Shannon–Cosine wavelet spectral method for solving fractional Fokker–Planck equations

Shuli Mei* and Wanlin Gao†

*College of Information and Electrical Engineering
China Agricultural University, Beijing 100083, P. R. China*

**meishuli@163.com*

†gaowlin@cau.edu.cn

Received 3 September 2017

Revised 14 December 2017

Accepted 15 December 2017

Published 26 January 2018

The windowed-Shannon wavelet is not recommended generally as the window function will destroy the partition of unity of Shannon mother wavelet. A novel windowing scheme is proposed to overcome the shortcoming of the general windowed-Shannon function, and then, a novel and efficient Shannon–Cosine wavelet spectral method is provided for solving the fractional PDEs. Taking full advantage of the waveform of sinc function to hold the partition of unity, Shannon–Cosine wavelet is constructed, which is composed of Shannon wavelet and the trigonometric polynomials. It was proved that the proposed wavelet function meets the requirements of being a trial function and possesses many other excellent properties such as normalization, interpolation, two-scale relations, compact support domain, and so on. Therefore, it is a real wavelet function instead of a general Shannon–Gabor wavelet which is a kind of quasi-wavelet. Next, by means of the Shannon–Cosine wavelet collocation method, the corresponding algebraic equation system of the fractional Fokker–Planck equation can be obtained. Approximate solutions of the fractional Fokker–Planck equations are compared with the exact solutions. These calculations illustrate that the accuracy of the Shannon–Cosine wavelet collocation solutions is quite high even using a small number of grid points.

Keywords: Shannon–Cosine wavelet; partition of unity; spectral method; Riesz basis; fractional PDEs.

AMS Subject Classification: 65Y04, 65T40

This is an Open Access article published by World Scientific Publishing Company. It is distributed under the terms of the Creative Commons Attribution 4.0 (CC-BY) License. Further distribution of this work is permitted, provided the original work is properly cited.

1. Introduction

In recent years, fractional calculus has been rediscovered by scientists and engineers and applied in an increasing number of fields, namely, in the areas of electromagnetism, control engineering, and signal processing.^{7,12,14} Besides, using the Caputo derivative in the Ortigueira sense, the fractional derivative of the Riemann zeta function has been computed recently,³ which plays an important role both in number theory and in several applications of quantum mechanics. Differential equations of fractional order appear more and more frequently in various research areas and engineering applications.^{5,8} An effective and easy-to-use method for solving such equations is needed. Compared with the finite difference method and the finite element method, the spectral method² has higher accuracy and low efficiency. The wavelet analysis theory has been developed to be a powerful tool to solve the partial differential equations in recent years,^{9,11} which overcomes the shortcoming of the spectral method by introducing the multi-scale analysis theory. Many typical orthogonal polynomials such as Legendre,^{4,6,15,18} Chebyshev^{19,20} polynomial expansions have been employed to construct the orthogonal wavelets to solve various integro-differential equations, which improve the numerical accuracy and efficiency of the corresponding spectral methods. It is well known that Daubechies wavelet is also an orthogonal one, which is complicated to be used in the spectral method as it has no analytical expression. Shannon wavelet^{10,13} is another orthogonal one with simple analytical expression, but it has no compact support domain, which limits its application in spectral method. The use of windowed or truncated sinc functions is not recommended because these fail to satisfy the partition of unity; this has the disturbing consequence that the reconstruction error will not vanish as the sampling step tends to zero.

The numerical accuracy of the wavelet spectral method on the fractional PDEs is sensitive to the performances of the wavelet. Encouraged by the achievements on the wavelet spectral method in recent years, we develop a Shannon–Cosine wavelet by introducing the trigonometric polynomials to improve the performance of Shannon wavelet. Shannon–Cosine wavelet has the compact support domain and holds all the excellent properties of Shannon wavelet,^{1,17} especially the partition of unity. And so, it breaks the curse of the windowed sinc function. In addition, we have also developed a spectral method for solving the fractional PDEs¹⁶ by means of Shannon–Cosine wavelet.

This paper is outlined as follows. In Sec. 2, we are concerned with the construction of Shannon–Cosine wavelet. In Sec. 3, we present and proof the properties of Shannon–Cosine wavelet, which can illustrate the kernel function constructed in Sec. 1 satisfy the definition of the wavelet. In Sec. 4, the wavelet approximation error formula is given, and based on it, the optimization model on the support domain parameter N is constructed. This actually provides a series of Shannon–Cosine wavelets with different support domain and smoothness.

2. Construction of Shannon–Cosine Wavelet

2.1. Definition of Shannon–Cosine scaling function

The representation of Shannon scaling function¹⁹ is based upon approximating the Dirac delta function as a band-limited function and is given by

$$\phi(x) = \frac{\sin(x)}{x}. \quad (2.1)$$

As mentioned above, Shannon–Gabor scaling function is not really a compact support function. Indeed, both of the exponential function and Shannon scaling function that appeared in Eq. (2.1) have no compact support property. In order to overcome the two shortcomings of the Shannon–Gabor scaling function, we introduce the cosine function instead of the exponential function to modulate Shannon scaling function as follows:

$$\begin{aligned} S_C(x) &= \frac{\sin(\pi x)}{\pi x} T_N^m(x) R_N(x), \\ T_N^m(x) &= \sum_{n=0}^m \left(a_n \cos \left(\frac{2n\pi x}{N} \right) \right), \\ R_N(x) &= \chi \left(x + \frac{N}{2} \right) - \chi \left(x - \frac{N}{2} \right), \end{aligned} \quad (2.2)$$

where $R_N(x)$ is a rectangular pulse function on $[-N/2, N/2]$, N is a constant related to the support domain and $\chi(x)$ is the Heaviside function defined as follows:

$$\chi(x) = \begin{cases} 0 & : x < 0 \\ \frac{1}{2} & : x = 0 \\ 1 & : x > 0. \end{cases}$$

The rectangular pulse function $R_N(x)$ forces the new modulate Shannon scaling function (2.2) to be a compact function while destroying the continuity at the endpoints $x = \pm N/2$ and failing to satisfy the normalization of $S_C(x)$. The continuity at the endpoints and the normalization can be satisfied by choosing the parameters $a_i (i = 0, 1, \dots, m)$ and N , respectively. The choice of the parameter N relates to the waveform of $S_C(x)$, which will be discussed in Sec. 3. We first give the choice strategy on the parameters a_i in Sec. 2.2.

2.2. Choice of the parameter

The parameters $a_i (i = 0, 1, \dots, m)$ in Eq. (2.2) are employed to characterize the smoothness and the compact support properties of the function at the boundary

points $x = \pm N/2$, that is,

$$\left. \frac{d^n}{dx^n} S_C(x) \right|_{x=\frac{N}{2}} = 0, \quad n = 0, 1, \dots, m+1. \quad (2.3)$$

It is obvious that $S_C(x)$ is a continuous and differentiable function on the interval $[-N/2, N/2]$, but is not always continuous at the endpoints of the interval. Since the $\text{sinc}(x)$ is infinite differentiable and $R_N(x)$ is just a truncation function, Eq. (4) is equivalent to Eq. (2.4) as follows:

$$\left. \frac{d^n}{dx^n} T_N^m(x) \right|_{x=\frac{N}{2}} = 0, \quad n = 0, 1, \dots, m+1. \quad (2.4)$$

Being an interpolation function, we have

$$S_C(0) = 1. \quad (2.5)$$

Substituting $x = N/2$ (or $x = -N/2$) and $x = 0$ into Eqs. (2.4) and (2.2), we can obtain a linear algebraic equation system with respect to the parameters a_i ($i = 0, 1, \dots, m$) as follows:

$$\mathbf{MA} = \mathbf{B}, \quad (2.6)$$

where

$$\mathbf{M} = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 & \dots & 1 \\ (-1)^0 & (-1)^1 & (-1)^2 & \dots & (-1)^{\text{col}(\text{col}^2)^0} & \dots & (-1)^m(m^2)^0 \\ 0 & (-1)^1 1^1 & (-1)^2(2^2)^1 & \dots & (-1)^{\text{col}(\text{col}^2)^1} & \dots & (-1)^m(m^2)^1 \\ 0 & (-1)^1 1^2 & (-1)^2(2^2)^2 & \dots & (-1)^{\text{col}(\text{col}^2)^2} & \dots & (-1)^m(m^2)^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & (-1)^1 1^{\text{row}-1} & (-1)^2(2^2)^{\text{row}-1} & \dots & (-1)^{\text{col}(\text{col}^2)^{\text{row}-1}} & \dots & (-1)^m(m^2)^{\text{row}-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & (-1)^1 1^{m-1} & (-1)^2(2^2)^{m-1} & \dots & (-1)^{\text{col}(\text{col}^2)^{m-1}} & \dots & (-1)^m(m^2)^{m-1} \end{bmatrix},$$

$\text{row}(0, 1, \dots, m)$ and $\text{col}(0, 1, \dots, m)$ denote the row and column number of the matrix $\mathbf{M}$, respectively. Both $\mathbf{A} = (a_0, a_1, \dots, a_m)^T$ and $\mathbf{B} = (1, 0, \dots, 0)^T$ are $(m+1)$ dimensional column vectors. The solution of the matrix equation (2.6) is the value of the parameters in Shannon–Cosine wavelet function (2.2), which can be calculated by the recurrence formula as follows:

$$a_n = \frac{(-1)^n \prod_{i=1}^{n-1} i^2 a_0 - \sum_{k=n+1}^m \left((-1)^k \prod_{i=1}^{n-1} (k^2 - i^2) a_k \right)}{(-1)^n \prod_{i=1}^{n-1} (n^2 - i^2)}, \quad (2.7)$$

where a_0 can be obtained by solving the equation $\sum_{i=0}^m a_i = 1$. The value of the parameter a_i ($i = 1, 2, \dots$) with respect to different m is given in Table 1.

