Asymptotic inequalities for alternating harmonics

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For $n \in \mathbb{N}$ the n th alternating harmonic number

$$H_n^* := \sum_{k=1}^n (-1)^{k-1} \frac{1}{k}$$

is given in the form

$$H_n^* = \ln 2 + \frac{(-1)^{n+1}}{4 \lfloor \frac{n+1}{2} \rfloor} + \sum_{i=1}^{q-1} \frac{(4^i - 1)B_{2i}}{(2i)(2 \lfloor \frac{n+1}{2} \rfloor)^{2i}} + r_q(n),$$

where $q \in \mathbb{N}$ is a parameter controlling the magnitude of the error term $r_q(n)$ estimated as

$$0 < (-1)^{q+1} r_q(n) < \frac{|B_{2q}|}{2q \cdot \lfloor \frac{n+1}{2} \rfloor^{2q}} < 2 \frac{\exp(\frac{1}{24q})}{1 - 2 \cdot 4^{-q}} \sqrt{\frac{\pi}{q}} \left(\frac{q}{e\pi \lfloor \frac{n+1}{2} \rfloor} \right)^{2q}.$$

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1. Introduction

Recently, the n th harmonic number (shortly the n th harmonic)[†]

$$H_n := \sum_{k=1}^n \frac{1}{k} \quad (n \geq 0) \tag{1}$$

[†]By definition, we have $\sum_{i=m}^n x_i = 0$ for $m > n$, consequently $H_0 := 0$.

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was accurately estimated in [2]. Twelve years ago, it was shown in [5] that for the n th alternating harmonic number (shortly the n th alternating harmonic),

$$H_n^* := \sum_{k=1}^n (-1)^{k-1} \frac{1}{k}, \quad (2)$$

the best constants a and b , satisfying the double inequality

$$\frac{1}{2n+a} \leq |H_\infty^* - H_n^*| \leq \frac{1}{2n+b}$$

for every $n \in \mathbb{N}$, are $a = \frac{1}{2n-\ln 2}$ and $b = 1$. Another proof with the best constants a and b can be found in [3]. Recently, the inequalities

$$\frac{1}{4n^2+2} < \left| H_n^* - (-1)^{n-1} \frac{1}{2n} - H_\infty^* \right| < \frac{1}{4n^2+1.177\dots} \quad (n \in \mathbb{N}), \quad (3)$$

were presented in [4] which enable one to estimate the alternating harmonic numbers. We shall give some further extremely accurate estimates based on results presented in [1, 2].

2. Preliminaries

Grouping positive and negative summands, for any $p \in \mathbb{N}$, we have

$$H_{2p}^* = \sum_{j=1}^p \frac{1}{2j-1} - \frac{1}{2} \sum_{j=1}^p \frac{1}{j} = \left(H_{2p} - \frac{1}{2} H_p \right) - \frac{1}{2} H_p = H_{2p} - H_p$$

and consequently

$$H_{2p+1}^* = H_{2p}^* + \frac{1}{2p+1} = H_{2p} - H_p + \frac{1}{2p+1} = H_{2p+1} - H_p.$$

Thus, using the floor (integer part) function $\lfloor \cdot \rfloor$, we have

$$H_n^* = H_n - H_{\lfloor n/2 \rfloor} \quad (n \geq 1). \quad (4)$$

Now, we are in a position to use the two basic results from [1, 2].

Lemma 1 ([2, Theorem 1]). *For positive integers m , n and q , where $m < n$, we have*

$$H_n = A_q(m) + B_q(n) + R_q(m, n), \quad (5)$$

where[†]

$$A_q(m) := H_m - \ln m - \frac{1}{2m} + \sum_{i=1}^{q-1} \frac{B_{2i}}{(2i)m^{2i}}, \quad (6)$$

$$B_q(n) := \ln n + \frac{1}{2n} - \sum_{i=1}^{q-1} \frac{B_{2i}}{(2i)n^{2i}}, \quad (7)$$

[†]See footnote [†].

