networks (ANNs).¹⁷ The aim is to provide a MFM that is of similar accuracy to the HFM over the required

domain, but at much less computational cost. A recent review of multi-fidelity modeling was undertaken by Simpson *et al.*¹⁸

There have been many publications on the topic of multi-fidelity modeling due to its many potential benefits. One notable example is Vitali *et al.*¹⁴ where a multi-fidelity approach was taken toward the calculation of the stress intensity factor in the 0° plies in a composite stiffened panel with a center crack. Coarse and fine FEM meshes were used as the LFM and HFM, respectively and the ratio and the difference between the stress intensity factors calculated for these models were used to create linear correction response surfaces with the ply thicknesses as independent variables. Results showed that the MFMs were of similar accuracy to the HFM. Sun *et al.*¹³ used a similar approach for the optimization of a sheet-metal forming process. FEM models were used for the LFM and the HFM with the ratio and difference of the responses between these models being approximated as second-order polynomial correction response surfaces. To avoid calling the LFM for each multi-fidelity approximation, the response of the LFM was approximated using moving least square modeling, reducing computation time. The most effective MFM obtained a low absolute percentage error of 1.06% with respect to the HFM.

The main objective of this work concerns the development of reliability analysis formulations incorporating the use of multi-fidelity modeling with the BEM. In this work, the computational cost of performing reliability analyses on 2D elastostatic structures using the BEM through the application of multi-fidelity modeling is investigated. The aim is to use MFMs to obtain similar accuracy to the HFM but at much less computational cost, greatly improving the efficiency of the reliability analysis procedures using MCS, FORM, and SORM. A second objective involves quantifying this improvement in efficiency for each of these three methods, determining the extent to which each of these methods benefit from the application of multi-fidelity modeling.

2. BEM Models

The BEM models used in this work consist of a 2D rectangular plate with a center circular hole, similar to that employed by Huang *et al.*¹ The plate has a width W of 1 unit, a length L of 2 units, and a radius R of 0.25 units. The material properties are the Young's modulus, $E = 1$ unit, and the Poisson's ratio, $\nu = 0.3$ units. The plate is subjected to a uniform far-field remote tensile stress, σ_T , of 1 unit along its top and bottom edges. The plate has roller constraints along the left and right edges and pinned supports at the middle of these edges. A diagram of the plate used in this work can be seen in Fig. 1.

It is necessary to define the failure condition of the plate in order to perform reliability analyses; this can be achieved through the use of a LSF. The LSF in this work is based on the maximum stress in the plate, which occurs as a tangential stress at the edge of the circular hole, and is marked as point A in Fig. 1. Let, $\mathbf{Z}$ denote a vector of random variables, $\mathbf{Z} = (\sigma_c, R, \sigma_T, \nu, L, W) = (Z_1, Z_2, Z_3, Z_4, Z_5, Z_6)$, and let

Fig. 1. The rectangular plate used for the BEM models.

$\mathbf{X}$ ($\mathbf{X} \subseteq \mathbf{Z}$) denote the random variables that influence the stress at point A, $\mathbf{X} = (R, \sigma_T, \nu, L, W)$. The LSF, $g(\mathbf{Z})$, in this case is given by:

$$g(\mathbf{Z}) = \sigma_c - \sigma_A(R, \sigma_T, \nu, L, W) = Z_1 - \sigma_A(\mathbf{X}), \quad (1)$$

where σ_c is material strength and σ_A is the tangential stress at point A. The structure fails when $\sigma_A > \sigma_c$ (or $g(\mathbf{Z}) < 0$). The random variables were given the probability distributions as seen in Table 1. They were assigned parameters based on their expected uncertainty, with σ_c and R having the most uncertainty in their values. To test the performance of the various models with MCS, FORM, and SORM under varying levels of uncertainty, the coefficient of variation ($\text{COV} = \sigma/\mu$, where σ and μ are the standard deviation and mean, respectively) of the center hole radius, COV_R , was varied uniformly in the closed interval $[0, 0.1] \Rightarrow \{\text{COV}_R \in \mathbb{R} : 0 \leq \text{COV}_R \leq 0.1\}$.

Creating the MFMs requires the definition of a LFM and a HFM. The LFM is a coarse model of the plate as seen in Fig. 1, consisting of 20 quadratic elements, while the HFM is a fine model of 100 elements of the same type. The BEM models can be

Table 1. The six random variables and their probability distribution parameters.

Notation	Symbol	Description	Probability distribution	Mean	Standard deviation
Z_1	σ_c	Material strength	Normal	5.00	0.75
Z_2	R	Hole radius	Normal	0.25	$[0, 0.025]$
Z_3	σ_T	Applied stress	Normal	1.00	0.05
Z_4	ν	Poisson's ratio	Normal	0.30	0.005
Z_5	L	Plate length	Normal	2.00	0.025
Z_6	W	Plate width	Normal	1.00	0.01

Fig. 2. The LFM (left) with 20 elements and the HFM (right) with 100 elements.

seen in Fig. 2. The ratio of the number of elements on the outer boundary to the number of elements on the hole was kept as $3/2$.