Table 1. Value of $a_i (i = 1, 2, \dots)$.

m	a_0	a_1	a_2	a_3	a_4	a_5	a_6	a_7
0	1							
1	1/2	1/2						
2	3/8	1/2	1/8					
3	5/16	15/32	3/16	1/32				
4	35/128	7/16	7/32	1/16	1/128			
5	63/256	105/256	15/64	45/512	5/256	1/512		
6	231/1024	99/256	495/2048	55/512	33/1024	3/512	1/2048	
7	429/2048	3003/8192	1001/4096	1001/8192	91/2048	91/8192	7/4096	1/8192

Fig. 1. Comparison between $\text{sinc}(x)$ and Shannon–Cosine scaling functions $S_C(x)$.

The differences between the functions $S_C(x)$ and $\text{sinc}(x)$ are shown in Fig. 1. Compared to the function $\text{sinc}(x)$, $S_C(x)$ possesses the compact support property and satisfies the partition of unity.

2.3. Fourier transform of Shannon–Cosine scaling function

In order to make Shannon–Cosine scaling function satisfy the partition of unity, we should choose the parameter N reasonably. The choice scheme is related to the Fourier transform of the scaling function.

Theorem 2.1 (Convolution theorem). Let $\mathcal{F}$ denote the Fourier transform operator, so $\mathcal{F}(f_1(x)) = F_1(\omega)$ and $\mathcal{F}(f_2(x)) = F_2(\omega)$, respectively. Then

$$\mathcal{F}(f_1(x)f_2(x)) = \frac{1}{2\pi} F_1(\omega) * F_2(\omega).$$

Based on the convolution theorem, we can deduce the Fourier transform of Shannon-Cosine function.

Let

$$t_N^n(x) = a_n \cos\left(\frac{2n\pi x}{N}\right) R_N(x),$$

the corresponding Fourier transform format is

$$\begin{aligned} \mathcal{F}(t_N^n(x)) &= \int_{-\infty}^{\infty} t_N^n(x) e^{-i\omega x} dx \\ &= a_n \int_{-\infty}^{\infty} \cos\left(\frac{2n\pi x}{N}\right) R_N(x) e^{-i\omega x} dx \\ &= \frac{(-1)^n 2a_n N^2 \omega}{N^2 \omega^2 - 4n^2 \pi^2} \sin\left(\frac{N\omega}{2}\right). \end{aligned}$$

According to the linear property of Fourier transform, we obtain

$$\begin{aligned} \mathcal{F}(T_N^m(x) R_N(x)) &= \sum_{n=0}^m \int_{-\infty}^{\infty} t_N^n(x) e^{-i\omega x} dx \\ &= \sum_{n=0}^m \frac{(-1)^n 2a_n N^2 \omega}{N^2 \omega^2 - 4n^2 \pi^2} \sin\left(\frac{N\omega}{2}\right). \end{aligned}$$

Fourier transform of $\text{sinc}(x)$ can be directly given as follows:

$$\mathcal{F}(\text{sinc}(\pi x)) = \chi(x + \pi) - \chi(x - \pi).$$

By means of the frequency domain convolution theorem, the Fourier transform format of the function $S_C(x)$ can be obtained as follows:

$$\begin{aligned} \hat{S}_C(\omega) &= \int_{-\infty}^{\infty} S_C(x) e^{-i\omega x} dx \\ &= \int_{-\infty}^{\infty} \left(\frac{\sin(\pi x)}{\pi x} T_N^m(x) R_N(x) \right) e^{-i\omega x} dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left((\chi(x + \pi) - \chi(x - \pi)) \right. \\ &\quad \left. \times \left(\sum_{n=0}^m \frac{(-1)^n 2a_n N^2 (\omega - w)}{N^2 (\omega - w)^2 - 4n^2 \pi^2} \sin\left(\frac{N(\omega - w)}{2}\right) \right) \right) dw \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{n=0}^m \frac{(-1)^n 2a_n N^2(\omega - w)}{N^2(\omega - w)^2 - 4n^2\pi^2} \sin\left(\frac{N(\omega - w)}{2}\right) \right) dw \\
 &= \frac{1}{\pi} \sum_{n=0}^m (-1)^n a_n \int_{-\pi}^{\pi} \left(\frac{N^2(\omega - w)}{N^2(\omega - w)^2 - 4n^2\pi^2} \sin\left(\frac{N(\omega - w)}{2}\right) \right) dw \\
 &= \frac{1}{\pi} \sum_{n=0}^m (-1)^n a_n \int_{-\pi-\omega}^{\pi-\omega} \left(\frac{N^2(w)}{N^2(w)^2 - 4n^2\pi^2} \sin\left(\frac{N(w)}{2}\right) \right) dw \\
 &= \frac{1}{2\pi} \sum_{n=0}^m a_n \left(\text{Si}\left(\frac{N(\omega + \pi)}{2} - n\pi\right) + \text{Si}\left(\frac{N(\omega + \pi)}{2} + n\pi\right) \right. \\
 &\quad \left. - \text{Si}\left(\frac{N(\omega - \pi)}{2} - n\pi\right) - \text{Si}\left(\frac{N(\omega - \pi)}{2} + n\pi\right) \right), \tag{2.8}
 \end{aligned}$$

where $\text{Si}(x)$ is the sine integral function which is defined for all complex x as follows:

$$\text{Si}(x) = \int_0^x \frac{\sin(t)}{t} dt. \tag{2.9}$$

In fact, the Shannon–Cosine scaling function has all the properties of the wavelet function, and possesses the interpolation property and compact support domain. According to the definition of interpolation wavelet, we can obtain the composite wavelets directly as follows:

$$\psi_C(t) = S_C\left(\frac{t-1}{2}\right) - 2S_C(2t-1).$$

3. Properties of Shannon–Cosine Scaling Function

There is no doubt that Shannon–Cosine scaling function can be viewed as the windowed or truncated sinc functions, which is not recommended because these fail to satisfy the partition of unity; this has the disturbing consequence that the reconstruction error will not vanish as the sampling step tends to zero. It is easy to understand that this can be overcome by means of the waveform of the sinc function. The sinc function has the waveform shown in Fig. 2, its support domain is $(-\infty, \infty)$, and it meets the normalization condition $\int_{-\infty}^{\infty} \phi(x) dx = 1$. It is easy to note that the integral value with respect to x of sinc function in domain I_1 is larger than 1 and smaller than 1 in domain I_2 . This reminds us that the reasonable choice of the support domain can ensure the Shannon–Cosine scaling function satisfied the partition of unity condition. In addition, according to the definition of the interpolative wavelets, the Shannon–Cosine scaling function $S_C \in L^2(\mathbb{R})$ should be a Riesz basis, and it should possess the regularity and orthogonality properties. In order to prove the above properties, we first discuss the properties of the sine integral function.

Fig. 2. Sinc function.

Theorem 3.1. According to the definition of the sine integral function $\text{Si}(x)$, the following statements hold:

- (1) $\text{Si}(x)$ is an odd function, i.e. $\text{Si}(-x) = -\text{Si}(x)$;
- (2) $\text{Si}(\infty) = \frac{\pi}{2}, 0 = 0$;
- (3) $\frac{\pi}{2} - \frac{1}{x} < \text{Si}(x) < \frac{\pi}{2} + \frac{1}{x}$, as $x > 0$
- (4) $\text{Si}(x) \in [\text{Si}(-\pi), \text{Si}(\pi)]$.

Proof. The proofs of the properties (1) and (2) are straightforward manipulations and we omit them. The proof of property (3) is complicated and we will prove it in Lemma 4. Here, we just give the proof of the property (4).

First, we prove that $x = (2k + 1)\pi, k \in \mathbb{Z}$ are the extreme points of $\text{Si}(x)$. Since

$$\text{Si}'(x) = \frac{d}{dx} \text{Si}(x) = \frac{\sin(x)}{x},$$

so, we have

$$\text{Si}'((2k + 1)\pi) = 0, \quad k \in \mathbb{Z},$$

and

$$\text{Si}'(x) > 0, \quad k \in (2k\pi, (2k + 1)\pi),$$

$$\text{Si}'(x) < 0, \quad k \in ((2k + 1)\pi, (2k + 2)\pi).$$

This indicates that $\text{Si}(2k + 1)\pi, k \in \mathbb{Z}$ is a sequence of maximum values of the function $\text{Si}(x)$.

