Cell-filtering-based multi-scale Shannon–Cosine wavelet denoising method for locust slice images

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The locust slice images have all the features such as strong self-similarity, piecewise smoothness and nonlinear texture structure. Multi-scale interpolation operator is an effective tool to describe such structures, but it cannot overcome the influence of noise on images. Therefore, this research designed the Shannon–Cosine wavelet which possesses all the excellent properties such as interpolation, smoothness, compact support and normalization, then constructing multi-scale wavelet interpolative operator, the operator can be applied to decompose and reconstruct the images adaptively. Combining the operator with the local filter operator (mean and median), a multi-scale Shannon–Cosine wavelet denoising algorithm based on cell filtering is constructed in this research. The algorithm overcomes the disadvantages of multi-scale interpolation wavelet, which is only suitable for describing smooth signals, and realizes multi-scale noise reduction of locust slice images. The experimental results show that the proposed method can keep all kinds of texture structures in the slice image of locust. In the experiments, the locust slice images with mixture noise of Gaussian and salt–pepper are taken as examples to compare the performances of the proposed method and other typical denoising methods. The experimental results show that the Peak Signal-To-Noise Ratio (PSNR) of the denoised images obtained by the proposed method is greater 27.3%, 24.6%, 2.94%, 22.9% than

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Weiner filter, wavelet transform method, median and average filtering, respectively; and the Structural Similarity Index (SSIM) for measuring image quality is greater 31.1%, 31.3%, 15.5%, 10.2% than other four methods, respectively. As the variance of Gaussian white noise increases from 0.02 to 0.1, the values of PSNR and SSIM obtained by the proposed method only decrease by 11.94% and 13.33%, respectively, which are much less than other 4 methods. This shows that the proposed method possesses stronger adaptability.

Keywords: Shannon–Cosine wavelet; partition of unity; spectral method; Riesz basis; fractional Partial Differential Equations (PDEs).

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1. Introduction

Locust slice images are usually applied to auxiliary analysis of the changes of the physiological structure of the infected locust in pesticide research fields.¹³ As the general biological section image, the texture structure of the locust slice images appeared as smooth curves with the multi-scale property, and its contour is often blurred.^{19,22} In addition, as the locust slice images are taken under the microscope, the light refraction caused by the dust in the air and the dust itself formed the mixed noises in the slice images,^{14,15} which consist of Gaussian white noise and the salt and pepper noise, as shown in Fig. 1.

This is the reason why it is difficult to get excellent denoised images by the common denoising methods such as the nonlinear PDEs method,¹⁶ shearlet transform,^{2,8,10,12,17,18} and the coupling technology of the above two methods.²⁶ It is well known that the nonlinear PDEs method is an effective denoising tool for biomedical images. It is an edge-preserving smoothing technology. But it does not have

Fig. 1. Slice image of locust.

multi-resolution property, and this results in the fine texture being destroyed in the denoised image.^{21,27}

The wavelet transform is a kind of multi-scale transformation which allows for an efficient representation of the image data since a significant number of the transform coefficients are small and can be quantized or discarded with little loss of real information. In general, wavelet function possesses almost all the excellent properties such as multi-scales, orthogonality, smoothness and the compact support domain, except directivity. Therefore, traditional wavelets are not very effective in dealing with images that contain orientated discontinuities (edges).

In Ref. 24, a discrete orthogonal transform from Faber–Schauder basis functions is synthesized. The method provides a good image interpolation quality and high performance in terms of signal-to-noise ratio when reconstructing the image from the portion of its spectrum. Another typical regular multi-scale function is Shannon wavelet,⁷ which is good at describing the images with periodic texture. Unfortunately, the first one does not possess the smoothness and the second one has no compact support property. These limit their application to some extent.

In order to overcome the shortcomings of the wavelets, several attempts for improvement of wavelet systems in higher dimensions have been made, including complex wavelet, contourlets, brushlets, curvelets, bandelets, shearlets and directionlets. Compared to the general wavelets, these modified wavelets can maintain the fine texture in the denoised images, but at the expense of significant artificial texture.¹⁹ Mei¹⁹ presents a coupling technique of the variational method and the shearlet transformation, which reduces the artifacts and keeps the advantage of the shearlet transform effectively. But it is ill-suited for the images with fine textures.

As a matter of fact, wavelet transform is defined by means of the convolution operation, and the wavelet function appears as the smooth curve. So, the wavelet transform cannot tell the difference between the noise points and the fine texture exactly. Compared to the traditional wavelets, the wavelet with interpolative property is sensitive to any non-smooth point-value of the image or signal. This leads to the result that the noise can be viewed as the fine texture and remains in the denoised image. In order to solve the problem, the cell-average interpolation scheme has been proposed by Donoho and Harten.^{3,6} In recent years, the “cell-average” has been developed into a more generalized concept “cell-filter” such as the cell-median, the cell-LPR (local polynomial regression),^{1,4} and so on. Consequently, the scheme becomes more flexible for dealing with different kinds of noises.