3. Multi-Fidelity Modeling with the BEM

In this work, response surface methodology (RSM) was used to create response surfaces that would be used to approximate the HFM using surrogate models. The ordinary least squares (OLS) method¹⁹ is used to approximate the unknown coefficients of the response surfaces. The response surfaces used in this work were second-order polynomial functions, with independent variables $\mathbf{X}$, of the following form:

$$\begin{aligned} \hat{y}(\mathbf{X}) &= \mathbf{f}(\mathbf{X})^T \hat{\boldsymbol{\beta}} \\ &= c_0 + c_R R + c_{\sigma_T} \sigma_T + c_\nu \nu + c_L L + c_W W + c_{RR} R^2 + c_{\sigma_T \sigma_T} \sigma_T^2 + c_{\nu \nu} \nu^2 \\ &\quad + c_{LL} L^2 + c_{WW} W^2 + c_{R \sigma_T} R \sigma_T + c_{R \nu} R \nu + c_{RL} RL + c_{RW} RW \\ &\quad + c_{\sigma_T \nu} \sigma_T \nu + c_{\sigma_T L} \sigma_T L + c_{\sigma_T W} \sigma_T W + c_{\nu L} \nu L + c_{\nu W} \nu W + c_{LW} LW, \end{aligned} \quad (2)$$

where $\hat{y}(\mathbf{X})$ is the response to be approximated and $\mathbf{f}(\mathbf{X})$ is a vector of p terms and is composed of the powers and cross-powers of the independent variables $\mathbf{X}$:

$$\begin{aligned} \mathbf{f}(\mathbf{x})^T &= (1, R, \sigma_T, \nu, L, W, R^2, \sigma_T^2, \nu^2, L^2, W^2, R \sigma_T, R \nu, \\ &\quad RL, RW, \sigma_T \nu, \sigma_T L, \sigma_T W, \nu L, \nu W, LW) \end{aligned} \quad (3)$$

$\hat{\boldsymbol{\beta}}$ is a vector of p coefficients:

$$\begin{aligned} \hat{\boldsymbol{\beta}}^T &= (c_0, c_R, c_{\sigma_T}, c_\nu, c_L, c_W, c_{RR}, c_{\sigma_T \sigma_T}, c_{\nu \nu}, c_{LL}, c_{WW}, \\ &\quad c_{R \sigma_T}, c_{R \nu}, c_{RL}, c_{RW}, c_{\sigma_T \nu}, c_{\sigma_T L}, c_{\sigma_T W}, c_{\nu L}, c_{\nu W}, c_{LW}). \end{aligned} \quad (4)$$

In this work, $p = 21$, this includes six linear terms, five squared terms, and 10 mixed terms.

Two types of MFMs were used to approximate the maximum stress in the HFM, $\sigma_{A,HF}(\mathbf{X})$. The direct type which involved directly calling the LFM, $\sigma_{A,LF}(\mathbf{X})$, each time a multi-fidelity approximation was required, and the indirect type which involved approximating the LFM as a polynomial response surface, $\sigma_{A,LF}^{\text{res}}(\mathbf{X})$. In this work, obtaining these MFMs involved obtaining correction response surfaces $\beta^{\text{res}}(\mathbf{X})$ and $\delta^{\text{res}}(\mathbf{X})$, where β is the ratio between $\sigma_{A,HF}(\mathbf{X})$ and $\sigma_{A,LF}(\mathbf{X})$ and δ is the difference between $\sigma_{A,HF}(\mathbf{X})$ and $\sigma_{A,LF}(\mathbf{X})$. β and δ were used in the MFMs instead of derivative-based approximations due to their ability to be applied to larger areas within the design domain and their ability to smooth out noise.¹⁴ β and δ were calculated at several design points $\mathbf{X}_i$ ($i = 1, 2, \dots, N$) to obtain $\beta^{\text{res}}(\mathbf{X})$ and $\delta^{\text{res}}(\mathbf{X})$. The design points were obtained from a Latin hypercube design^{6,13,14} of 30 design points ($N = 30$), requiring a computation time of 18.3s. In this work, the Latin hypercube design was modified to improve its application to reliability analysis; the design points were obtained by uniformly sampling from each random variable's cumulative distribution function (CDF) between the probabilities 2.5% and 97.5%. This ensured that the design points were highly concentrated in the highly-likely areas of each variable's domain, and less concentrated in the less-likely areas. The response surfaces were created with the parameters in Table 1 with $\text{COV}_R = 0.05$ to provide good accuracy over the entire range of COV_R values investigated. A fifth surrogate model for the HFM was obtained by approximating $\sigma_{A,HF}(\mathbf{X})$ as a response surface, $\sigma_{A,HF}^{\text{res}}(\mathbf{X})$. The details of the seven models are shown in Table 2.