$$0 \leq (-1)^{q+1} R_q(m, n) < \frac{|B_{2q}|}{2q \cdot m^{2q}} < 2 \frac{\exp(\frac{1}{24q})}{1 - 2 \cdot 4^{-q}} \sqrt{\frac{\pi}{q}} \left(\frac{q}{e\pi m} \right)^{2q}. \quad (8)$$

Next, setting $a = 1$ in [1, Theorem 1], we obtain for the Euler–Mascheroni constant $C := \gamma(1)$ the expression[§]

$$C = H_n - \ln(1+n) + \frac{1}{2(1+n)} + \sum_{j=1}^{q-1} \frac{B_{2j}}{2j(1+n)^{2j}} + R_n^*(1, q) \quad (n, q \in \mathbb{N}) \quad (9)$$

with the inequalities

$$\frac{|B_{2q}|}{2q(2+n)^{2q}} < (-1)^{q+1} R_n^*(1, q) < (1 - 4^{-q}) \frac{|B_{2q}|}{q(1+n)^{2q}}. \quad (10)$$

Now, considering the identity $H_{n+1} = H_n + \frac{1}{n+1}$, replacing $1+n$ by n , and introducing the notation $r_n^*(q) := -R_n^*(1, q)$, we get from relations (9) and (10) the following crucial lemma.

Lemma 2 ([1, Theorem 1]). *For every integer $n \geq 1$ there holds the equality*

$$H_n = C + \ln n + \frac{1}{2n} - \sum_{j=1}^{q-1} \frac{B_{2j}}{(2j)n^{2j}} + r_n^*(q), \quad (11)$$

with the estimates

$$-(1 - 4^{-q}) \frac{|B_{2q}|}{qn^{2q}} < (-1)^{q+1} r_n^*(q) < -\frac{|B_{2q}|}{2q(n+1)^{2q}}. \quad (12)$$

3. Approximating the Alternating Harmonics

For any integer $n \geq 2$, using (4) and setting $m = \lfloor \frac{n}{2} \rfloor$ in Lemma 1, we obtain

$$\begin{aligned} H_n^* &= \left(A_q \left(\left\lfloor \frac{n}{2} \right\rfloor \right) + B_q(n) + R_q \left(\left\lfloor \frac{n}{2} \right\rfloor, n \right) \right) - H_{\lfloor n/2 \rfloor} \\ &= \left(\overline{H_{\lfloor n/2 \rfloor}} - \ln \left\lfloor \frac{n}{2} \right\rfloor - \frac{1}{2 \lfloor \frac{n}{2} \rfloor} + \sum_{i=1}^{q-1} \frac{B_{2i}}{(2i) \lfloor \frac{n}{2} \rfloor^{2i}} \right. \\ &\quad \left. + \ln n + \frac{1}{2n} - \sum_{i=1}^{q-1} \frac{B_{2i}}{(2i)n^{2i}} + R_q \left(\left\lfloor \frac{n}{2} \right\rfloor, n \right) \right) - \overline{H_{\lfloor n/2 \rfloor}} \\ &= \ln \frac{n}{\lfloor n/2 \rfloor} + \frac{1}{2} \left(\frac{1}{n} - \frac{1}{\lfloor n/2 \rfloor} \right) + \sum_{i=1}^{q-1} \frac{B_{2i}}{(2i)} \left(\frac{1}{\lfloor n/2 \rfloor^{2i}} - \frac{1}{n^{2i}} \right) \\ &\quad + R_q \left(\left\lfloor \frac{n}{2} \right\rfloor, n \right), \end{aligned} \quad (13)$$

[§]Consider footnote [†].

where using (8),

$$0 \leq (-1)^{q+1} < R_q \left(\left\lfloor \frac{n}{2} \right\rfloor, n \right) < \frac{|B_{2q}|}{2q \cdot \lfloor n/2 \rfloor^{2q}}. \quad (14)$$