The purpose of this research is to construct a novel denoising algorithm for the locust slice images by means of the “cell-filter” and the Shannon–Cosine wavelet which has been proposed by Mei and Gao²⁰ recently. The Shannon–Cosine wavelet possesses many excellent numerical properties such as the interpolation, the compact support, symmetry, smoothness and so on. This paper is organized as follows. We review the properties of Shannon–Cosine wavelet first. And then present the multi-scale interpolative operator based on the Shannon–Cosine wavelet. Furthermore, we design a multi-scale interpolative wavelet transform matrix using the

“cell-filter” concept, and some properties are proved. Finally, we perform some numerical examples in denoising image context.

2. Shannon–Cosine Wavelet Transform

2.1. Shannon–Cosine wavelet and its properties

Shannon wavelet possesses almost all the excellent numerical properties such as smoothness, interpolation and orthogonality, except the compact support domain which limits its application greatly.²³ It is for this reason that Hoffman *et al.*¹¹ have suggested using the Shannon–Gabor wavelet, which introduces the Gaussian window function to improve the compact support property of Shannon wavelet function in required precision range. However, the presence of the Gaussian window destroys the normative property possessed by the Shannon wavelet, effectively worsening the approximation to a Dirac delta function. So, it is just a quasi-wavelet and not a real wavelet function.

To overcome the shortcoming of Shannon–Gabor wavelet function,^{28,30} we take the linear combination of the cosine functions instead of Gaussian window to improve the compact support property of Shannon wavelet, that is²⁰

$$\begin{aligned} S_C(x) &= \frac{\sin(\pi x)}{\pi x} T_N^m(x) R_N(x), \\ T_N^m(x) &= \sum_{n=0}^m \left(a_n \cos\left(\frac{2n\pi x}{N}\right) \right), \\ W_N(x) &= \chi\left(x + \frac{N}{2}\right) - \chi\left(x - \frac{N}{2}\right), \end{aligned} \quad (2.1)$$

where $R_N(x)$ is a rectangular pulse function on $[-N/2, N/2]$; N is a constant related to the support domain, $\chi(x)$ is the Heaviside function defined as follows:

$$\chi(x) = \begin{cases} 0 & : x < 0, \\ \frac{1}{2} & : x = 0, \\ 1 & : x > 0. \end{cases} \quad (2.2)$$

Obviously, it is a function with compact support domain $[-N/2, N/2]$. $a_i (i = 0, 1, \dots, m)$ are applied to define the smoothness near the boundary of the function, which can be obtained by solving the equation defined as follows:

$$\left. \frac{d^n}{dx^n} S_C(x) \right|_{x=\frac{N}{2}} = 0, \quad n = 0, 1, \dots, m. \quad (2.3)$$

It has been proved that Shannon–Cosine function has the interpolative property,²⁸ that is $S_C(0) = 1$. Substituting $x = N/2$ into Eqs. (2.1) and (2.3), a system of linear algebra equations with respect to $a_n (n = 0, 1, \dots, m)$ can be obtained. For

example, as $m = 3$, the corresponding system of equations is

$$\begin{cases} a_0 + a_1 + a_2 + a_3 = 1, \\ a_0 - a_1 + a_2 - a_3 = 0, \\ -a_1 + 4a_2 - 9a_3 = 0, \\ -a_1 + 16a_2 - 81a_3 = 0. \end{cases} \quad (2.4)$$

Its solution can be obtained directly as follows:

$$a_0 = \frac{5}{16}, \quad a_1 = \frac{15}{32}, \quad a_2 = \frac{3}{16}, \quad a_3 = \frac{1}{32}. \quad (2.5)$$

Compared to Shannon wavelet, Shannon–Cosine is a real compact support function as shown in Fig. 2.

Fig. 2. Comparison between $\text{sinc}(x)$ and Shannon–Cosine scaling functions $S_C(x)$. (a) Shannon scaling function and (b) Shannon–Cosine scaling function $S_C(x)$, $m = 3$.

Table 1. Value of the compact support parameter N .

19.149560460820794	21.135248098522425	23.123447719961405
25.113547757267952	27.105121575295925	29.097861737012863
31.091541469097137	33.085988700389862	35.081073641777039
37.076687097549438	39.072752714157104	41.069202184677124

The compact support parameter N is employed to meet the requirement of the partition of unity, that is $\int_{-\infty}^{\infty} \phi(x)dx = 1$. Its solution can be solved by the interval bisection method, as shown in Table 1. Thus, compared to Shannon–Gabor wavelet, Shannon–Cosine wavelet is a real compact support wavelet and meets the requirement of the partition of unity.

It should be pointed out that the presence of the compact support domain destroys the orthogonality of the Shannon scaling function.