3.1. Model performance evaluation

To test the robustness of these response surfaces, they were run with the design points from a second Latin hyper cube design, the “test” dataset. This dataset consisted of 100 design points sampled from each variable's CDF between a wider range of probabilities, 0.5% to 99.5%. The “design” dataset consisted of the 30 design points used to create each response surface. Various statistical parameters evaluated for the response surfaces $\beta^{\text{res}}(\mathbf{X})$, $\delta^{\text{res}}(\mathbf{X})$, $\sigma_{A,LF}^{\text{res}}(\mathbf{X})$, and $\sigma_{A,HF}^{\text{res}}(\mathbf{X})$ with the design and test datasets can be seen in Table 3. These were the coefficient of determination

Table 2. The σ_A approximations and LSFs of the seven models investigated.

Model	σ_A approximation	LSF
MF1	$\sigma_{A,MF1}(\mathbf{X}) = \beta^{\text{res}}(\mathbf{X})\sigma_{A,LF}(\mathbf{X})$	$g_{MF1}(\mathbf{Z}) = Z_1 - \sigma_{A,MF1}(\mathbf{X})$
MF2	$\sigma_{A,MF2}(\mathbf{X}) = \delta^{\text{res}}(\mathbf{X}) + \sigma_{A,LF}(\mathbf{X})$	$g_{MF2}(\mathbf{Z}) = Z_1 - \sigma_{A,MF2}(\mathbf{X})$
MF3	$\sigma_{A,MF3}(\mathbf{X}) = \beta^{\text{res}}(\mathbf{X})\sigma_{A,LF}^{\text{res}}(\mathbf{X})$	$g_{MF3}(\mathbf{Z}) = Z_1 - \sigma_{A,MF3}(\mathbf{X})$
MF4	$\sigma_{A,MF4}(\mathbf{X}) = \delta^{\text{res}}(\mathbf{X}) + \sigma_{A,LF}^{\text{res}}(\mathbf{X})$	$g_{MF4}(\mathbf{Z}) = Z_1 - \sigma_{A,MF4}(\mathbf{X})$
HFM ^{res}	$\sigma_{A,HF}^{\text{res}}(\mathbf{X})$	$g_{HF, \text{res}}(\mathbf{Z}) = Z_1 - \sigma_{A,HF}^{\text{res}}(\mathbf{X})$
LFM	$\sigma_{A,LF}(\mathbf{X}) = \sigma_{A,E20}(\mathbf{X})$	$g_{LF}(\mathbf{Z}) = Z_1 - \sigma_{A,LF}(\mathbf{X})$
HFM	$\sigma_{A,HF}(\mathbf{X}) = \sigma_{A,E100}(\mathbf{X})$	$g_{HF}(\mathbf{Z}) = Z_1 - \sigma_{A,HF}(\mathbf{X})$

Table 3. Various statistical parameters evaluated for the response surfaces $\beta^{\text{res}}(\mathbf{X})$, $\delta^{\text{res}}(\mathbf{X})$, $\sigma_{A,\text{LF}}^{\text{res}}(\mathbf{X})$, and $\sigma_{A,\text{HF}}^{\text{res}}(\mathbf{X})$ with the design (D) and test (T) datasets.

Statistical parameter	$\beta^{\text{res}}(\mathbf{X})$	$\delta^{\text{res}}(\mathbf{X})$	$\sigma_{A,\text{LF}}^{\text{res}}(\mathbf{X})$	$\sigma_{A,\text{HF}}^{\text{res}}(\mathbf{X})$
R_D^2	1.0000	1.0000	1.0000	1.0000
$R_{\text{adj},D}^2$	1.0000	1.0000	1.0000	1.0000
SE_D	9.54×10^{-5}	3.00×10^{-4}	2.67×10^{-2}	2.25×10^{-2}
MAPE_D (%)	1.16×10^{-4}	5.78×10^{-1}	4.52×10^{-3}	2.27×10^{-3}
R_T^2	0.9999	0.9992	1.0000	1.0000
$R_{\text{adj},T}^2$	0.9998	0.9989	1.0000	1.0000
SE_T	6.04×10^{-5}	1.88×10^{-4}	1.69×10^{-2}	1.57×10^{-2}
MAPE_T (%)	4.03×10^{-4}	6.57	1.53×10^{-2}	1.14×10^{-2}

R^2 , the adjusted coefficient of determination R_{adj}^2 , the standard error (SE), and the mean absolute percentage error (MAPE).

The data in Table 3 suggests that the created response surfaces are able to provide highly accurate approximations for the responses of the LFM and the HFM over the required domain. $\beta^{\text{res}}(\mathbf{X})$ proved to be more accurate than $\delta^{\text{res}}(\mathbf{X})$, this suggests that MFMs using $\beta^{\text{res}}(\mathbf{X})$ could prove to be more accurate.