Similarly, for every integer $n \geq 2$, considering (4) and Lemma 2,

$$\begin{aligned} H_n^* &= \left(\mathfrak{X} + \ln n + \frac{1}{2n} - \sum_{j=1}^{q-1} \frac{B_{2j}}{(2j)n^{2j}} + r_n^*(q) \right) \\ &\quad - \left(\mathfrak{X} + \ln \lfloor n/2 \rfloor + \frac{1}{2\lfloor n/2 \rfloor} - \sum_{j=1}^{q-1} \frac{B_{2j}}{(2j)\lfloor n/2 \rfloor^{2j}} + r_{\lfloor n/2 \rfloor}^*(q) \right) \\ &= \ln \frac{n}{\lfloor n/2 \rfloor} + \frac{1}{2} \left(\frac{1}{n} - \frac{1}{\lfloor n/2 \rfloor} \right) + \sum_{j=1}^{q-1} \frac{B_{2j}}{2j} \left(\frac{1}{\lfloor n/2 \rfloor^{2j}} - \frac{1}{n^{2j}} \right) + r_n^{**}(q), \end{aligned} \quad (15)$$

with $r_n^{**}(q) := r_n^*(q) - r_{\lfloor n/2 \rfloor}^*(q)$.

Referring to (12),

$$-(1 - 4^{-q}) \frac{|B_{2q}|}{q n^{2q}} < (-1)^{q+1} r_n^*(q) < -\frac{|B_{2q}|}{2q(n+1)^{2q}}$$

and

$$\frac{|B_{2q}|}{2q(\lfloor n/2 \rfloor + 1)^{2q}} < -(-1)^{q+1} r_{\lfloor n/2 \rfloor}^*(q) < (1 - 4^{-q}) \frac{|B_{2q}|}{q \lfloor n/2 \rfloor^{2q}}.$$

Consequently, for $n \geq 2$,

$$(-1)^{q+1} (r_n^*(q) - r_{\lfloor n/2 \rfloor}^*(q)) > \frac{|B_{2q}|}{2q} \left(\frac{1}{(\lfloor n/2 \rfloor + 1)^{2q}} - (1 - 4^{-q}) \cdot \frac{2}{n^{2q}} \right) \quad (16)$$

and

$$(-1)^{q-1} (r_n^*(q) - r_{\lfloor n/2 \rfloor}^*(q)) < \frac{|B_{2q}|}{2q} \left((1 - 4^{-q}) \frac{2}{\lfloor n/2 \rfloor^{2q}} - \frac{1}{(n+1)^{2q}} \right). \quad (17)$$

Comparing (13) and (15), we conclude that

$$r_q(n) := R_q \left(\left\lfloor \frac{n}{2} \right\rfloor, n \right) = r_n^*(q) - r_{\lfloor n/2 \rfloor}^*(q) \quad (n \geq 2). \quad (18)$$

This way we obtained several expressions for $r_q(n)$ in Theorem 1 below. Simple calculation shows that the estimate (14) is more accurate for $r_q(n)$ than (17), which will not be considered in the sequel. Additionally, (16) tell us that $(-1)^{q+1} r_q(n) > 0$, for all $q \geq 1$ and $n \geq 3$. Indeed, we consider $\delta_q(n) := \frac{1}{(\lfloor n/2 \rfloor + 1)^{2q}} - (1 - 4^{-q}) \cdot \frac{2}{n^{2q}}$, figuring in (16). Obviously, $\delta_q(2) < 0$, for $q \geq 1$. But, for any $q \geq 1$ and $n = 3$ we have

$$\delta_q(3) > 0 \Leftrightarrow \frac{1}{4^q} > \frac{(1 - 4^{-q}) \cdot 2}{9^q} \Leftrightarrow \left(\frac{9}{4} \right)^q > (1 - 4^{-q}) \cdot 2,$$

where $(\frac{9}{4})^q = (1 + \frac{5}{4})^q > 1 + q \cdot \frac{5}{4} > 2$, for $q \geq 1$. Similarly we confirm also the estimate $\delta_q(4) > 0$, true for $q \geq 1$.[¶]

In case $n \geq 5$ and $q \geq 1$ we obviously have

$$\delta_q(n) \geq \frac{1}{(n/2+1)^{2q}} - (1 - 4^{-q}) \cdot \frac{2}{n^{2q}} =: \delta_q^*(n),$$

where

$$\delta_q^*(n) > 0 \Leftrightarrow \left[2 \left(1 - \frac{2}{n+2} \right) \right]^{2q} > (1 - 4^{-q}) \cdot 2,$$

with

$$\left[2 \left(1 - \frac{2}{n+2} \right) \right]^{2q} \geq \left[2 \left(1 - \frac{2}{7} \right) \right]^2 = \frac{100}{49} > 2, \quad \text{for } n \geq 5.$$

Considering (13), (15), (18), (14) and (17), we confirm the following result.