Lemma. *Sum-cosine window function*²⁵ $T_N^m(x) \geq 0$, that is

$$T_N^m(x) = \sum_{q=0}^m \left(a_q \cos\left(\frac{2q\pi x}{N}\right) \right) \geq 0.$$

Proof. Let $y = \frac{2\pi x}{N}$, then $y \in [-\pi, \pi]$. The sum-cosine windows function $T_N^m(x)$ can be rewritten as

$$\sum_{q=0}^m \left(a_q \cos\left(\frac{2q\pi x}{N}\right) \right) = \sum_{q=0}^m (a_n \cos(qy)). \quad (2.6)$$

By means of the multiple-angle formulas,²⁹ we have

$$\cos(qy) = q \sum_{p=0}^{\lfloor q/2 \rfloor} \frac{(-1)^p (q-p-1)! 2^{q-2p-1} \cos^{q-2p}(x)}{p!(q-2p!)}.$$

Substituting Eq. (2.7) into Eq. (2.6), we obtain

$$\sum_{q=0}^m \left(a_q \cos\left(\frac{2q\pi x}{N}\right) \right) = \sum_{q=0}^m (a_n \cos(qy)) = \frac{1}{2^m} (1 + \cos(y))^m \geq 0. \quad (2.8)$$

This completes the proof. $\square$

Proposition. *The Shannon–Cosine scaling function (2.1) is not a shift-orthonormal one.*

Proof. According to the definition of the shift-orthonormal function, we know

$$\int_{-\infty}^{\infty} S_C(x) S_C(x-k) dx = 0, \quad (2.9)$$

where $k \in \mathbb{Z}$. Based on the definition of the Shannon–Cosine function, we know that the compact support domains of $S_C(x)$ and $S_C(x-k)$ are $[-N/2, N/2]$ and

$[k - N/2, k + N/2]$, respectively. The intersection of the above two intervals can be obtained as $[k - N/2, N/2]$.

Without loss of generality, let $k = \lfloor N \rfloor$, that is, $k \in (N - 1, N]$. It is obvious that $k > x$ as $N \geq 2$. And so, we have

$$\begin{aligned} & \int_{-\infty}^{\infty} S_C(x) S_C(x - k) dx \\ &= \int_{k-N/2}^{N/2} \frac{\sin(\pi x)}{\pi x} \left(\sum_{q=0}^m a_q \cos \frac{2q\pi x}{N} \right) \frac{\sin(\pi(x - k))}{\pi(x - k)} \left(\sum_{n=0}^m a_n \cos \frac{2n\pi(x - k)}{N} \right) dx \\ &= \int_{k-N/2}^{N/2} \frac{\sin(\pi x)}{\pi x} \frac{\sin(\pi(x - k))}{\pi(x - k)} \left(\sum_{q=0}^m a_q \cos \frac{2q\pi x}{N} \right) \left(\sum_{n=0}^m a_n \cos \frac{2n\pi(x - k)}{N} \right) dx \\ &= \frac{(-1)^k}{\pi^2} \int_{k-N/2}^{N/2} \frac{\sin^2(\pi x)}{x(k - x)} \left(\sum_{q=0}^m a_q \cos \frac{2q\pi x}{N} \right) \left(\sum_{n=0}^m a_n \cos \frac{2n\pi(x - k)}{N} \right) dx. \end{aligned} \quad (2.10)$$

Substituting Eq. (2.8) into Eq. (2.10), we obtain

$$\int_{-\infty}^{\infty} S_C(x) S_C(x - k) dx = \begin{cases} > 0 : k = 2t, \quad t \in \mathbb{Z}, \\ = 0 : k = \frac{N}{2}, \\ < 0 : k = 2t + 1, \quad t \in \mathbb{Z}, \end{cases} \quad (2.11)$$

where $k = \lfloor N \rfloor \neq N/2$. This means that the Shannon–Cosine scaling function $S_C(x)$ is not the shift-orthonormal function. This completes the proof. $\square$

2.2. Multi-scale Shannon–Cosine wavelet transform matrix

Let us start with the interpolating multi-resolution analysis⁹ that is necessary for a detailed discussion of sparse grids for the purposes of interpolation or approximation, respectively. Let $\phi(x)$ be any of the interpolating basis functions such as the Shannon–Cosine. This mother function of all basis functions can be used to generate an arbitrary $\phi^{j,k}(x)$ by dilation and translation, that is

Definition 1. For any $j \in \mathbb{N}_0$, the translation format of Shannon–Cosine function is defined as

$$\begin{aligned} \phi_{j,k}(x) &= \frac{\sin\left(\frac{\pi(x - x_{j,k})}{\Delta x_j}\right)}{\frac{\pi(x - x_{j,k})}{\Delta x_j}} \left(\cos\left(\frac{\pi(x - x_{j,k})}{N \Delta x_j}\right) \right)^2 \\ &\quad \times \left[\chi\left(\frac{x - x_{j,k}}{\Delta x_j} + \frac{N}{2}\right) - \chi\left(\frac{x - x_{j,k}}{\Delta x_j} - \frac{N}{2}\right) \right], \end{aligned}$$

where $k \in \{0, 1, 2, \dots, 2^j\}$, $n \in \{0, 1, 2, \dots, 2^J\}$, $0 \leq j \leq J-1$, $j, J \in \mathbb{Z}$, $x \in [x_{\min}, x_{\max}]$, $\Delta x_j = \frac{x_{\max} - x_{\min}}{2^j}$, $x_{j,k} = x_{\min} + k\Delta x_j$.