To determine the performance of these surrogate models they were compared to the performance of several BEM models of a range of fidelities. σ_A was calculated at the mean values of the parameters seen in Table 1 for each of the models. The average computation time for each was calculated over 1000 runs. The results are shown in Table 4.

It can be seen from Table 4 that the indirect MFMs utilizing $\sigma_{A,\text{LF}}^{\text{res}}$ have similar accuracy to $\sigma_{A,\text{HF}}^{\text{res}}$. The most accurate MFMs involved directly calling the LFM, and of those the most accurate approximations were those that utilized β^{res} , while those

Table 4. MAPE (%) of the various surrogate models and BEM models with respect to the HFM. Also shown are the average CPU wall-times over 1000 runs for each model. “X” in $\sigma_{A,\text{EX}}$ denotes the number of elements in a BEM model.

Model type	Model	MAPE (%)	Average computation time per run (ms)
Direct MFMs	$\sigma_{A,\text{MF1}}$	6.38×10^{-4}	142
	$\sigma_{A,\text{MF2}}$	8.43×10^{-4}	138
Indirect MFMs	$\sigma_{A,\text{MF3}}$	1.57×10^{-2}	0.402
	$\sigma_{A,\text{MF4}}$	1.62×10^{-2}	0.412
HFM response surface	$\sigma_{A,\text{HF}}^{\text{res}}$	1.14×10^{-2}	0.247
BEM models	$\sigma_{A,E10}$	3.40	66
	$\sigma_{A,E20}$ (LFM)	6.71×10^{-2}	110
	$\sigma_{A,E40}$	4.88×10^{-2}	155
	$\sigma_{A,E60}$	1.15×10^{-2}	255
	$\sigma_{A,E80}$	3.07×10^{-3}	328
	$\sigma_{A,E100}$ (HFM)	0.00	499

utilizing δ^{res} proved to be less accurate. This agrees well with observations made of the data presented previously in Table 3. The best approximation was given by the multi-fidelity model MF1, which achieved the lowest MAPE of $6.38 \times 10^{-4}\%$. This model is 4.81 times more accurate than the 80-element mesh, $\sigma_{A,E80}$, while also having a CPU wall-time 2.65 times less. $\sigma_{A,\text{HF}}^{\text{res}}$ proved to be the least accurate approximation, achieving similar accuracy to the 60-element mesh, $\sigma_{A,E60}$, but with a computation time about 1032 times less.

Results suggest that MF1 and MF2 both act as good substitutes for the HFM, achieving very similar levels of accuracy and with computation times similar to that of the LFM. On the other hand, the indirect models, MF3 and MF4, as well as HFM^{res} proved to be less accurate, but showed much lower computation times.

4. Multi-Fidelity Modeling-Based Structural Reliability Analysis

In this section, multi-fidelity modeling is used in combination with reliability analyses performed using MCS, FORM, and SORM. The models shown in Table 2 are investigated, this includes the two BEM models, the LFM and the HFM, the four MFMs (MF1–MF4), and the response surface approximation of the high-fidelity BEM model, HFM^{res} .

The reliability, P_R , of a structure can be determined by evaluating the following integral:

$$P_R = 1 - P_F = P\{g(\mathbf{Z}) > 0\} = \int_{g(\mathbf{Z}) > 0} f_{\mathbf{Z}}(\mathbf{Z}) d\mathbf{Z}, \quad (5)$$

where $f_{\mathbf{Z}}(\mathbf{Z})$ is the joint probability density function of $\mathbf{Z}$, and P_F is the probability of failure. P_F and P_R involve integrating $f_{\mathbf{Z}}(\mathbf{Z})$ over the regions defined by $g(\mathbf{Z}) < 0$ (the failure region) and $g(\mathbf{Z}) > 0$ (the safe region), respectively. All the random variables are assumed to be mutually independent. The direct evaluation of the above integral is usually very difficult since it can be multidimensional if many random variables are involved. The integration boundary $g(\mathbf{Z}) = 0$ can also be multidimensional and is usually a nonlinear function. The LSF, $g(\mathbf{Z})$, may also be a black-box model (in this case a BEM model) and so it may be computationally expensive to evaluate.

There are several methods that can be used to evaluate this integral; MCS, which involves sampling from each random variable's distribution, and FORM and SORM, which attempt to simplify the integration.

4.1. Monte Carlo simulation (MCS)

MCS involves randomly sampling from known probability distributions to determine an unknown probability distribution.³ If $\mathbf{Z} = (Z_1, Z_2, \dots, Z_n)$ is a vector of known random variables, and if $Y = h(\mathbf{Z})$, where Y is a random variable with an unknown probability distribution and h is some process or function, then we can randomly

sample the variables in $\mathbf{Z}$ per their probability distributions, input them into h and obtain a value for Y . This is repeated many times until a histogram can be created of Y , allowing us to estimate its probability distribution.