Next, we are going to prove that $x = \pi$ is the global maximum value point of the function $\text{Si}(x)$.

As

$$\left| \frac{\sin(x + (2k + 1)\pi)}{x + (2k + 1)\pi} \right| > \left| \frac{\sin(x + (2k + 2)\pi)}{x + (2k + 2)\pi} \right|, \quad x \in [0, \pi], \quad k \in \mathbb{Z}. \quad (3.1)$$

So,

$$\int_0^\pi \left| \frac{\sin(x + (2k + 1)\pi)}{x + (2k + 1)\pi} \right| dx > \int_0^\pi \left| \frac{\sin(x + (2k + 2)\pi)}{x + (2k + 2)\pi} \right| dx, \quad x \in [0, \pi], \quad k \in \mathbb{Z}. \quad (3.2)$$

According to the definition of the sine integral function $\text{Si}(x)$, we obtain

$$\int_0^\pi \left| \frac{\sin(x + (2k + 1)\pi)}{x + (2k + 1)\pi} \right| dx = \text{Si}((2k + 1)\pi) - \text{Si}((2k + 2)\pi), \quad k \in \mathbb{Z}, \quad (3.3)$$

$$\int_0^\pi \left| \frac{\sin(x + (2k + 2)\pi)}{x + (2k + 2)\pi} \right| dx = \text{Si}((2k + 3)\pi) - \text{Si}((2k + 2)\pi), \quad k \in \mathbb{Z}. \quad (3.4)$$

Substituting Eqs. (3.3) and (3.4) into Eq. (3.2), we obtain

$$\text{Si}((2k + 3)\pi) - \text{Si}((2k + 2)\pi) < \text{Si}((2k + 1)\pi) - \text{Si}((2k + 2)\pi), \quad k \in \mathbb{Z}.$$

It can be simplified as

$$\text{Si}((2k + 1)\pi) > \text{Si}((2k + 3)\pi), \quad k \in \mathbb{Z}. \quad (3.5)$$

The inequality (3.5) implies that $\text{Si}(\pi)$ is the global maximum value of $\text{Si}(x)$.

As the sine integral function is an even one, so $\text{Si}(\pi)$ is in the value range $[\text{Si}(-\pi), \text{Si}(\pi)]$. This completes the proof. $\square$

Lemma 1. *Let*

$$f(x) = \text{Si}(x) - \left(\frac{\pi}{2} - \frac{1}{x} \right), \quad x > 0, \quad x_q \in ((2k + 1/2)\pi, (2k + 1)\pi), \quad k \in \mathbb{Z}, \quad k > 0.$$

The sequences of minimum and maximum points of the function $f(x)$ are $\{x_{2k} \in (2k\pi, x_q)\}$ and $\{x_{2k+1} \in (x_q, (2k + 1)\pi)\}$, respectively.

Proof. The extreme points $\{x_n, n \in \mathbb{Z}\}$ of $f(x)$ can be obtained by solving the equation as follows:

$$f'(x) = \frac{\sin(x)}{x} - \frac{1}{x^2} = 0. \quad (3.6)$$

That is, the extreme points x_n satisfy the equation $\sin(x_n) = \frac{1}{x_n}$. Let

$$q(x) = \sin - \frac{1}{x} = 0. \quad (3.7)$$

As $\sin(x)$ is a periodic function with the period 2π , and $f_1(x) = \frac{1}{x}$ is an odd function, we can consider only the case $x > 0$ without loss of generality. In this case, the function $f_1(x) = \frac{1}{x} > 0$ is a monotonically decreasing function. So, the sequence real roots of the equation $q(x) = 0$ must be included in each standard open interval of the sine function $\sin(x)$, i.e. $x \in (2k\pi, (2k + 1/2)\pi) \cup ((2k + 1/2)\pi, (2k + 1)\pi)$,

$k \geq 0$. Based on the property of the sine function, the solution interval can be divided into two open subintervals furthermore as follows:

Case 1. $x \in (2k\pi, (2k + 1/2)\pi)$

$$q'(x) = \cos(x) + \frac{1}{x^2} > 0. \quad (3.8)$$

This implies that $q(x)$ is a monotonically increasing function. In addition

$$q(2k\pi) = \sin(2k\pi) - \frac{1}{2k\pi} < 0,$$

and

$$q((2k + 1/2)\pi) = 1 - \frac{1}{\left(\frac{2k+1}{2}\right)\pi} > 0.$$

Therefore, the equation $q(x) = 0$ has a unique root as $x \in (2k\pi, (2k + 1/2)\pi)$. We denote this root as x_{2k} .

Case 2. $x \in ((2k + 1/2)\pi, (2k + 1)\pi)$.

It is obvious that $q'(x)$ is monotonically decreasing function $q''(x) = -\sin(x) - \frac{2}{x^3} < 0$ in this case. In addition,

$$q'(x) \Big|_{x=(2k+1/2)\pi} = \frac{1}{\left(\left(\frac{2k+1}{2}\right)\pi\right)^2} > 0, \quad (3.9)$$

$$q'(x) \Big|_{x=(2k+1)\pi} = \frac{1}{((2k+1)\pi)^2} - 1 < 0. \quad (3.10)$$

So, there exists a unique solution $x_q \in ((2k + 1/2)\pi, (2k + 1)\pi)$ to the equation $q'(x_q) = 0$. This implies that $q'(x) > 0$ as $x \in ((2k + 1/2)\pi, x_q)$. We have proved in case 1 that $q'(x) > 0$ as $x \in (2k\pi, (2k + 1/2)\pi)$.

In order to prove that the equation $q(x) = 0$ has a unique root in the open interval $((2k + 1/2)\pi, (2k + 1)\pi)$, we should prove $q(x_q) > 0$ in advance. From $q'(x_q) = 0$, we can obtain

$$\cos(x_q) = -\frac{1}{x_q^2},$$

$$q(x_q) = \sin(x_q) - \frac{1}{x_q} = \sqrt{1 - \frac{1}{x_q^4}} - \frac{1}{x_q}.$$

Since $x_q > (2k + 1/2)\pi > 1$, $q(x_q)$ can be rewritten as

$$q(x_q) = \frac{\sqrt{x_q^4 - 1} - x_q}{x_q^2}.$$

This implies that the inequality $q(x_q) > 0$ is established for any $x_q > \frac{1}{2}\sqrt{2+2\sqrt{5}}$. Indeed, $x_q > (2k + 1/2)\pi > \frac{1}{2}\sqrt{2+2\sqrt{5}}$, and so $q(x_q) > 0$ is established as $x \in (2k\pi, x_q)$. This implies that the function is a monotonically increasing function $q(x)$

in this open interval. In addition,

$$\begin{aligned} q(x) \Big|_{x=2k\pi} &= \sin(2k\pi) - \frac{1}{2k\pi} < 0, \\ q(x) \Big|_{x=x_q} &> 0. \end{aligned}$$

Therefore, the equation $q(x) = 0$ has a unique root as $x \in (2k\pi, x_q)$. We denote the root as x_{2k} . As $x \in (x_q, (2k+1)\pi)$, $q'(x_q) = 0$ and $q'(x)$ is a monotonically decreasing function. In addition,

$$q(x_q) > 0$$

$$q((2k+1)\pi) = \frac{1}{((2k+1)\pi)^2} - 1 < 0.$$

Therefore, the equation $q(x) = 0$ has a unique root as $x \in (x_q, (2k+1)\pi)$, which is denoted as x_{2k+1} .

So far, we have proved that the extreme sequence of the function $f(x)$ attained at x_{2k} and x_{2k+1} . At the extreme points, we have

$$f'(x) = \frac{\sin(x)}{x} - \frac{1}{x^2} = 0. \quad (3.11)$$

And so,

$$\begin{aligned} f''(x) &= \frac{\cos(x)}{x} - \frac{\sin(x)}{x^2} + \frac{2}{x^3} \\ &= \frac{\cos(x)}{x} + \frac{1}{x^3}. \end{aligned} \quad (3.12)$$

Since $q'(x_{2k}) = \cos(x_{2k}) + \frac{1}{x_{2k}^3} > 0$, we have

$$f''(x_{2k}) = \frac{\cos(x_{2k})}{x_{2k}} + \frac{1}{x_{2k}^3} > 0.$$

Therefore, $f(x_{2k}) = \text{Si}(x_{2k}) - (\frac{\pi}{2} - \frac{1}{x_{2k}})$ is a sequence of minimum of the function $f(x)$, $x \in (0, \infty)$.

Similarly, $f(x_{2k+1}) = \text{Si}(x_{2k+1}) - (\frac{\pi}{2} + \frac{1}{x_{2k+1}})$ is a sequence of maximum of the function $f(x)$, $x \in (0, \infty)$.

