Erratum

Curvature properties of Gödel metric

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We present some corrections of a part of the original paper [5, Sec. 4, lines 14⁹–15⁹].
 In addition, we give examples of manifolds related to the presented corrections.

Further, we denote by S and κ the Ricci tensor and the scalar curvature of the product manifold $(\overline{M} \times \tilde{N}, g = \overline{g} \times \tilde{g})$, respectively. It is obvious that $\text{rank } S = \text{rank } \overline{S}$ and $\kappa = \overline{\kappa}$ on $U_C \subset \overline{M} \times \tilde{N}$. Now [5, Eq. (24)] yields on this set

$$\text{rank} \left(S - \left(\frac{\kappa}{n-1} - (n-2)L_C \right) g \right) \leq 2. \quad (0.1)$$

We note that $\text{rank}(S - (\frac{\kappa}{n-1} - (n-2)L_C)g) = 1$ if and only if $\text{rank}\bar{S} = 1$ and $L_C = \frac{\kappa}{(n-2)(n-1)}$. Therefore the modified version of [5, Theorem 3] is the following theorem.

Theorem 0.1. *Let $(\bar{M} \times \tilde{N}, g = \bar{g} \times \tilde{g})$ be the product manifold of an $(n-1)$ -dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$, $n \geq 4$, and a one-dimensional manifold $(\tilde{N}, \tilde{g})$. Moreover, let $(\bar{M}, \bar{g})$ be a conformally flat manifold, provided that $n \geq 5$. If on $\bar{M}$ we have $\text{rank}(\bar{S} - \rho\bar{g}) = 1$, for some function ρ , then $\text{rank}(S - \rho g) \leq 2$ and [5, Eq. (12)], i.e. $C \cdot C = L_C Q(g, C)$, with $L_C = \frac{1}{n-2}(\frac{\kappa}{n-1} - \rho)$, hold on $\bar{M} \times \tilde{N}$. In particular, if the rank of the Ricci tensor $\bar{S}$ of $(\bar{M}, \bar{g})$ is one, then the rank of the Ricci tensor S of $\bar{M} \times \tilde{N}$ is also one and [5, Eq. (12)], with $L_C = \frac{\kappa}{(n-2)(n-1)}$, or equivalently, $\text{con } h(R) \cdot \text{con } h(R) = 0$ holds on $\bar{M} \times \tilde{N}$.*

We recall that semi-Riemannian manifolds satisfying the condition $\text{rank}(S - \rho g) = 2$ are called 2-quasi-Einstein manifolds (see, e.g. [2–4]). We present now an application of the last theorem. From Theorem 4.1 of the paper [67] (cited in [5]), it follows that a hypersurface M immersed isometrically in a semi-Riemannian space of constant curvature N , $\dim N \geq 5$, is a quasi-umbilical hypersurface if and only if it is a conformally flat manifold. Furthermore, using the Gauss equation of M in N , we can easily prove that if M is quasi-umbilical hypersurface then it is also a quasi-Einstein manifold. These facts, together with the last theorem, lead to the following theorem.

Theorem 0.2. *Let $(\bar{M}, \bar{g})$ be a manifold which is isometric with a quasi-umbilical hypersurface immersed isometrically in a semi-Riemannian space of constant curvature N , $\dim N \geq 5$. Let $(\tilde{N}, \tilde{g})$ be a one-dimensional manifold. Then the product manifold $(\bar{M} \times \tilde{N}, g = \bar{g} \times \tilde{g})$ is a manifold with pseudosymmetric Weyl conformal curvature tensor. Moreover, on the subset U_C of this manifold we have $\text{rank}(S - \rho g) \leq 2$, where ρ is some function on U_C .*

In this way, we obtain examples of quasi-Einstein and 2-quasi-Einstein manifolds [9, 10] with pseudosymmetric Weyl conformal curvature tensor. We mention that quasi-Einstein and 2-quasi-Einstein hypersurfaces with pseudosymmetric Weyl conformal curvature tensor immersed isometrically in semi-Riemannian spaces of constant curvature were investigated in the paper [33] (cited in [5]) and [4], respectively.