MCS can be used to evaluate the LSFs in Table 2 to obtain the reliability of the structure with the random variables $\mathbf{Z} = (Z_1, Z_2, \dots, Z_6) = (\sigma_c, R, \sigma_T, \nu, L, W)$ described in Table 1. For each simulation, the variables in $\mathbf{Z}$ are randomly sampled from their probability distributions, with the samples of the last five variables (R, σ_T, ν, L, W) being used as inputs to a BEM model to evaluate σ_A . If $g(\mathbf{Z}) < 0$ ($\sigma_A(R, \sigma_T, \nu, L, W) > \sigma_c$) for a simulation, then the structure is assumed to have failed. After a certain number of simulations have been carried out, N_{MCS} , the total number of simulations in which the structure failed, N_F , can be determined and the failure probability P_F can be calculated as:

$$P_F = \frac{N_F}{N_{\text{MCS}}}. \quad (6)$$

Reliability can be calculated as:

$$P_R = 1 - P_F. \quad (7)$$

Because MCS is a simple brute-force sampling method, it can be used as a benchmark to validate the results obtained from the LSF approximation methods, FORM and SORM.

4.2. First-order reliability method (FORM)

FORM used in this work refers to the AFOSM method for nonlinear LSFs. AFOSM is an improvement over previous FORM methods such as the mean value first-order second-moment (MVFOSM) method and the Hasofer–Lind (HL) AFOSM method in that it can be used with better accuracy with non-normal random variables and nonlinear LSFs.²

There are two steps that FORM takes to make the integration in Eq. (5) more manageable. The first step involves simplifying $f_{\mathbf{Z}}(\mathbf{Z})$ such that its contours are more regular and symmetric; this is achieved by transforming the random variables from $\mathbf{Z}$ -space into $\mathbf{U}$ -space, the standard normal space. The second step involves approximating the integration boundary $g(\mathbf{U}) = 0$ as a first-order Taylor expansion:

$$g(\mathbf{U}) \approx g(\mathbf{U}^*) + \nabla g(\mathbf{U}^*)(\mathbf{U} - \mathbf{U}^*)^T, \quad (8)$$

where $\mathbf{U}$ is created by transforming $\mathbf{Z}$ into $\mathbf{U}$ -space, $\mathbf{U}^*$ is the expansion point, and $\nabla g(\mathbf{U}^*)$ is:

$$\nabla g(\mathbf{U}^*) = \left(\frac{\partial g(\mathbf{U})}{\partial U_1}, \frac{\partial g(\mathbf{U})}{\partial U_2}, \dots, \frac{\partial g(\mathbf{U})}{\partial U_n} \right) \bigg|_{\mathbf{U}^*}. \quad (9)$$

To minimize the accuracy lost by this approximation, it is necessary to expand $g(\mathbf{U})$ at the point $\mathbf{U}^*$ that contributes the most to the integrations as seen in Eq. (5) and so it will be the point that corresponds to the highest probability density. This

point is termed as the *most probable point* (MPP)³ and is the point along the integration boundary $g(\mathbf{U}) = 0$ that is closest to the origin of U-space. The distance between the MPP and the origin of U-space is termed the reliability index β . β is related to the probability of failure P_F and reliability P_R in the following manner:

$$P_R = 1 - P_F = 1 - \Phi(-\beta) = \Phi(\beta), \quad (10)$$

where Φ denotes the standard normal CDF. In this work, a Newton–Raphson type recursive algorithm developed by Rackwitz and Fiessler²⁰ is used to determine the MPP.

4.3. Second-order reliability method (SORM)

The SORM involves approximating $g(\mathbf{U}) = 0$ as a second-order Taylor expansion:

$$g(\mathbf{U}) \approx g(\mathbf{U}^*) + \nabla g(\mathbf{U}^*)(\mathbf{U} - \mathbf{U}^*)^T + \frac{1}{2}(\mathbf{U} - \mathbf{U}^*)\mathbf{H}(\mathbf{U}^*)(\mathbf{U} - \mathbf{U}^*)^T, \quad (11)$$

where $\mathbf{H}(\mathbf{U}^*)$ is the Hessian matrix evaluated at $\mathbf{U}^*$ and contains the second-order derivatives of the LSF with respect to the random variables in U-space. Since SORM is of a higher-order than FORM, it is expected that it will more accurately approximate nonlinear LSFs, such as those involved in this work. The SORM used in this work is Breitung's asymptotic approximation²¹:

$$P_F = P(g(\mathbf{X}) < 0) = \Phi(-\beta_{\text{FORM}}) \prod_{i=1}^{n-1} (1 + \beta_{\text{FORM}}\kappa_i)^{-1/2} \quad (12)$$

where β_{FORM} is the converged value of the reliability index calculated from FORM and κ_i ($i = 1, \dots, n-1$) are the principal curvatures of the LSF at the converged MPP location $\mathbf{U}^*$ from FORM. This approximation is accurate only for large values of β , which is the case for practical high-reliability problems.^{2,22}