This completes the proof. $\square$

Lemma 2. At the sequence minimum points x_{2k} of the function $f(x) = \text{Si}(x) - (\frac{\pi}{2} - \frac{1}{x})$, the following inequality is established:

$$\int_{2k\pi}^{x_{2k}} \frac{\sin(x)}{x} dx < \frac{1}{2k\pi} - \frac{1}{x_{2k}}.$$

Proof. According to Lemma 1, we have $x_{2k} \in (2k\pi, 2k\pi + \pi/2)$, so

$$\int_{2k\pi}^{x_{2k}} \frac{\sin(x)}{x} dx < \frac{1}{2k\pi} \int_{2k\pi}^{x_{2k}} \sin(x) dx = \frac{1 - \cos(x_{2k})}{2k\pi} < \frac{1}{2k\pi} - \frac{\cos(x_{2k})}{x_{2k}}. \quad (3.13)$$

From Eq. (3.13), we obtain

$$\sin(x_{2k}) - \frac{1}{x_{2k}} = 0,$$

and so, we have

$$\cos(x_{2k}) = \sqrt{1 - \frac{1}{x_{2k}^2}}. \quad (3.14)$$

Substituting Eq. (3.14) into Eq. (3.13), we obtain

$$\int_{2k\pi}^{x_{2k}} \frac{\sin(x)}{x} dx < \frac{1}{2k\pi} - \frac{\cos(x_{2k})}{x_{2k}} = \frac{1}{2k\pi} - \frac{\sqrt{x_{2k}^2 - 1}}{x_{2k}^2} < \frac{1}{2k\pi} - \frac{1}{x_{2k}}.$$

This completes the proof. $\square$

Lemma 3. *The sine integration function $\text{Si}(x)$ satisfies the inequality as follows:*

$$\text{Si}((2k+2)\pi) - \text{Si}(2k\pi) < \frac{1}{2\pi} \left(\frac{1}{k} - \frac{1}{k+1} \right), \quad k \in \mathbb{Z}, \quad k > 0.$$

Proof.

$$\text{Si}((2k+2)\pi) - \text{Si}(2k\pi) = \int_{2k\pi}^{(2k+1)\pi} \frac{\sin(x)}{x} dx + \int_{(2k+1)\pi}^{(2k+2)\pi} \frac{\sin(x)}{x} dx, \quad (3.15)$$

$$\begin{aligned} \int_{2k\pi}^{(2k+1)\pi} \frac{\sin(x)}{x} dx &= \sum_{n=0}^{N-1} \left(\int_{(2k+n/N)\pi}^{(2k+(n+1)/N)\pi} \frac{\sin(x)}{x} dx \right) \\ &< \sum_{n=0}^{N-1} \left(\frac{1}{2k\pi + \frac{n\pi}{N}} \int_{n\pi/N}^{(n+1)\pi/N} \sin(x) dx \right), \quad N \in \mathbb{Z} \end{aligned} \quad (3.16)$$

$$\begin{aligned} \int_{(2k+1+n/N)\pi}^{(2k+1+(n+1)/N)\pi} \frac{\sin(x)}{x} dx &= \sum_{n=0}^{N-1} \left(\int_{(2k+n/N)\pi}^{(2k+(n+1)/N)\pi} \frac{\sin(x)}{x} dx \right) \\ &< \sum_{n=0}^{N-1} \left(\frac{-1}{(2k+1)\pi + \frac{(n+1)\pi}{N}} \int_{n\pi/N}^{(n+1)\pi/N} \sin(x) dx \right), \end{aligned}$$

$N \in \mathbb{Z}.$

(3.17)

Substituting Eqs. (3.17) and (3.16) into Eq. (3.15), we obtain

$$\begin{aligned} \text{Si}((2k+2)\pi) - \text{Si}(2k\pi) &< \sum_{n=0}^{N-1} \left(\frac{1}{2k\pi + \frac{n\pi}{N}} \int_{n\pi/15}^{(n+1)\pi/N} \sin(x) dx \right. \\ &\quad \times \left. \frac{1}{(2k+1)\pi + \frac{(n+1)\pi}{N}} \int_{n\pi/N}^{(n+1)\pi/N} \sin(x) dx \right) \\ &< \frac{1}{2\pi} \left(\frac{1}{k} - \frac{1}{k+1} \right), \quad (N \geq 15). \end{aligned} \quad (3.18)$$

Inequality (3.18) can be proved easily by mathematical induction with respect to the parameter “ k ”

This completes the proof. $\square$

Lemma 4. *The sine integration function $\text{Si}(x)$ satisfies the inequality as follows:*

$$\frac{\pi}{2} - \frac{1}{x} < \text{Si}(x) < \frac{\pi}{2} + \frac{1}{x}.$$

Proof. The values of the function $f(x) = \text{Si}(x) - (\frac{\pi}{2} - \frac{1}{x})$ at the sequence minimum points $\{x_{2k}, k \in \mathbb{Z}\}$ form a sequence $\{f(x_{2k}), k \in \mathbb{Z}\}$.

$$\begin{aligned} f(x_{2k+2}) &= \text{Si}(x_{2k+2}) - \left(\frac{\pi}{2} - \frac{1}{x_{2k+2}} \right) \\ &= \int_0^{x_{2k}} \frac{\sin(x)}{x} dx + \left(\int_{2k\pi}^{(2k+2)\pi} \frac{\sin(x)}{x} dx - \int_{2k\pi}^{x_{2k}} \frac{\sin(x)}{x} dx \right) \\ &\quad + \int_{(2k+2)\pi}^{x_{2k+2}} \frac{\sin(x)}{x} dx - \left(\frac{\pi}{2} - \frac{1}{x_{2k+2}} \right) \\ &= f(x_{2k}) + \left(\frac{\pi}{2} - \frac{1}{x_{2k}} \right) + \left(\int_{2k\pi}^{(2k+2)\pi} \frac{\sin(x)}{x} dx - \int_{2k\pi}^{x_{2k}} \frac{\sin(x)}{x} dx \right) \\ &\quad + \int_{(2k+2)\pi}^{x_{2k+2}} \frac{\sin(x)}{x} dx - \left(\frac{\pi}{2} - \frac{1}{x_{2k+2}} \right). \end{aligned} \quad (3.19)$$

It can be deduced from Lemma 2 that

$$- \int_{2k\pi}^{x_{2k}} \frac{\sin(x)}{x} dx < - \left(\frac{1}{2k\pi} - \frac{1}{x_{2k}} \right), \quad (3.20)$$

$$\int_{(2k+2)\pi}^{x_{2k+2}} \frac{\sin(x)}{x} dx < \frac{1}{(2k+2)\pi} - \frac{1}{x_{2k+2}}. \quad (3.21)$$

According to Lemma 3, we have

$$\int_{2k\pi}^{(2k+2)\pi} \frac{\sin(x)}{x} dx < \frac{1}{2\pi} \left(\frac{1}{k} - \frac{1}{k+1} \right). \quad (3.22)$$

Substituting inequalities (3.20)–(3.22) into Eq. (3.19), we obtain

$$\begin{aligned} f(x_{2k+2}) &< f(x_{2k}) + \frac{1}{2\pi} \left(\frac{1}{k} - \frac{1}{k+1} \right) \\ &\quad - \left(\frac{1}{2k\pi} - \frac{1}{x_{2k}} \right) + \frac{1}{(2k+2)\pi} - \frac{1}{x_{2k+2}} \\ &\quad + \left(\frac{\pi}{2} - \frac{1}{x_{2k}} \right) - \left(\frac{\pi}{2} - \frac{1}{x_{2k+2}} \right) = f(x_{2k}). \end{aligned}$$

So, $f(x_{2k})$ is a decreasing sequence of numbers as $x_{2k} > 0$, $k \in \mathbb{Z}$.

According to the definition of $f(x)$, we have

$$\lim_{x \rightarrow \infty} f(x) = \text{Si}(\infty) - \left(\frac{\pi}{2} \right) = 0.$$

This indicates that the sequence $\{f(x_{2k}), k \in \mathbb{Z}\}$ converges to 0, that is, $\lim_{k \rightarrow \infty} f(x_{2k}) = 0$. So, $f(x) > 0$, that is,

$$\frac{\pi}{2} - \frac{1}{x} < \text{Si}(x).$$

Similarly, it is easy to prove that

$$\text{Si}(x) < \frac{\pi}{2} + \frac{1}{x}.$$

This completes the proof. $\square$

In fact, it is easy to obtain the position of the extreme points by the interval bisection method, which is shown in Table 2.