Theorem 1. *The multi-scale interpolative wavelet transform matrix based on Shannon-Cosine wavelet can be defined as*

$$\begin{aligned} \mathbf{C}_{k,n}^{j,J} &= \mathbf{R}_{2k+1,n}^{j+1,J} - \sum_{k_0=0}^{2^j} \mathbf{R}_{k_0,n}^{j_0,J} \phi_{j_0,k_0}(x_{j+1,2k+1}) - \sum_{j_1=j_0}^{j-1} \sum_{k_1=0}^{2^{j_1}-1} \mathbf{C}_{k_1,n}^{j_1,J} \psi_{j_1,k_1}(x_{j+1,2k+1}), \\ \mathbf{C}_{k,n}^{j_0,J} &= \mathbf{R}_{2k+1,n}^{j_0+1,J} - \sum_{k_0=0}^{2^{j_0}} \mathbf{R}_{k_0,n}^{j_0,J} \phi_{j_0,k_0}(x_{j_0+1,2k+1}), \\ \mathbf{R}_{k,n}^{j,J} &= \begin{cases} 1, & x_{j,k} = x_{J,n}, \\ 0, & \text{otherwise,} \end{cases} \end{aligned} \quad (2.12)$$

where $0 \leq j_0 \leq J-1$, $j_0 \in \mathbb{Z}$, $\Delta x_J = \frac{x_{\max} - x_{\min}}{2^J}$, $x_{J,n} = x_{\min} + n\Delta x_J$.

Proof. Let $\mathbf{C}_{k,n}^{j,J}$ denote the multi-scale interpolative wavelet transform matrix, then the discrete interpolative wavelet transform coefficient can be expressed as

$$\alpha_{j,k} = \sum_{n=0}^{2^J} \mathbf{C}_{k,n}^{j,J} f(x_{J,n}), \quad (2.13)$$

where $f(x)$ is the approximated function. Based on the definition of the interpolative wavelet transform coefficient,⁹ we have

$$\alpha_{j,k} = f(x_{j,k}) - \left[\sum_{k_0=0}^{2^{j_0}} f(x_{j_0,k_0}) \phi_{j_0,k_0}(x_{j,k}) \sum_{j_1=j_0}^{j-1} \sum_{k_1=0}^{2^{j_1}-1} \alpha_{j_1,k_1} \psi_{j_1,k_1}(x_{j,k}) \right], \quad (2.14)$$

where $\psi_{j,k} = \phi_{j+1,2k+1}$. By means of the definition of the restricted operator, we have

$$\begin{cases} f(x_{j,k}) = \sum_{n=0}^{2^J} \mathbf{R}_{2k+1,n}^{j+1,J} f(x_{J,n}), \\ f(x_{j_0,k_0}) = \sum_{n=0}^{2^J} \mathbf{R}_{k_0,n}^{j_0,J} f(x_{J,n}). \end{cases} \quad (2.15)$$

Substituting Eq. (2.15) into Eq. (2.14), we obtain

$$\begin{aligned} \alpha_{j,k} &= \sum_{n=0}^{2^J} \left[\mathbf{R}_{2k+1,n}^{j+1,J} - \sum_{k_0=0}^{2^{j_0}} \mathbf{R}_{k_0,n}^{j_0,J} \phi_{j_0,k_0}(x_{j+1,2k+1}) \right] f(x_{J,n}) \\ &\quad - \sum_{j_1=j_0}^{j-1} \sum_{k_1=0}^{2^{j_1}-1} \alpha_{j_1,k_1} \psi_{j_1,k_1}(x_{j+1,2k+1}). \end{aligned} \quad (2.16)$$

Substituting Eq. (2.13) into Eq. (2.16), we have

$$\begin{aligned}
 \sum_{n=0}^{2^J} f(x_{J,n}) &= \sum_{n=0}^{2^J} \left[\mathbf{R}_{2k+1,n}^{j+1J} - \sum_{k_0=0}^{2^{j_0}} \mathbf{R}_{k_0,n}^{j_0,J} \phi_{j_0,k_0}(x_{j+1,2k+1}) \right] f(x_{J,n}) \\
 &\quad - \sum_{n=0}^{2^J} \sum_{j_1=j_0}^{j-1} \sum_{k_1=0}^{2^{j_1}-1} \mathbf{C}_{k_1,n}^{j_1,J} f(x_{J,n}) \psi_{j_1,k_1}(x_{j+1,2k+1}) \\
 &= \sum_{n=0}^{2^J} \left[\mathbf{R}_{2k+1,n}^{j+1J} - \sum_{k_0=0}^{2^{j_0}} \mathbf{R}_{k_0,n}^{j_0,J} \phi_{j_0,k_0}(x_{j+1,2k+1}) \right. \\
 &\quad \left. - \sum_{j_1=j_0}^{j-1} \sum_{k_1=0}^{2^{j_1}-1} \mathbf{C}_{k_1,n}^{j_1,J} \psi_{j_1,k_1}(x_{j+1,2k+1}) \right] f(x_{J,n}). \quad (2.17)
 \end{aligned}$$

Obviously, taking off the sum operator of both sides of Eq. (2.17), Eq. (2.12) is obtained. This completes the proof. $\square$

3. Cell-Filter Based Multi-Scale Shannon–Cosine Wavelet Transform

According to the definition of the interpolative function, the multi-scale interpolative wavelet is just suitable for the approximation of the smooth function. In other words, it is easy to mistake the “salt & pepper” noise as the fine texture in images, this is illustrated in Fig. 3. Figure 3(a) is the image added in the mixed noises of Gaussian and “salt & pepper” noises. As the noise density is smaller, the median filter (Fig. 3(b)) and the average filter (Fig. 3(c)) have better performances in removing the “salt & pepper” noise compared to the “symlet” wavelet transform (Fig. 3(d)) and Wiener filter (Fig. 3(e)). It should be noted that the multi-scale Shannon–Cosine wavelet transform can remove Gaussian noise clearly, although it is helpless to the “salt & pepper” noise (Fig. 3(e)).