4.4. Numerical example

FORM and SORM require the derivatives of the LSF under investigation, which can be directly calculated for the response surfaces β^{res} , δ^{res} , $\sigma_{A,\text{LF}}^{\text{res}}$, and $\sigma_{A,\text{HF}}^{\text{res}}$. The calculation of the derivatives of $\sigma_{A,\text{LF}}$ and $\sigma_{A,\text{HF}}$ is more complex as they require the differentiation of the response of the BEM models. Methods used in the past include the IDM^{1,23} and the FDM.⁵ In this work, the FDM is used with the first-order forward scheme and second-order central difference scheme. The step size in U-space, ΔU , was set as 1×10^{-5} for FORM and 1×10^{-3} for SORM as these were found to provide accurate results. The first-order derivatives were evaluated in the following manner (for $i \neq 1$):

$$\frac{\partial g(\mathbf{U})}{\partial U_i} = \sigma_{z_i} \frac{\partial g(\mathbf{Z})}{\partial Z_i} = \sigma_{z_i} \left(-\frac{\sigma_A(Z_i + \Delta Z_i, \mathbf{Z}_{-i}) - \sigma_A(Z_i, \mathbf{Z}_{-i})}{\Delta Z_i} \right). \quad (13)$$

For $i = 1$:

$$\frac{\partial g(\mathbf{U})}{\partial U_1} = \sigma_{z_1} \frac{\partial g(\mathbf{Z})}{\partial Z_1} = \sigma_{z_1}, \quad (14)$$

where σ_{z_i} is the standard deviation of the i 'th random variable and $\Delta Z_i = \sigma_{z_i} \Delta U$. $\mathbf{Z}_{-i}$ is a vector that includes all the random variables in $\mathbf{Z}$ that influence σ_A but excludes Z_i . The second-order derivatives were calculated as (for $i, j \neq 1$ and $i = j$):

$$\frac{\partial^2 g(\mathbf{U})}{\partial U_i^2} = \sigma_{z_i}^2 \left(-\frac{\sigma_A(Z_i + \Delta Z_i, \mathbf{Z}_{-i}) - 2\sigma_A(Z_i, \mathbf{Z}_{-i}) + \sigma_A(Z_i - \Delta Z_i, \mathbf{Z}_{-i})}{(\Delta Z_i)^2} \right). \quad (15)$$

For $i, j \neq 1$ and $i \neq j$:

$$\frac{\partial^2 g(\mathbf{U})}{\partial U_i \partial U_j} = \sigma_{z_i} \sigma_{z_j} \left(-\frac{\sigma_{A,(i+1,j+1)} - \sigma_{A,(i+1,j-1)} - \sigma_{A,(i-1,j+1)} + \sigma_{A,(i-1,j-1)}}{4\Delta Z_i \Delta Z_j} \right), \quad (16)$$

where

$$\sigma_{A,(i+1,j+1)} = \sigma_A(Z_i + \Delta Z_i, Z_j + \Delta Z_j, \mathbf{Z}_{-i,-j}), \quad (17)$$

$$\sigma_{A,(i+1,j-1)} = \sigma_A(Z_i + \Delta Z_i, Z_j - \Delta Z_j, \mathbf{Z}_{-i,-j}), \quad (18)$$

$$\sigma_{A,(i-1,j+1)} = \sigma_A(Z_i - \Delta Z_i, Z_j + \Delta Z_j, \mathbf{Z}_{-i,-j}), \quad (19)$$

$$\sigma_{A,(i-1,j-1)} = \sigma_A(Z_i - \Delta Z_i, Z_j - \Delta Z_j, \mathbf{Z}_{-i,-j}), \quad (20)$$

where $\mathbf{Z}_{-i,-j}$ is a vector that includes all the random variables in $\mathbf{Z}$ that influence σ_A but excludes Z_i and Z_j . For $i = 1$ or $j = 1$:

$$\frac{\partial^2 g(\mathbf{U})}{\partial U_i \partial U_j} = 0. \quad (21)$$

The reliability of each of the seven models was evaluated using FORM and SORM at 101 uniformly-spaced values of COV_R in the closed interval $[0, 0.1]$. For MCS, 11 uniformly-spaced values were investigated. A total of 2.2×10^9 MCS simulations were carried out for each of the seven models, 2×10^8 for each of the 11 reliability indices corresponding to the 11 values of COV_R .