Theorem 1. *For every integer $n \geq 2$ and any integer $q \geq 1$ we have*

$$H_n^* = \ln 2 + \frac{(-1)^{n+1}}{4 \lfloor \frac{n+1}{2} \rfloor} + \sum_{i=1}^{q-1} \frac{(4^i - 1) B_{2i}}{(2i)(2 \lfloor \frac{n+1}{2} \rfloor)^{2i}} + r_q(n), \quad (19)$$

where

$$0 \leq (-1)^{q+1} r_q(n) < \frac{|B_{2q}|}{2q \cdot \lfloor \frac{n+1}{2} \rfloor^{2q}} < 2 \frac{\exp(\frac{1}{24q})}{1 - 2 \cdot 4^{-q}} \sqrt{\frac{\pi}{q}} \left(\frac{q}{e\pi \lfloor \frac{n+1}{2} \rfloor} \right)^{2q}. \quad (20)$$

For $n \geq 3$ and $q \geq 1$ there holds the following improved estimate:

$$(-1)^{q+1} r_q(n) > \frac{|B_{2q}|}{2q} \left(\frac{1}{(\lfloor \frac{n+1}{2} \rfloor + 1)^{2q}} - (1 - 4^{-q}) \cdot \frac{2}{(2 \lfloor \frac{n+1}{2} \rfloor)^{2q}} \right) > 0. \quad (21)$$

Setting $q = 1$ and considering the equality $B_2 = \frac{1}{6}$, from Theorem 1 we obtain the first simple formula given in the next corollary.

Corollary 1. *For any integer $n \geq 2$,*

$$\begin{aligned} \ln 2 + \frac{(-1)^{n+1}}{4 \lfloor \frac{n+1}{2} \rfloor} + \frac{1}{12} \left(\frac{1}{(\lfloor \frac{n+1}{2} \rfloor + 1)^2} - \frac{3}{8 \lfloor \frac{n+1}{2} \rfloor^2} \right) \\ < H_n^* < \ln 2 + \frac{(-1)^{n+1}}{4 \lfloor \frac{n+1}{2} \rfloor} + \frac{1}{12 \lfloor \frac{n+1}{2} \rfloor^2}. \end{aligned}$$

Corollary 2. *For an integer $n \geq 15$,*

$$\ln 2 + \frac{(-1)^{n+1}}{4 \lfloor \frac{n+1}{2} \rfloor} + \frac{1}{24 \lfloor \frac{n+1}{2} \rfloor^2} < H_n^* < \ln 2 + \frac{(-1)^{n+1}}{4 \lfloor \frac{n+1}{2} \rfloor} + \frac{1}{12 \lfloor \frac{n+1}{2} \rfloor^2}.$$

Inspecting the inequalities directly for $n \leq 15$ and using Corollary 2 we confirm the next estimates.

[¶]For $q = 1$ we can prove it directly and for $q \geq 2$ using the Bernoulli inequality.

Corollary 3. For any integer $n \geq 1$ we have

$$\begin{aligned} \frac{3}{5} &< \ln 2 + \frac{1}{2(n+1)} + \frac{1}{6(n+1)^2} \\ &< H_n^* < \ln 2 + \frac{1}{2(n+1)} + \frac{1}{3(n+1)^2} < \frac{6}{5}, \quad \text{for odd } n, \\ \frac{2}{5} &< \ln 2 - \frac{1}{2n} + \frac{1}{6n^2} < H_n^* < \ln 2 - \frac{1}{2n} + \frac{1}{3n^2} < \frac{4}{5}, \quad \text{for even } n. \end{aligned}$$

Putting $q = 2$ in Theorem 1 and considering the equalities $B_2 = \frac{1}{6}$ and $B_4 = -\frac{1}{30}$, and examining the estimates for $n = 1$ directly, we get the following result.