It is well known that averaging is more robust to noise than point evaluation, and so the cell-average⁵ discretization can be introduced to the multi-scale Shannon–Cosine wavelet transform to improve the performance of the multi-scale Shannon–Cosine wavelet in removing noise in images. Cell-average based interpolative wavelet coefficient can be defined as

$$\alpha_{j,k} = c(x_{j,k}) - \left[\sum_{k_0=0}^{2^{j_0}} c(x_{j_0,k_0}) \phi_{j_0,k_0}(x_{j,k}) \sum_{j_1=j_0}^{j-1} \sum_{k_1=0}^{2^{j_1}-1} \alpha_{j_1,k_1} \psi_{j_1,k_1}(x_{j,k}) \right], \quad (3.1)$$

where c is the cell operator with respect to the signal f , which can be viewed as any operator such as the average, the weight average, the median, the wavelet, the Wiener filter and so on. Here, we take the average operator as an example to illustrate the relation between $c(x_{j,k})$ and $f(x_{j,k})$.

Fig. 3. Comparison of denoising with different filters (mixture noises of Gaussian white noise of mean 0 and variance 0.02 with “salt & pepper” noise of density 0.05). (a) Noisy image, (b) median filtering, (c) average filtering, (d) wiener filter, (e) symlet wavelet filter and (f) multi-scale Shannon-Cosine wavelet filter.

To develop the new method, we need to introduce the uniform partition points. Suppose $x \in [x_{\min}, x_{\max}]$, the scale parameter j is the positive integer, the space between two adjacent points is defined as

$$\Delta x_j = \frac{x_{\max} - x_{\min}}{2^j}.$$

And then the partition points are defined as

$$x_{j,k} = x_{\min} + k\Delta x_j, \quad k = 0, 1, 2, \dots, 2^j.$$

The weighted average value at the partition point $x_{j,k}$ is

$$\begin{aligned}
 c(x_{j,k}) &= \frac{\sum_{n=(k-1)2^{J-j}}^{(k+1)2^{J-j}} w(x_{J,n} - x_{j,k}) f(x_{J,n})}{\sum_{n=(k-1)2^{J-j}}^{(k+1)2^{J-j}} w(x_{J,n} - x_{j,k})} \\
 &= \sum_{n=(k-1)2^{J-j}}^{(k+1)2^{J-j}} \frac{w(x_{J,n} - x_{j,k})}{\sum_{n=(k-1)2^{J-j}}^{(k+1)2^{J-j}} w(x_{J,n} - x_{j,k})} f(x_{J,n}) \\
 &= \sum_{n=(k-1)2^{J-j}}^{(k+1)2^{J-j}} \omega(x_{J,n} - x_{j,k}) f(x_{J,n}), \tag{3.2}
 \end{aligned}$$

where $x_{J,n} \in [x_{j,k} - \Delta x_j, x_{j,k} + \Delta x_j]$, ω is the normalizing weighted function, which is usually defined as the filter window function such as:

(1) Gaussian window function

$$\omega(x - x_{j,k}) = \begin{cases} \exp\left(-\frac{(x - x_{j,k})^2}{2(r\Delta x_j)^2}\right), & \|x\| < \|x_{j,k}\|^2, \\ 0, & \text{others,} \end{cases} \tag{3.3}$$

where r is a real constant, which is usually taken as 3.2.¹¹

(2) Average window function:

$$\omega(x - x_{j,k}) = \begin{cases} 1, & \|x\| < \|x_{j,k}\|, \\ 0, & \text{others.} \end{cases} \tag{3.4}$$

(3) Other nonlinear window functions

Given a set of random variables $X = (x_{j,0}, x_{j,1}, \dots, x_{j,N})$, the order statistics $(x_{(j,0)} \leq x_{(j,1)} \leq \dots \leq x_{(j,N)})$ are random variables, defined by sorting the values of $x_{j,i}$ in an increasing order. The median value is then given as

$$\text{median}(X) = \begin{cases} x_{(j,k+1)} = x_{(j,m)}, & \text{for } N = 2k + 1, \\ 0.5(x_{(j,k)} + x_{(j,k+1)}), & \text{for } N = 2k, \end{cases} \tag{3.5}$$

where $m = 2K + 1$ is the median rank. The median is considered to be a robust estimator of the location parameter of a distribution and has found numerous applications in smoothing and denoising, especially for signals contaminated by impulsive noise.