4.5. Results and discussion

Figure 3 shows the reliability indices obtained from MCS, FORM, and SORM for the LFM and the HFM. It can be seen from this figure that SORM follows MCS much more closely than FORM, suggesting that SORM provides a much more accurate approximation for the failure domain than FORM. This is because SORM approximates each of these nonlinear LSFs in Table 2 using a second-order Taylor expansion, taking into account the curvature of the LSFs, therefore providing a more accurate evaluation of the failure domain. The consequence of this is that SORM is more computationally expensive than FORM, as can be seen in Fig. 4. For a low level of uncertainty ($\text{COV}_R = 0$), the MAPE between MCS and FORM for the HFM is

Fig. 3. Reliability indices calculated using MCS, FORM, and SORM for the LFM and the HFM. Also shown is the MAPE between MCS and FORM and MCS and SORM for the HFM at each COV_R value.

Fig. 4. The ratio of the average CPU wall-time of each model for MCS, FORM, and SORM to the average CPU wall-time of the HFM for MCS. Average CPU wall-time is defined as the time required to obtain a reliability index for a model.

$9.80 \times 10^{-4}\%$ while it is smaller at $2.13 \times 10^{-4}\%$ for SORM. In Fig. 4 and Table 5, it can be seen that FORM is on average 3.41×10^6 times faster than MCS, while SORM is on average 1.05×10^6 times faster. SORM is also on average 4.98 times more accurate than FORM, but has a computation time that is on average 3.20 times greater. A significant portion of this time is used to evaluate the entries of the Hessian matrix $H(U^*)$ which requires a total of $2n^2$ calls to a BEM model, where n is the number of random variables being investigated. Because of the symmetry of $H(U^*)$, it is only necessary to directly evaluate half of its non-diagonal entries. FORM, by comparison, requires $n + 1$ calls to the BEM model for each iteration of the MPP search algorithm. Given that in this work $n = 6$ and that on average the MPP search algorithm required six iterations, FORM required a total of 42 calls to

Table 5. MAPE (%) of the various models with respect to the HFM and the average CPU wall-time (s) required to obtain a reliability index value. A total of 2×10^8 MCS simulations were required for each data point/reliability index value for each model.

Model	Average MAPE (%)			Average computation time (s) per data point		
	MCS	FORM	SORM	MCS	FORM	SORM
LFM	1.56×10^{-1}	1.29×10^{-1}	1.48×10^{-1}	9.06×10^5	5.59×10^{-1}	7.16×10^{-1}
MF1	5.82×10^{-4}	1.27×10^{-4}	7.71×10^{-4}	9.79×10^5	6.11×10^{-1}	2.83
MF2	2.81×10^{-3}	5.02×10^{-4}	1.50×10^{-3}	9.45×10^5	6.09×10^{-1}	1.69
MF3	4.60×10^{-2}	1.02×10^{-2}	2.66×10^{-2}	4.26×10^4	8.21×10^{-3}	3.85×10^{-2}
MF4	4.47×10^{-2}	1.02×10^{-2}	2.54×10^{-2}	8.08×10^4	7.84×10^{-3}	2.59×10^{-2}
HFM ^{res}	7.36×10^{-2}	7.34×10^{-3}	4.53×10^{-2}	1.35×10^4	4.00×10^{-3}	1.77×10^{-2}
HFM	0.00	0.00	0.00	3.26×10^6	1.31×10^1	1.68×10^1

the BEM model on average to obtain a reliability index value. SORM required 72 for the evaluation of $H(\mathbf{U}^*)$, and since SORM relies on the results from FORM, it therefore required a total of 114 calls on average to obtain a reliability index value. Both FORM and SORM become less accurate as COV_R increases, this is due to the increasing level of uncertainty.

Given the data shown in Table 4 it was expected that the direct MFMs (MF1 and MF2) would prove to be the most accurate surrogate models for reliability analysis. This seems to be the case when looking at Figs. 5–8 where they are shown to be significantly superior in accuracy to the LFM while also being of similar computational cost, suggesting that they could act as very good substitutes for the HFM. The indirect MFMs (MF3 and MF4) and the response surface approximation of the HFM, HFM^{res}, by comparison were shown to be less accurate than the direct models but significantly faster in terms of computation time, achieving wall-times that were around 10–20 times less than the direct MFMs for MCS, with HFM^{res} proving to be the fastest at 70 times less. Despite this, the accuracy of the indirect MFMs is similar to that of the direct MFMs for low levels of uncertainty ($\text{COV}_R < 0.05$), but worsens

Fig. 5. Reliability indices calculated using FORM for the models LFM, HFM, MF1, MF3, and MF5. Also shown is the MAPE of the models with respect to the HFM with FORM at each COV_R value.

Fig. 6. Reliability indices calculated using SORM for the models LFM, HFM, MF1, MF3, and MF5. Also shown is the MAPE of the models with respect to the HFM with SORM at each COV_R value.

Fig. 7. Reliability indices calculated using MCS for the models LFM, HFM, MF1, MF3, and MF5. Also shown is the MAPE of the models with respect to the HFM with MCS at each COV_R value.