Theorem 3.2. *The family of functions $\{\phi_k(x) = S_C(x - k)\}_{k \in \mathbb{Z}}$ forms a Riesz basis of $V(S_C)$.*

Proof. The definition of a Riesz basis is that there must exist two strictly positive constants $A > 0$ and $B < +\infty$ such that

$$\forall c(k) \in l_2, \quad A \cdot \|c\|_{l_2}^2 \leq \left\| \sum_{k \in F} c(k) \phi_k \right\|^2 \leq B \cdot \|c\|_{l_2}^2,$$

where $\|c\|_{l_2}^2 = \sum_{k \in \mathbb{Z}} |c(k)|^2$ is the squared l_2 -norm of $c(k)$. Because of the translation-invariant structure of the construction, the equivalent expression in Fourier domain of the Riesz basis requirement is²⁰

$$A \leq \sum_{k \in \mathbb{Z}} \left| \hat{\phi}(\omega + 2k\pi) \right|^2 \leq B, \quad \omega \in [0, 2\pi], \quad (3.23)$$

Table 2. Coordinate value of the extreme points.

k	x_{2k}	k	x_{2k}
0	$\left(\frac{11621}{32768}\pi, \frac{11622}{32768}\pi\right)$	17	$\left(34\pi + \frac{48}{16384}\pi, 34\pi + \frac{49}{16384}\pi\right)$
1	$\left(2\pi + \frac{813}{16384}\pi, 2\pi + \frac{814}{16384}\pi\right)$	18	$\left(36\pi + \frac{46}{16384}\pi, 36\pi + \frac{47}{16384}\pi\right)$
2	$\left(4\pi + \frac{412}{16384}\pi, 4\pi + \frac{413}{16384}\pi\right)$	19	$\left(38\pi + \frac{43}{16384}\pi, 38\pi + \frac{44}{16384}\pi\right)$
3	$\left(6\pi + \frac{276}{16384}\pi, 6\pi + \frac{277}{16384}\pi\right)$	20	$\left(40\pi + \frac{41}{16384}\pi, 40\pi + \frac{42}{16384}\pi\right)$
4	$\left(8\pi + \frac{207}{16384}\pi, 8\pi + \frac{208}{16384}\pi\right)$	21	$\left(42\pi + \frac{39}{16384}\pi, 42\pi + \frac{40}{16384}\pi\right)$
5	$\left(10\pi + \frac{165}{16384}\pi, 10\pi + \frac{166}{16384}\pi\right)$	22	$\left(44\pi + \frac{37}{16384}\pi, 44\pi + \frac{38}{16384}\pi\right)$
6	$\left(12\pi + \frac{138}{16384}\pi, 12\pi + \frac{139}{16384}\pi\right)$	23	$\left(46\pi + \frac{36}{16384}\pi, 46\pi + \frac{37}{16384}\pi\right)$
7	$\left(14\pi + \frac{118}{16384}\pi, 14\pi + \frac{119}{16384}\pi\right)$	24	$\left(48\pi + \frac{34}{16384}\pi, 48\pi + \frac{35}{16384}\pi\right)$
8	$\left(16\pi + \frac{103}{16384}\pi, 16\pi + \frac{104}{16384}\pi\right)$	25	$\left(50\pi + \frac{33}{16384}\pi, 50\pi + \frac{34}{16384}\pi\right)$
9	$\left(18\pi + \frac{92}{16384}\pi, 18\pi + \frac{93}{16384}\pi\right)$	26	$\left(52\pi + \frac{31}{16384}\pi, 52\pi + \frac{32}{16384}\pi\right)$
10	$\left(20\pi + \frac{82}{16384}\pi, 20\pi + \frac{83}{16384}\pi\right)$	27	$\left(54\pi + \frac{30}{16384}\pi, 54\pi + \frac{31}{16384}\pi\right)$
11	$\left(22\pi + \frac{75}{16384}\pi, 22\pi + \frac{76}{16384}\pi\right)$	28	$\left(56\pi + \frac{29}{16384}\pi, 56\pi + \frac{30}{16384}\pi\right)$
12	$\left(24\pi + \frac{69}{16384}\pi, 24\pi + \frac{70}{16384}\pi\right)$	29	$\left(58\pi + \frac{28}{16384}\pi, 58\pi + \frac{29}{16384}\pi\right)$
13	$\left(26\pi + \frac{63}{16384}\pi, 26\pi + \frac{64}{16384}\pi\right)$	30	$\left(60\pi + \frac{27}{16384}\pi, 60\pi + \frac{28}{16384}\pi\right)$
14	$\left(28\pi + \frac{59}{16384}\pi, 28\pi + \frac{60}{16384}\pi\right)$	31	$\left(62\pi + \frac{26}{16384}\pi, 62\pi + \frac{27}{16384}\pi\right)$
15	$\left(30\pi + \frac{55}{16384}\pi, 30\pi + \frac{56}{16384}\pi\right)$	32	$\left(64\pi + \frac{25}{16384}\pi, 64\pi + \frac{26}{16384}\pi\right)$
16	$\left(32\pi + \frac{51}{16384}\pi, 32\pi + \frac{52}{16384}\pi\right)$		

where $\hat{\phi}(\omega)$ is the Fourier transform of $\phi(x)$. By means of Eq. (2.8), we obtain

$$\begin{aligned} \hat{\phi}(2k\pi + \omega) = & \frac{1}{2\pi} \sum_{n=0}^m a_n \left(\text{Si} \left(\frac{N}{2} \cdot ((2k+1)\pi + \omega) + n\pi \right) \right. \\ & + \text{Si} \left(\frac{N}{2} \cdot ((2k+1)\pi + \omega) - n\pi \right) \\ & - \text{Si} \left(\frac{N}{2} \cdot ((2k-1)\pi + \omega) - n\pi \right) \\ & \left. - \text{Si} \left(\frac{N}{2} \cdot ((2k-1)\pi + \omega) + n\pi \right) \right), \quad m \in \mathbb{Z}. \end{aligned}$$

Let

$$\begin{aligned} p_1 &= \frac{N}{2} \cdot ((2k+1)\pi + \omega) + n\pi, & p_2 &= \frac{N}{2} \cdot ((2k+1)\pi + \omega) - n\pi, \\ p_3 &= \frac{N}{2} \cdot ((2k-1)\pi + \omega) - n\pi, & p_4 &= \frac{N}{2} \cdot ((2k-1)\pi + \omega) + n\pi. \end{aligned}$$

It is obvious that $p_i > 0$ ($i = 1, 2, 3, 4$) as $k \geq 1$ and $p_i < 0$ ($i = 1, 2, 3, 4$) as $k \leq -2$.

Case 1. $k \geq 1$

$$\begin{aligned}\pi - \frac{1}{p_1} - \frac{1}{p_2} &< \text{Si}(p_1) + \text{Si}(p_2) < \pi + \frac{1}{p_1} + \frac{1}{p_2}, \\ \pi - \frac{1}{p_3} - \frac{1}{p_4} &< \text{Si}(p_3) + \text{Si}(p_4) < \pi + \frac{1}{p_3} + \frac{1}{p_4}.\end{aligned}$$

Adding above two inequalities together, it can be obtained that

$$-\left(\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4}\right) < \text{Si}(p_1) + \text{Si}(p_2) - \text{Si}(p_3) - \text{Si}(p_4) < \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4}.$$

Case 2. $k \leq -2$

$$\begin{aligned}-\pi + \frac{1}{p_1} + \frac{1}{p_2} &< \text{Si}(p_1) + \text{Si}(p_2) < -\pi - \frac{1}{p_1} - \frac{1}{p_2}, \\ \pi + \frac{1}{p_3} + \frac{1}{p_4} &< -\text{Si}(p_3) - \text{Si}(p_4) < \pi - \frac{1}{p_3} - \frac{1}{p_4}.\end{aligned}$$

Adding above two inequalities together, it can be obtained that

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} < \text{Si}(p_1) + \text{Si}(p_2) - \text{Si}(p_3) - \text{Si}(p_4) < -\left(\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4}\right).$$

So, the inequality (3.23) holds if and only if the series

$$\sum_{k \in \mathbb{Z}, k \neq 0, \pm 1, \pm 2} \left(\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4}\right),$$

is convergent, which can be proved as follows:

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} = \frac{1}{N\pi} \left(\frac{1}{k+c_1} + \frac{1}{k+c_2} + \frac{1}{k+c_3} + \frac{1}{k+c_4} \right),$$

where

$$\begin{aligned}c_1 &= \frac{\omega}{2\pi} + \frac{n}{N} + \frac{1}{2}, & c_2 &= \frac{\omega}{2\pi} + \frac{n}{N} - \frac{1}{2} \\ c_3 &= \frac{\omega}{2\pi} - \frac{n}{N} + \frac{1}{2}, & c_4 &= \frac{\omega}{2\pi} - \frac{n}{N} - \frac{1}{2},\end{aligned}$$