The above presented theorems together with Theorem 4.2 and Example 4.1 of the paper [68] (cited in [5]) imply

Theorem 0.3. *Let $(\bar{M}, \bar{g})$ be a manifold which is isometric with a quasi-umbilical hypersurface immersed isometrically in a semi-Euclidean space N , $\dim N \geq 5$. Let $(\tilde{N}, \tilde{g})$ be a one-dimensional manifold. Then the manifold $(\bar{M} \times \tilde{N}, g = \bar{g} \times \tilde{g})$ is a manifold with pseudosymmetric Weyl conformal curvature tensor satisfying $R \cdot R = Q(S, R)$. Moreover, on the subset U_C of this manifold we have $\text{rank}(S - \rho g) \leq 2$, where ρ is some function on U_C .*

Examples. (i) Let $\overline{S}$ and $\overline{\kappa}$ be the Ricci tensor and the scalar curvature of a Riemannian manifold $(\overline{M}, \overline{g})$, $\dim \overline{M} = 3$. Moreover, let $\overline{S}$ has at every point of $\overline{M}$ exactly two distinct eigenvalues $\rho_1 = \rho_2 = \rho$ and $\rho_3 = \overline{\kappa} - 2\rho$. Thus at every point of $\overline{M}$, we have $\text{rank}(\overline{S} - \rho \overline{g}) = 1$. We mention that 3-dimensional Riemannian manifolds with the Ricci tensor having two distinct eigenvalues were investigated among others in [6, 7]. We consider now the product manifold $(M, g) = (\overline{M} \times \mathbb{R}, g = \overline{g} \times \widetilde{g})$, where $\widetilde{g}$ is the standard metric on $\mathbb{R}$. Let S and κ be the Ricci tensor and the scalar curvature of (M, g) , respectively. If we set $L_C = \frac{1}{2}(\frac{\kappa}{3} - \rho)$ then we get on M : $\text{rank}(S - \rho g) = \text{rank}(S - (\frac{\kappa}{3} - 2L_C)g) = 2$, provided that ρ is nonzero at every point of M . Thus, in view of the presented above results, (M, g) is a manifold with pseudosymmetric Weyl conformal curvature tensor and $C \cdot C = L_C Q(g, C)$ on M . (ii) Let $(\overline{M}, \overline{g})$, $\dim \overline{M} \geq 4$, be a Riemannian manifold defined in [1, Sec. 5(a.ii)]. That manifold is a conformally flat manifold satisfying $\text{rank } \overline{S} = 1$ on $\overline{M}$ ([1, p. 160]). Thus, the product manifold of $(\overline{M}, \overline{g})$ and a one-dimensional manifold is a manifold with pseudosymmetric Weyl conformal curvature tensor and its Ricci tensor S satisfies $\text{rank } S = 1$. (iii) In Example 4.1(i) given in the paper [31] (cited in [5]) an example of a conformally flat manifold $(\overline{M}, \overline{g})$, $\dim \overline{M} \geq 4$, satisfying $\text{rank } \overline{S} = 1$ and $\overline{\kappa} = 0$ is given. (iv) In [2, Example 7.6(i)] (cf. [4, Remark 3.1(ii)]) an example of a 2-quasi-Einstein hypersurface M in the standard unit $(n+1)$ -sphere $S^{n+1}(1)$ is given. Moreover, if $n \geq 4$ then M is a manifold with pseudosymmetric Weyl conformal curvature tensor. (v) In [3, Theorem 6.1] it was stated that every warped product manifold $\overline{M} \times_F \widetilde{N}$ with a 2-dimensional base manifold $(\overline{M}, \overline{g})$ and a 2-dimensional manifold $(\widetilde{N}, \widetilde{g})$ or an $(n-2)$ -dimensional Einstein semi-Riemannian manifold $(\widetilde{N}, \widetilde{g})$, when $n \geq 5$, and a warping function F is a 2-quasi-Einstein manifold. In particular, if $(\widetilde{N}, \widetilde{g})$ is a space of constant curvature then $\overline{M} \times_F \widetilde{N}$ is a manifold with pseudosymmetric Weyl conformal curvature tensor [3, Theorem 7.1]. (vi) The Robinson-Trautman metric is a 4-dimensional Lorentzian warped product manifold with 2-dimensional fiber of non-constant curvature. In [8] it was showed that the Robinson-Trautman metric is 2-quasi-Einstein manifold and a manifold with pseudosymmetric Weyl conformal curvature tensor. (vii) Recently, in [11] the curvature properties of Vaidya metric, which is a 4-dimensional warped product manifold with 2-dimensional fiber of constant curvature were studied. Among other things it was showed that the Vaidya metric is a Ricci simple manifold with pseudosymmetric Weyl conformal curvature tensor.

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