Fig. 8. The inverse of the ratio of each model's average MAPE to the average MAPE of the LFM for MCS, FORM, and SORM.

after this. This is due to the fact that the response surfaces were designed at $\text{COV}_R = 0.05$ in order to minimize the average error over the range of COV_R values investigated. A method to mitigate this would be to design a set of response surfaces for each COV_R under investigation. The most accurate MFM proved to be MF1, which provided the greatest accuracy improvement over the LFM for MCS, FORM, and SORM. This was expected from the results shown previously in Table 4. Although it provides very high accuracy over the entire range of COV_R values investigated, achieving a very small MAPE for MCS with respect to the HFM of $2.7 \times 10^{-3}\%$ for $\text{COV}_R = 0.10$, it is most effective when used for low levels of uncertainty. MF1 provided very similar computation times to the LFM, being only 7.95% slower for MCS and 9.27% for FORM. MF1 is significantly slower than the LFM for SORM, providing computation times that are 295.47% greater, this is because the second derivative of its LSF contains six terms and requires three calls to the LFM in the case where $i = j$ and in the case $i \neq j$ 8 terms need to be evaluated and six calls to the LFM are required. In contrast, the second derivative of the LFM's LSF only contains one term that requires two calls for $i = j$ or four calls for $i \neq j$.

From Fig. 5, we can see that all the MFMs perform similarly well with FORM. This can also be seen from the sensitivities in Table 6 which are used by FORM. These sensitivities quantify the influence that a change in each of the random variables in $\mathbf{X}$ has on σ_A . It is also worth mentioning that the sensitivities are in-line with what is expected, with R and σ_T having the largest influence on σ_A , and ν and L having the least amount of influence. The MFMs with SORM in Fig. 6, by comparison, perform very differently, with the indirect models proving to be less accurate than the direct models, especially at high levels of uncertainty ($\text{COV}_R > 0.05$). This can be explained by the fact that the response surfaces used in this work are second-order polynomials, their second-order derivatives are therefore constant. The effect of this is maximized in the case of the indirect models since they are purely constructed of response surfaces. A means of mitigating this would be to use higher-order polynomials, possibly third or fourth order, at the expense of increased computation time. From Fig. 8, we can see that the reliability analysis method that showed the greatest improvement in accuracy over the LFM when used in conjunction with the MFMs was FORM, which experienced a 1011 times reduction in

Table 6. Sensitivities of σ_A for the seven models investigated with respect to the random variables $\mathbf{Z}$ evaluated at their mean values.

$\mathbf{Z}_i$	Sensitivity $\delta\sigma_A/\delta Z_i$						
	LFM	MF1	MF2	MF3	MF4	HFM ^{res}	HFM
R	5.85	5.69	5.80	5.73	5.73	5.72	5.70
σ	3.08	3.08	3.08	3.08	3.08	3.08	3.08
ν	-8.19×10^{-1}	-7.86×10^{-1}	-8.08×10^{-1}	-7.64×10^{-1}	-7.62×10^{-1}	-7.74×10^{-1}	-7.85×10^{-1}
L	-1.93×10^{-3}	-1.42×10^{-3}	-1.76×10^{-3}	-3.77×10^{-3}	-3.75×10^{-3}	-1.01×10^{-3}	-1.37×10^{-3}
W	-1.46	-1.42	-1.45	-1.44	-1.44	-1.43	-1.42

error when used with MF1. SORM and MCS experienced an improvement that was less pronounced but still significant, with MF1 providing an average MAPE that was 192 and 268 times less for SORM and MCS respectively than the LFM.

In conclusion, the MFMs were found to provide superior accuracy to the LFM for each of the reliability analysis methods investigated with computation times very similar to those of the LFM, or in the case of the indirect models significantly less. The direct MFMs provided reliability indices that were of the order of 50–100 times more accurate than the indirect MFMs and the response surface approximation of the high-fidelity model, HFM^{res} . The best overall performance was achieved by MF1 which involved coupling the LFM with the ratio response surface $\beta^{\text{res}}(\mathbf{X})$. Results suggest that MF1 could act as a very good substitute for the HFM when used for reliability analysis with MCS, FORM, and SORM.

5. Conclusions

In this work, a method for the application of multi-fidelity modeling to the reliability analysis of 2D elastostatic structures using the BEM is proposed. The reliability of several MFMs created using second-order polynomial response surfaces was evaluated alongside that of a LFM and a HFM using MCS, FORM, and SORM. Results show that, MF1, the MFM that directly coupled the LFM with a ratio correction response surface proved to be the most accurate, providing errors relative to the HFM that were on average 1011, 192, and 268 times smaller than the LFM for FORM, SORM, and MCS, respectively. It was also of similar computational cost, being only 7.95% slower in the case of MCS. These results suggest that the multi-fidelity models involving directly calling the LFM could act as good substitutes for the HFM for reliability analysis using MCS, FORM, and SORM.

Future work will seek to apply the proposed methodology to more complex structures, such as anisotropic plates and shells. The IDM will also be investigated.

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