Besides, the Wiener filter window function which is involved in this paper is also a nonlinear one. It is too complicated to express in an explicit window function. So, the window function $\omega(x - x_{j,k})$ can be viewed as an abstract symbol of the nonlinear filters.

Substituting Eq. (3.2) into Eq. (3.1), we obtain

$$\begin{aligned} \alpha_{j,k} = & \sum_{n=(k-1)2^{J-j}}^{(k+1)2^{J-j}} \omega(x_{J,n} - x_{j,k}) f(x_{J,n}) \\ & - \left[\sum_{k_0=0}^{2_j^j} \left(\sum_{n=(k_0-1)2^{J-j_0}}^{(k_0+1)2^{J-j_0}} \omega(x_{J,n} - x_{j_0,k_0}) f(x_{J,n}) \right) \right. \\ & \left. \cdot \phi_{j_0,k_0}(x_{j,k}) + \sum_{j_1=j_0}^{j-1} \sum_{k_1=0}^{2^{j_1-1}} \alpha_{j_1,k_1} \psi_{j_1,k_1}(x_{j,k}) \right]. \end{aligned} \quad (3.6)$$

From Eq. (3.6), Theorem 2 can be deduced as follows:

Theorem 2. *Cell-filtering-based multi-scale Shannon–Cosine wavelet transform matrix can be defined as*

$$\begin{aligned} \mathbf{C}_{k,n}^{j,J} &= \mathbf{W}_{k,n}^{j,J} - \sum_{k_0=0}^{2^{j_0}} \mathbf{W}_{k_0,n}^{j_0,J} \phi_{j_0,k_0}(x_{j,k}) - \sum_{j_1=j_0}^{j-1} \sum_{k_1=0}^{2^{j_1-1}} \mathbf{C}_{k_1,n}^{j_1,J} \psi(x_{j,k}), \\ \mathbf{C}_{k,n}^{j_0,J} &= \mathbf{W}_{k,n}^{j_0,J} - \sum_{k_0=0}^{2^{j_0}} \mathbf{W}_{k_0,n}^{j_0,J} \phi_{j_0,k_0}(x_{j_0,k}), \end{aligned} \quad (3.7)$$

where $\mathbf{W}_{k,n}^{j,J} = \omega(x_{J,n} - x_{j,k})$, $k \in \{0, 1, 2, \dots, 2^j\}$, $n \in \{0, 1, 2, \dots, 2^J\}$, $0 \leq j \leq J-1$, $0 \leq j_0 \leq J-1$, $x \in [x_{\min}, x_{\max}]$.

Proof. It is because the weight function $\omega(x - x_{j,k})$ possesses the compact support property, we have

$$\sum_{n=(k-1)2^{J-j}}^{(k+1)2^{J-j}} \omega(x_{J,n} - x_{j,k}) f(x_{J,n}) = \sum_{n=-2^J}^{2^J} \omega(x_{J,n} - x_{j,k}) f(x_{J,n}). \quad (3.8)$$

Substituting Eq. (3.8) into Eq. (3.7), and taking the same approach as the proof in Theorem 1, we obtain Theorem 2. This completes the proof. $\square$

Substituting $\mathbf{C}_{k,n}^{j,J}$ defined in Theorem 2 into Eq. (2.12), we can obtain a novel wavelet transform for images. The new “cell-filter” wavelet transform possesses properties both of the “cell-filter” and the multi-scale Shannon–Cosine wavelet transform.

For instance, taking the median operator as the “cell-filter”, it is easy to understand that the proposed wavelet transform can remove both the “salt & pepper” noise and Gaussian noise in Fig. 3(a). This is illustrated in Fig. 4. Indeed, if we combine the median operator with the filter suited for removing Gaussian noise such as the wavelet filter or Wiener filter, we can obtain better denoised results as shown

Fig. 4. Comparisons between the proposed method (4 window functions) with other two kinds of combinatorial methods. (a) Performs a coupling de-noising process with median and Wiener filter. SSIM = 0.6025, PSNR = 12.6663. (b) Performs a coupling de-noising process with median and wavelet filter. SSIM = 0.7391, PSNR = 14.2185. (c) Performs a de-noising process with the proposed method (average window) SSIM = 0.5284, PSNR = 11.8274. (d) Performs a de-noising process with the proposed method (Wiener window) SSIM = 0.7577, PSNR = 15.1626. (e) Performs a de-noising process with the proposed method (wavelet window) SSIM = 0.6758, PSNR = 15.5583. (f) Performs a de-noising process with the proposed method (median window) SSIM = 0.78, PSNR = 14.2746.

Fig. 4. (Continued)

in Figs. 4(a) and 4(b). It is well known that the multi-scale interpolative wavelet transform matrix $\mathbb{C}_{k,n}^{j,J}$ is different from the traditional wavelet transform,²⁰ which can identify the piecewise smooth regions in images. Indeed, it is a kind of multi-scale interpolative operator, and so it is good at describing the biomedical images and the cartoon images. This is the reason why it can remove the mixed noises of the Gaussian and ‘salt & pepper’. The shortcoming of the proposed method is that it may mistake the ‘salt & pepper’ points as the contour points in images, which can be overcome by means of the window functions. This is also illustrated in Figs. 4(c)–4(f). As the median filter is good at removing the ‘salt & pepper’, taking the median window function as the normalizing weighted function in Eq. (3.2) is better than others such as the average (Fig. 4(c)), Wiener filter (Fig. 4(d)), and wavelet (Fig. 4(e)) window functions.