Corollary 4. For every integer $n \geq 1$,

$$\ln 2 + \frac{(-1)^{n+1}}{4 \lfloor \frac{n+1}{2} \rfloor} + \frac{1}{16 \lfloor \frac{n+1}{2} \rfloor^2} - \frac{1}{120 \lfloor \frac{n+1}{2} \rfloor^4} < H_n^* < \ln 2 + \frac{(-1)^{n+1}}{4 \lfloor \frac{n+1}{2} \rfloor} + \frac{1}{16 \lfloor \frac{n+1}{2} \rfloor^2}.$$

Using $q = 6$ in Theorem 1 and considering the equalities $B_2 = \frac{1}{6}$, $B_4 = B_8 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$, $B_{10} = \frac{5}{66}$ and $B_{12} = -\frac{691}{2730}$, and inspecting the inequalities for $n = 1$ directly, we obtain the next result.

Corollary 5. For any integer $n \geq 1$,

$$\begin{aligned} H_n^* &= \ln 2 + \frac{(-1)^{n+1}}{4 \lfloor \frac{n+1}{2} \rfloor} + \frac{1}{16 \lfloor \frac{n+1}{2} \rfloor^2} - \frac{1}{128 \lfloor \frac{n+1}{2} \rfloor^4} \\ &\quad + \frac{1}{256 \lfloor \frac{n+1}{2} \rfloor^6} - \frac{17}{4096 \lfloor \frac{n+1}{2} \rfloor^8} + \frac{31}{4096 \lfloor \frac{n+1}{2} \rfloor^{10}} + r_6(n), \end{aligned}$$

where

$$-\frac{691}{32760 \lfloor \frac{n+1}{2} \rfloor^{12}} < r_6(n) < 0.$$

Directly from Theorem 1 we get the following consequence.

Corollary 6 (Estimating the tail of the Mercator series for $\ln 2$). For integers $n \geq 2$ and $q \geq 1$,

$$\sum_{k=n+1}^{\infty} \frac{(-1)^{k-1}}{k} = \frac{(-1)^n}{4 \lfloor \frac{n+1}{2} \rfloor} - \sum_{i=1}^{q-1} \frac{(4^i - 1)B_{2i}}{(2i)(2 \lfloor \frac{n+1}{2} \rfloor)^{2i}} + \varepsilon_q(n)$$

where

$$0 < (-1)^q \varepsilon_q(n) < 2 \frac{\exp(\frac{1}{24q})}{1 - 2 \cdot 4^{-q}} \sqrt{\frac{\pi}{q}} \left(\frac{q}{e\pi \lfloor \frac{n+1}{2} \rfloor} \right)^{2q}.$$

Referring to Corollary 3 we have $H_n^* > \frac{2}{5}$ for $n \geq 1$. Consequently, using (19) and (20) in Theorem 1, we obtain the next corollary.

Corollary 7. For integers $n \geq 2$ and $q \geq 1$, the approximation

$$H_n^* \approx H_q^*(n) := \ln 2 + \frac{(-1)^{n+1}}{4 \lfloor \frac{n+1}{2} \rfloor} + \sum_{i=1}^{q-1} \frac{(4^i - 1)B_{2i}}{(2i)(2 \lfloor \frac{n+1}{2} \rfloor)^{2i}}$$

has the relative error

$$\rho_q(n) := \frac{H_n^* - H_q^*(n)}{H_n^*}$$

estimated as

$$0 < (-1)^{q+1} \rho_q(n) < 5 \frac{\exp(\frac{1}{24q})}{1 - 2 \cdot 4^{-q}} \sqrt{\frac{\pi}{q}} \left(\frac{q}{e\pi \lfloor \frac{n+1}{2} \rfloor} \right)^{2q}.$$

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