Let $c = \min\{c_1, c_2, c_3, c_4\}$, we have

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} < \frac{4}{N\pi} \left(\frac{1}{k+c} \right) < \frac{4}{N\pi} \left(\frac{1}{k-1} \right),$$

and so, we have

$$\begin{aligned}
 |\hat{\phi}(2k\pi + \omega)|^2 &= \left| \frac{1}{2\pi} \sum_{n=0}^m a_n \left(\text{Si} \left(\frac{1}{2} N \cdot ((2k+1)\pi + \omega) + n\pi \right) \right. \right. \\
 &\quad \left. \left. + \text{Si} \left(\frac{1}{2} N \cdot ((2k+1)\pi + \omega) - n\pi \right) - \text{Si} \left(\frac{1}{2} N \cdot ((2k-1)\pi + \omega) - n\pi \right) \right. \right. \\
 &\quad \left. \left. - \text{Si} \left(\frac{1}{2} N \cdot ((2k-1)\pi + \omega) + n\pi \right) \right) \right|^2 \\
 &< \left| \frac{1}{2\pi} \sum_{n=0}^m a_n \left(\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} + \frac{1}{p_4} \right) \right|^2 \\
 &< \left(\frac{4}{N\pi^2} \right)^2 \left(\frac{1}{k-1} \right)^2,
 \end{aligned}$$

and so we can obtain the inequation as follows:

$$\begin{aligned}
 0 &\leq \sum_{k \in \mathbb{Z}} |\hat{\phi}(2k\pi + \omega)|^2 \\
 &\leq \sum_{k=-2}^2 |\hat{\phi}(2k\pi + \omega)|^2 \\
 &\quad + \left(\frac{4}{N\pi^2} \right)^2 \sum_{k \in \mathbb{Z}, k \neq 0, \pm 1, \pm 2} \left(\frac{1}{k-1} \right)^2, \quad \omega \in [0, 2\pi].
 \end{aligned}$$

It is obvious that the series $\sum_{k \in \mathbb{Z}, k \neq 0, \pm 1, \pm 2} \left(\frac{1}{k-1} \right)^2$ is convergence, and so the inequation (3.23) holds.

This completes the proof. $\square$

The Fourier transform of Shannon–Cosine function is shown in Fig. 3. It should be pointed out that it is not a strict compact support function although it decays rapidly. This results in some numerical errors as the Shannon–Cosine scaling function $S_C(x)$ is used to represent the signals to some extent.

In order to prove the interpolative property of Shannon–Cosine function, the definition of the discrete points should be given firstly. The nodes for the interpolation processes of this section are equally spaced on the interval $[x_{\min}, x_{\max}]$ with a dyadic step size, i.e.

$$x_n^j = x_{\min} + n \frac{x_{\max} - x_{\min}}{2^j}, \quad \forall j \in \mathbb{N}_0, \quad n = 0, 1, 2, \dots, 2^j.$$

In the following, the relations between the nodes on consecutive levels such as $x_n^j = x_{2n}^{j+1}$ will be exploited frequently.

Fig. 3. $\hat{\phi}(\omega)$, $m = 3$.

Definition 3.1. For any $j \in \mathbb{N}_0$, the translation format of Shannon–Cosine function is defined as

$$\phi_n^j(x) = \frac{\sin\left(\frac{\pi(x-x_n^j)}{\Delta}\right)}{\frac{\pi(x-x_n)}{\Delta}} \left(\cos\left(\frac{\pi(x-x_n^j)}{N\Delta}\right) \right)^2 \left[\chi\left(\frac{x-x_n^j}{\Delta} + \frac{N}{2}\right) - \chi\left(\frac{x-x_n^j}{\Delta} - \frac{N}{2}\right) \right],$$

where

$$x \in [x_{\min}, x_{\max}], \quad \Delta = \frac{x_{\max} - x_{\min}}{2^j}.$$

Definition 3.2. For any $j \in \mathbb{N}_0$, the spaces V_j are defined by

$$V_j := \text{span}\{\phi_n^j, n = 0, 1, \dots, 2^j\}$$

Definition 3.3. For any $j \in \mathbb{N}_0$, the interpolation operator I_j mapping any real-valued L^1 -integrable function f into the space V_j is defined as

$$I_j f(x) = \sum_{n=0}^{2^j} f(x_n^j) \phi_n^j(x).$$

The following properties of the operators I_j are therefore obvious:

- (1) $I_j f(x_n^j) = f(x_n^j)$,
- (2) $I_j f = f$, $\forall f \in V_j$.

Theorem 3.3 (Interpolative property of Shannon–Cosine function). *The following interpolation properties hold for each $k, n = 0, 1, 2, \dots, 2^j$:*

$$\phi_n^j(x_k^j) = \delta_{k,n}, \quad \left. \frac{d}{dx} \phi(x) \right|_{x=x_n^j} = 0.$$

Proof. By evaluating the definition of the function $\phi_n^j(x)$, the above interpolation properties can be established easily. $\square$

Besides the interpolative property, Shannon–Cosine function has the two-scale relations, which is the same as the sinc function. This has been proved in Theorem 3.4.

Theorem 3.4 (Two-scale relations). *Shannon–Cosine function $\phi(x)$ can be represented as a linear combination of dilates and translates of itself:*

$$\phi(x) = \sum_{n \in \mathbb{Z}} \phi\left(\frac{n}{2}\right) \phi(2x - n).$$

Proof. Using once again the fact that the interpolation operator I_j in Definition 3.3 is a projection for V_{j+1} , we find

$$\begin{aligned} \phi_n^j(x) &= I_{j+1}(\phi_n^j(x)) \\ &= \sum_{k=0}^{2^{j+2}+1} \phi_n^j(x_n^{j+1}) \phi_k^{j+1}(x). \end{aligned}$$

The interpolation properties of theorem 3.3 now imply the desired result. $\square$

All the above theorems ensure that the Shannon–Cosine scaling function $S_C(x)$ meets the requirements of the multi-scale interpolation function.

4. Approximation Error Analysis and Choice of Parameter N

Equation (2.2) is the definition of the composite interpolative wavelet function, the support domain is $[-N/2, N/2]$. The choice of the parameter N should keep that the composite function holds the normalization condition, so that the projection representation with the function $S_C(x)$ has smaller error.

As $T \rightarrow 0$ for all $f \in W_2^1$ (the set of L_2 -functions that are differentiable once), the conditions on ϕ such that the approximation error vanishes can be derived by

the asymptotic form of the error as follows:

$$\begin{aligned}\lim_{T \rightarrow 0} \|f - Q_T\{f\}\|^2 &= \lim_{T \rightarrow 0} \int_{-\infty}^{\infty} E_{\tilde{\phi}, \phi}(T\omega) |\hat{f}(\omega)|^2 \frac{d\omega}{2\pi} \\ &= \int_{-\infty}^{\infty} \lim_{T \rightarrow 0} E_{\tilde{\phi}, \phi}(T\omega) |\hat{f}(\omega)|^2 \frac{d\omega}{2\pi} \\ &= E_{\tilde{\phi}, \phi}(0) \|f\|^2.\end{aligned}\quad (4.1)$$

Equation (4.1) illustrates that the requirement $\forall f \in W_2^1, \lim_{T \rightarrow 0} \|f - Q_T\{f\}\|^2 = 0$ is equivalent to $E_{\tilde{\phi}, \phi}(0) = 0$. $E_{\tilde{\phi}, \phi}(\omega)$ is the error kernel given by

$$E_{\tilde{\phi}, \phi}(\omega) = |1 - \hat{\tilde{\phi}}(\omega)\hat{\phi}(\omega)|^2 + |\hat{\tilde{\phi}}(\omega)|^2 \sum_{k \neq 0} |\hat{\phi}(\omega + 2k\pi)|^2, \quad (4.2)$$

where $k \in \mathbb{Z}$, $\hat{\phi}(\omega)$ and $\hat{\tilde{\phi}}(\omega)$ are the Fourier transform of $\phi(x)$ and $\tilde{\phi}(x)$, respectively. And so,

$$E_{\tilde{\phi}, \phi}(0) = |1 - \hat{\tilde{\phi}}(0)\hat{\phi}(0)|^2 + |\hat{\tilde{\phi}}(0)|^2 \sum_{k \neq 0} |\hat{\phi}(2k\pi)|^2, \quad (4.3)$$

$\hat{\phi}(x)$ is the dual function of $\phi(x)$, that is,

$$\tilde{\phi}(x - m)\phi(x - n) = \delta_{m,n}, \quad m, n \in \mathbb{Z}.$$

It is obvious that $\tilde{\phi}(x) = \phi(x)$ if $\phi(x)$ denotes the Shannon–Cosine function, as it has the interpolative property. Equation (4.3) is a sum of positive terms, and it can obviously vanish only if all are zero. This means, the necessary and sufficient condition of the partition of unity should be

$$\hat{\phi}(0) = 1 \quad \text{and} \quad \hat{\phi}(2k\pi) = \delta_k, \quad k \in \mathbb{Z}. \quad (4.4)$$

Taking $\phi(x) = S_C(x)$, the condition $\hat{\phi}(2k\pi) = \delta_k$ is satisfied according to Eq. (4.4). Theorem 4.1 reveals the relation between the support domain parameter N and the first condition in Eq. (4.3).