4. Experiments on Locust Slice Image Denoising

4.0.1. Experiments on locust slice image denoising

The purpose of the experiment is to prepare the clear locust pictures, which play an important role in analysis of the infecting mechanism of the biological pesticide to the locust. Therefore, we take the infected locust to prepare the 0.05 μm serial ultrathin slices with LKB-3 ultramicrotome. The cutting direction is vertical to the long axis of the locust. Then, the slices are stained with uranyl acetate followed by the lead citrate so that they can be observed, photographed by means of H-7500 microscopy. The enhanced and grayed slice images are shown in Figs. 5(a1)–5(a4). It is easy to observe that the noises that appeared in the slice images are composed of more complex ingredients compared to the artificial noises such as the salt & pepper noises, Gaussian noises and so on. This will be illustrated in Sec. 4.2.

Fig. 5. Comparison of different denoising methods on slice images of locust.

4.1. Comparison of denoising effect of locust slice images with different methods

The common denoising method for the biomedical image includes: Wiener filter, wavelet transform, variational method, shearlet transform, average and median operators, etc. The average and median operators are just suited for Gaussian noises and the salt & pepper noises, respectively. In addition, it is difficult to identify the noises and the fine texture accurately by the variational model, and the solution should be solved by iterative algorithm. Therefore, we just compare the proposed method with Wiener filter and the wavelet transform in this section. Wiener filter is an adaptive method based on statistics estimated from a local neighborhood of each pixel, which is a classical tool for removing the noises in images. Symlets is an improved Daubechies wavelet, which is a nearly symmetrical wavelet and keeps all the merits of Daubechies wavelets. And so, Symlets is well suited to deal with the images compared to Daubechies wavelets.

The numerical experiments with respect to the wavelet transform apply the image denoising function 'wdencomp' and the wavelet 'sym4' embedded in MATLAB R2014, and the embedded function 'ddencomp' is applied to calculate the threshold and the approximation coefficients. The embedded function 'wiener2' is applied in the experiments on Wiener filters in this section.

The denoised slice images by means of Wiener filter are shown in Figs. 5(b1)–5(b4), using neighborhoods of size 6×6 to estimate the local image mean and standard deviation. The denoised images obtained by the wavelet transform are shown in Figs. 5(c1)–5(c4). In the proposed method, the multi-scale interpolative operator is constructed with Shannon–Cosine wavelet ($N = 23.123447719961405$), and the cell filter is the median operator, in which each output pixel contains the median value in the 6×6 neighborhood around the corresponding pixel in the input image. The denoised image obtained by the proposed method is shown in Figs. 5(d1)–5(d4), and the corresponding sparse points grid is shown in Figs. 5(e1)–5(e5). It can be observed that the density of the sparse grid points varies with the abundance of the texture in local region. In other words, the grid points decrease in the smooth area of the images, and increase in the non-smooth area adaptively. This is helpful for removing noises from the noised images when it employs reconstructed images with fine textures.

It should be noted that any of the four denoised slice images obtained by the proposed method is clearer than that obtained by other two methods. In order to analyze the influence of the noise type and noise density on the denoising effect in detail, the denoising effect should be quantized with some parameters.

4.1.1. Influence of noise type on denoising effect

It has been mentioned above that the variational method is well suited for dealing with the biomedical images. We can obtain an over-denoised image from the original figure shown in Fig. 5(a1) by this method, which has been shown in Fig. 6(a). We

Fig. 6. Comparison of different denoising methods on slice image of locust with mixed noises. (a) Original image, (b) image with mixed noises of Gaussian and salt & pepper, (c) wavelet denoising, (d) Wiener filtering, (e) proposed method and (f) sparse points/amount of pixels.

take the over-denoised image (Fig. 6(a)) as the original clean image, and then add the mixed noises which include Gaussian noises of mean 0 and variance 0.02 and the salt & pepper noises of density 0.05 to the original image. The result is shown in Fig. 6(b), which is close to the real image shown in Fig. 5(a1).

It is well known that both the wavelet transform and the Wiener filtering are not well suited for removing the salt & pepper noises from the images. The denoised

images obtained by the two methods are shown in Figs. 6(c) and 6(d), respectively. The denoised image obtained by the proposed method is shown in Fig. 6(e), and the corresponding sparse grid points are shown in Fig. 6(f). In this section, we take the parameters PSNR and SSIM (Structural Similarity Index) to evaluate the image quality of the denoised images obtained by the different methods. And we take the functions 'psnr.m' and 'ssim.m' embedded in MATLAB R14a to calculate the two parameters directly.

As shown in Table 1, compared to classical methods such as Wiener filtering, the wavelet transform, the median filtering and the average filtering, PSNR of the denoised image obtained by the proposed method increases 27.3%, 24.6%, 2.94% and 22.9%, respectively; and SSIM increases 31.1%, 31.3%, 15.5% and 10.2%, respectively. This illustrates that the proposed method possesses excellent performance in removing the mixed noises in images.