Theorem 4.1. *Shannon–Cosine scaling function $S_C(x)$ satisfies the normalization condition, that is,*

$$\int_{-\infty}^{\infty} S_C(x) dx = 1.$$

Proof. Substituting $\omega = 0$ into Eq. (2.8), we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} S_C(x) dx &= \hat{S}_C(0) \\ &= \frac{1}{2\pi} \sum_{n=0}^m a_n \left(\text{Si} \left(\frac{(N\pi)}{2} - n\pi \right) + \text{Si} \left(\frac{(N\pi)}{2} + n\pi \right) \right)\end{aligned}$$

$$\begin{aligned} & -\operatorname{Si}\left(\frac{-N\pi}{2}-n\pi\right)-\operatorname{Si}\left(\frac{(-N\pi)}{2}+n\pi\right) \\ & =\frac{1}{\pi} \sum_{n=0}^m a_n\left(\operatorname{Si}\left(\frac{(N \pi)}{2}-n \pi\right)+\operatorname{Si}\left(\frac{(N \pi)}{2}+n \pi\right)\right) . \end{aligned}$$

It can be noticed that $\hat{S}_C(0)$ can be viewed as a continuous function with respect to the real number “ N ”, which is a parameter related to the length of the support domain. Let

$$\begin{aligned} P(N) & =\hat{S}_C(0) \\ & =\frac{1}{\pi} \sum_{n=0}^m a_n\left(\operatorname{Si}\left(\frac{(N \pi)}{2}-n \pi\right)+\operatorname{Si}\left(\frac{(N \pi)}{2}+n \pi\right)\right), \end{aligned}$$

and then, we have

$$\begin{aligned} P'(N) & =\frac{\mathrm{d} P(N)}{\mathrm{d} N} \\ & =\frac{1}{\pi} \sum_{n=0}^m a_n \frac{\mathrm{d}}{\mathrm{d} N}\left(\operatorname{Si}\left(\frac{(N \pi)}{2}-n \pi\right)+\operatorname{Si}\left(\frac{(N \pi)}{2}+n \pi\right)\right) \\ & =\frac{2 N}{\pi} \sin \left(\frac{N \pi}{2}\right) \sum_{n=0}^m a_n(-1)^n \frac{1}{N^2-4 n^2} . \end{aligned}$$

Let

$$g(N)=\sum_{n=0}^m a_n(-1)^n \frac{1}{N^2-4 n^2} .$$

According to Eq. (2.7), we know that $g(N) > 0$. So, the sign of $P'(N)$ is in accordance with one of the functions $\sin(N\pi/2)$. This implies that $P(N)$ is an oscillator function with respect to parameter ‘ N ’ around the constant 1, that is,

$$P(N)\left\{\begin{array}{l} >1: N=4 n, n \geq 2 \\ <1: N=4 n+2, n \geq 2 . \end{array}\right.$$

Obviously,

$$\frac{\mathrm{d} P(N)}{\mathrm{d} N}\left\{\begin{array}{l} <0: N \in(4 n, 4 n+2), n > 3 \\ =0: N=4 n, n \geq 3 \\ >0: N \in(4 n+2, 4 n+4), n > 3 . \end{array}\right.$$

That is, $N=4 n, N \in \mathbb{Z}, n \geq 3$ is in correspondence with the extreme point of the function $P(N)$. $P(N)$ is a monotone function on the intervals $(2 n, 2 n+1)$ and

Table 3. Solution of $P(N) = 1$.

7.437330346546332	9.323969185188616	11.261199765911442
13.219709116325248	15.189881678437814	17.167295939754695
19.149560460820794	21.135248098522425	23.123447719961405
25.113547757267952	27.105121575295925	29.097861737012863
31.091541469097137	33.085988700389862	35.081073641777039
37.076687097549438	39.072752714157104	41.069202184677124
43.065983057022095	45.063048839569092	47.060366630554199
49.057900428771973	51.055631637573242	53.053528785705566
55.051587104797363	57.049772262573242	59.048089981079102
61.046504974365234	63.045036315917969	65.043693542480469
67.042350769042969	69.041099548339844	

$(2n + 1, 2n + 2)$. Based on the mean value theorem, there must exist unique points $N/2 \in (2n, 2n + 1)$ and $N/2 \in (2n + 1, 2n + 2)$ such that to $P(N) = 1$, i.e.

$$P(N) = \hat{S}_C(0) = \int_{-\infty}^{\infty} S_C(x) dx = 1.$$

This completes the proof. □

As a matter of fact, by means of the interval bisection method, it is easy to obtain the value of N which is in correspondence with $P(N) = 1$ as shown in Table 3. In this section, a choice scheme of the parameter N is proposed. In fact, the partition of unity of Shannon–Cosine function is ensured by the scheme. So, it overcomes the shortcoming of other windowed *Sinc* function completely.

5. Shannon–Cosine Spectral Method on Fractional Fokker–Planck Equations

In order to testify the validity of Shannon–Cosine function constructed in this paper, a spectral method for the fractional PDEs is constructed by means of Shannon–Cosine function in this section. The solution of the fractional PDEs is sensitive to the numerical algorithm, and so it can be employed to test the numerical performance of Shannon–Cosine function.

Fractional Fokker–Planck equation is a typical fractional PDE, which is often used to describe a sub-diffusive behavior of a particle under the combined influence of external nonlinear force field, and a Boltzmann thermal heat bath. This section considers the accuracy and efficiency of the proposed method for a fractional Fokker–Planck equation. Comparisons are made with results obtained with Shannon wavelet, Shannon–Gabor wavelet solutions and the exact analytic solution.

Consider the Fokker–Planck equation as follows:

$$\frac{\partial u(x, t)}{\partial t} = {}_0 D_t^{1-\alpha} \left[\frac{\partial}{\partial x} (-1) + \frac{\partial^2}{\partial x^2} \right] u(x, t), \quad 0 \leq x \leq 1, \quad t > 0. \quad (5.1)$$

With the initial condition

$$u(x, 0) = x(1 - x), \quad 0 \leq x \leq 1,$$

and the boundary conditions

$$u(0, t) = -\frac{3t^\alpha}{\Gamma(1 + \alpha)} - \frac{2t^{2\alpha}}{\Gamma(1 + 2\alpha)}, \quad t > 0,$$

$$u(1, t) = -\frac{t^\alpha}{\Gamma(1 + \alpha)} - \frac{2t^{2\alpha}}{\Gamma(1 + 2\alpha)}, \quad t > 0,$$

the exact analytic solution is

$$u(x, t) = x(1 - x) + (2x - 3)\frac{t^\alpha}{\Gamma(1 + \alpha)} - \frac{2t^{2\alpha}}{\Gamma(1 + 2\alpha)}.$$

All the comparisons in this section are made qualitatively by comparing the calculation precision in the same time step and space mesh grid size. The measure of error e is given by

$$e = \frac{\sqrt{\frac{1}{2^j + 1} \sum_{i=0}^{2^j} (u(x_i) - u_{\text{exact}}(x_i))^2}}{u_{\text{exact}}(x_i)},$$

which provides a general measure of the accuracy of the solution over the main body of the distribution and is often used to investigate the accuracy of the FEM.

It has been pointed out that the finite difference approximation formats are not perfect as they don't take the boundary problems into account. According to the wavelet collocation method, the fractional Fokker–Planck equation (5.1) can be approximately represented as

$$\begin{aligned} & \tau^{-\alpha} \left[u_j(x_i, t_n) + \sum_{k=1}^{n-1} g_k u_j(x_i, t_{n-k}) - \sum_{k=0}^{n-1} g_k u_j(x_i, t_0) \right] \\ &= f'(x_i) u_j(x_i, t_n) + \sum_{m=0}^{2^j} u_j(x_m, t_n) [w'(x_i - x_m) + K_\alpha w''(x_i - x_m)], \end{aligned} \quad (5.2)$$

where w can be the Shannon–Cosine function $Sc(x)$ or Shannon wavelet and Shannon–Gabor wavelet, $i = 0, 1, 2, \dots, 2^j$. Let

$$\begin{aligned} V_j^n &= (u_j(x_0, t_n), u_j(x_1, t_n), \dots, u_j(x_{2^j}, t_n))^T \\ F &= \text{diag}(f'(x_0), f'(x_1), \dots, f'(x_{2^j})) \\ W_1 &= \begin{bmatrix} w'(x_0 - x_0) & w'(x_0 - x_1) & \cdots & w'(x_0 - x_{2^j}) \\ w'(x_1 - x_0) & w'(x_1 - x_1) & \cdots & w'(x_1 - x_{2^j}) \\ \dots & \dots & \dots & \dots \\ w'(x_{2^j} - x_0) & w'(x_{2^j} - x_1) & \cdots & w'(x_{2^j} - x_{2^j}) \end{bmatrix}, \end{aligned}$$

$$W_2 = \begin{bmatrix} w''(x_0 - x_0) & w''(x_0 - x_1) & \cdots & w''(x_0 - x_{2j}) \\ w''(x_1 - x_0) & w''(x_1 - x_1) & \cdots & w''(x_1 - x_{2j}) \\ \cdots & \cdots & \cdots & \cdots \\ w''(x_{2j} - x_0) & w''(x_{2j} - x_1) & \cdots & w''(x_{2j} - x_{2j}) \end{bmatrix}.$$

Then the system of equations (5.2) can be expressed as the matrix format:

$$(W_1 + K_\alpha W_2 + F - \tau^{-\alpha} I) V_j^n = \sum_{k=1}^{n-1} g_k V_j^{n-k} - \sum_{k=0}^{n-1} g_k V_j^0. \quad (5.3)$$

Next, we are discussing the numerical performances of the proposed composite function in this paper compared to Shannon wavelet and Shannon–Gabor wavelet methods. we use the spectral methods (Eq. (5.3)) constructed by these three functions to solve Eqs. (5.2), respectively. It is easy to see that only the proposed function possesses the compact support domain, this ensures that it has higher numerical efficiency compared to other two wavelets. So, we just discuss the numerical precision of these three methods in this section. Table 4 illustrates that Shannon–Gabor wavelet⁶ and the Shannon–Cosine wavelet numerical methods have better numerical stability compared to the Shannon wavelet. As $N = 11.261199765911442$, the numerical error obtained by the Shannon–Cosine spectral method is smaller than Shannon–Gabor wavelet method with the same parameters j and α . It has been proved that the decay rate of the Fourier transform of Shannon–Cosine wavelet increases with the increasing of the N value. This implies that the precision of proposed method possesses better numerical performance if we take the larger N value listed in Table 3. In addition, comparing with the Shannon–Gabor wavelet function, the Shannon–Cosine wavelet possesses the compact support domain, and this improves its numerical precision and efficiency.