To evaluate the performance of the proposed method in removing the single noise, PSNR and SSIM of the denoised images from the images that added only one type of noise are also given in Table 2. It is easy to find that the proposed method possesses the best performance in removing Gaussian noises. If the image includes only the salt & pepper noises, the performance of the proposed method is close to the median filtering. Indeed, PSNR and SSIM obtained by the median filtering are just larger 1.1db and 0.01 than the proposed method, respectively. It is well known that the median method is the best tool in removing the salt & pepper noises, and this illustrates that the proposed method has wider areas of application.

4.1.2. Influence of noise density on denoising effect

As a matter of fact, the median filtering has greater advantage in removing the salt & pepper noises from the images, and the proposed method is close to it, no matter how much the density. So, the discussion focuses on the influence of the variance of Gaussian on the denoising effect. As the variance decreases from 0.1 to 0.04, both PSNR and SSIM increase accordingly with any of the denoising methods. Anyway, the proposed method has the best performance always. This has been illustrated in Table 3.

The decay rate of the parameters PSNR & SSIM with the increasing of Gaussian noises variance are shown in Fig. 7. With the increasing of Gaussian noises variance from 0.02 to 0.1, the value of PSNR of the denoised image obtained by the proposed method decreased 11.94%. Ones obtained by Wiener filtering, the wavelet transform, the median filter and average operator decreased 26.97%, 26.42%, 16.84% and 25.08%, respectively. Correspondingly, the values of SSIM obtained by above 5 methods decreased 13.33%, 32.62%, 33.70%, 27.50%, 22.56%, respectively. The decay rates of both of PSNR & SSIM obtained by the proposed method are the smallest ones in these 5 methods. This indicates that the proposed method is robust to the noises type and the density compared to other methods.

Table 2. Influence of noises type on denoising effect.

	Mixed noises with Salt & Pepper noise (noise density $d = 0.05$) and Gaussian noise (mean $m = 0$, variance $v = 0.02$)				Gaussian noise (mean $m = 0$, variance $v = 0.02$)			
	PSNR	SSIM	PSNR	SSIM	PSNR	SSIM	PSNR	SSIM
Average filter	22.8683118856210	0.7682444292039	27.6882051370250	0.7993551252087	25.5450362054294	0.87868553181969		
Median filter	28.8023486467965	0.8193967652387	32.7711462791110	0.9593260213446	29.2717740473653	0.83847975997081		
Wavelet transform	22.3615705346022	0.6268587042195	16.6883476079290	0.3221020642826	25.4950003395944	0.80506677891017		
Wiener filter	21.5809035959691	0.6278907004701	23.3198450325333	0.65777314391386	25.0403940986408	0.83414398646212		
Proposed method	29.6749364290415	0.9120252590000	31.6969143827506	0.9496932478738	29.8390107636970	0.91859162335531		

Table 3. Influence of Gaussian noise variance on denoising effect (mean $m = 0$).

Variance	Average filtering		Median filtering		Wavelet method		Wiener filtering		Proposed method	
0.1	PSNR	19.138534572707	24.343577041886	18.759028073832	18.286164277370	26.275392544275				
	SSIM	0.680436731558	0.607880277674	0.533762025187	0.562028165327	0.796159309284				
0.08	PSNR	20.068149468949	25.221591767703	19.710154060520	19.115340110798	27.172239445105				
	SSIM	0.718530853670	0.650504061113	0.579003223242	0.600625025791	0.826313575207				
0.06	PSNR	21.328068665862	26.183767133836	21.008323903117	20.353624065080	27.915179982848				
	SSIM	0.762710110529	0.697006493700	0.635453688341	0.655525815599	0.853605095718				
0.04	PSNR	22.875047366674	27.427519153212	22.589525640174	21.955321021386	28.829395993052				
	SSIM	0.814519788933	0.757440971159	0.703036669592	0.725315279559	0.888135024949

Fig. 7. Variation trends of PSNR and SSIM with increasing of Gaussian noises variance.

5. Conclusion

In this paper, the multi-scale interpolative operator is constructed with Shannon-Cosine wavelet, and the cell-filter could be any normalization operator such as the median, the average, the wavelet or Wiener filter and so on. Combining above two operators together, we can obtain a novel multi-scale denoising method for the locust slice images, which has excellent performance in removing the mixed noises

compared to the common denoising methods. The coupling operator theory has been proved and the experiments illustrate its correctness. And then, we can draw the conclusions as follows:

- (1) The multi-scale interpolative operator constructed with Shannon–Cosine wavelet is sensitive to the texture and the contour that appears in the images. Based on this performance, the sparse feature points can be obtained, which is the basis for the texture-preserved image denoising.
- (2) The cell-filter based multi-scale interpolative operator possesses all the merits of the multi-scale interpolative operator and the cell-filter. The coupling operator is well suited for dealing with the images with mixed noises, especially for the locust slices images.
- (3) It is well known that the variational method is an effective tool for dealing with the biomedical images. Compared to this method, the proposed method has higher efficiency as it does not have to employ the iterative method.

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