It has been mentioned before that the choice of the support domain parameter N should meet the normalization requirement of the Shannon–Cosine function. Table 5 illustrates this is helpful to improve the numerical precision. If the parameter N is taken from Table 3, the numerical error is smaller evidently than taking the nearest integers, especially as N is a smaller real number.

Table 4. Comparison of error e among three wavelet spectral methods ($t = 0.1$, $\tau = 4.e - 5$).

α	j	Shannon wavelet	Shannon–Gabor wavelet	Shannon–Cosine wavelet ($N = 11.261199765911442$)
0.6	5	1.21169916725e–2	3.78197241630e–3	3.120464370709e–3
	6	3.19109092234e–2	3.95608831306e–3	2.883228914143e–3
	7	9.02960920008e–2	4.020250761399e–3	2.703047202213e–3
0.4	5	8.05655412813e–2	1.7424567613492e–2	1.7734367831187e–2
	6	1.86922737125e–1	1.7782829332785e–2	1.7214726118372e–2
	7	3.77208140261e–1	1.7965849038444e–2	1.6797420179495e–2

Table 5. Relation between the error e and the parameters α and $N(j = 6, t = 0.1, \tau = 4.e - 5)$.

N	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$
15	$1.1167799405158e-1$	$3.0970032906686e-2$	$9.728993145760e-3$
15.189881678437814	$6.1805437103529e-2$	$1.1690694727071e-2$	$1.768541025469e-3$
16	$7.0080443574596e-2$	$1.5193075905760e-2$	$3.362808203568e-3$
11	$2.5933136529558e-1$	$9.8219799292471e-2$	$4.792686304517e-2$
11.261199765911442	$9.8414740415106e-2$	$1.7214726118372e-2$	$2.883228914143e-3$
12	$2.2627667076402e-1$	$8.7669346423611e-2$	$5.451190258077e-2$
7	$7.07487344361926e-1$	$6.1282729694081e-1$	$4.5923502693985e-1$
7.437330346546332	$1.00047365307431e-1$	$1.2649825246784e-2$	$3.3429501895605e-2$
8	1.011767995338374	NaN	$1.82683661656795e15$

6. Conclusions

The Shannon–Cosine kernel function is actually a real wavelet function, which is different from Shannon–Gabor wavelet which can only be viewed as a quasi-wavelet as it destroys the partition of unity. The proposed kernel function is a compact support function, and preserves the interpolation property and satisfies the wavelet normalization requirement. Therefore, the proposed wavelet function overcomes the disadvantage of the general windowed or truncated *Sinc* functions, that is, the reconstruction error will not vanish as the sampling step tends to zero. In addition, the choice scheme of the support domain parameter N proposed in this paper is helpful to generalize window function to multi-scale cases and can be used to decrease the sampling errors, especially as the window width is smaller. The corresponding spectral solution of the fractional Fokker–Planck equation illustrates this point. Compared with the Shannon wavelet and the Shannon–Gabor wavelet, the proposed Shannon–Cosine spectral method has higher precision and efficiency. Obviously, the method can be developed to solve various PDEs in engineering.

In addition, sinc is a common sampling function.^{10,13} The scheme proposed in this paper also provided a novel multi-scale window for the windowed Fourier transform in signal processing. It is able to overcome the shortcoming of the general window function, that is, hold the partition of unity of sinc function and the Fourier kernel function.

Acknowledgments

This work is supported by the National Key Technology R & D Program of China (no. 2015BAH28F0103) and Beijing Natural Science Foundation (no. 4172034).

References

1. A. Aldroubi and M. Unser, Sampling procedures in function spaces and asymptotic equivalence with Shannon's sampling theory, *Numer. Funct. Anal. Optim.* **15**(1–2) (1994) 1–21.

2. C. Cattani, Fractional calculus and Shannon wavelet, *Math. Probl. Eng.* **2012** (2012) 1–26.
3. C. Cattani and E. Guariglia, Fractional derivative of the Hurwitz Zeta-Function and chaotic decay to zero, *J. King Saud Univ. Sci.* **28**(1) (2016) 75–81.
4. I. Celik, Chebyshev wavelet collocation method for solving generalized Burgers-Huxley equation, *Math. Methods Appl. Sci.* **39**(3) (2016) 366–377.
5. Y. Chen, X. Ke and Y. Wei, Numerical algorithm to solve system of nonlinear fractional differential equations based on wavelets method and the error analysis, *Appl. Math. Comput.* **251** (2015) 475–488.
6. Y. M. Chen, Y. Q. Wei and D. Y. Liu, Numerical solution for a class of nonlinear variable order fractional differential equations with Legendre wavelets, *Appl. Math. Lett.* **46** (2015) 83–88.
7. E. Guariglia, Entropy and fractal antennas, *Entropy* **18**(3) (2016) 1–17.
8. E. Guariglia, Spectral analysis of the Weierstrass-Mandelbrot function, in *Proc. 2nd Int. Multidisciplinary Conf. on Computer and Energy Science, Split, Croatia* (IEEE, Piscataway, NJ, USA, 2017), 6pp.
9. E. Guariglia and S. Silvestrov, Fractional-Wavelet analysis of positive definite distributions and wavelets on $D'(C)$, in *Engineering Mathematics II: Algebraic, Stochastic and Analysis Structures for Networks, Data Classification and Optimization*, Silvestrov, Rancic (Eds.), Vol. 179 (Springer Proceedings in Mathematics & Statistics, 2016), pp. 337–353.
10. D. K. Hoffman, G. W. Wei, D. S. Zhang and D. J. Kouri, Shannon wavelet distributed approximating functional, *Chem. Phys. Lett.* **287** (1998) 119–224.
11. S. Kestler, K. Steih and K. Urban, An efficient space-time adaptive wavelet Galerkin method for time-periodic parabolic partial differential equations, *Math. Comput.* **85**(299) (2016) 1309–1333.
12. S. Mashayekhi and M. Razzaghi, Numerical solution of the fractional Bagley-Torvik equation by using hybrid functions approximation, *Math. Methods Appl. Sci.* **39**(3) (2016) 353–365.
13. R. Metzler, E. Barkai and J. Klafter, Anomalous diffusion and relaxation close to thermal equilibrium: A fractional Fokker-Planck equation approach, *Phys. Rev. Lett.* **82** (1999) 3563–3567.
14. M. D. Ortigueira, A coherent approach to non-integer order derivatives, *Signal Process.* **10**(86) (2006) 2505–2515.
15. P. K. Sahu and S. Saha Ray, Legendre wavelets operational method for the numerical solutions of nonlinear Volterra integro-differential equations system, *Appl. Math. Comput.* **256** (2015) 715–723.
16. M. Tahami, A. Askari Hemmat and S. A. Yousefi, Numerical solution of two-dimensional first kind Fredholm integral equations by using linear Legendre wavelet, *Int. J. Wavelets Multiresolut. Inf. Process.* **14**(1) (2016) Article ID 1650004 (20 pages), doi: 10.1142/S0219691316500041.
17. M. Unser, Sampling — 50 years after Shannon, *Proc. IEEE* **88**(4) (2000) 3563–3567.
18. F. K. Yin, J. Q. Song and F. S. Lu, A coupled method of Laplace transform and Legendre wavelets for nonlinear Klein-Gordon equations, *Math. Methods Appl. Sci.* **37**(6) (2016) 781–792.
19. S. Zhi and F. Li, Numerical solution of high-order differential equations by using periodized Shannon wavelets, *Appl. Math. Model.* **38**(7–8) (2014) 2235–2248.
20. F. Zhou and X. Xu, Numerical solution of the convection diffusion equations by the second kind Chebyshev wavelets, *Appl. Math. Comput.* **247** (2014) 353–367.