Cubic arc-transitive Cayley graphs on Frobenius groups

Hailin Liu

*School of Mathematics and Statistics, Yunnan University
 Kunming 650091, P. R. China
 hailinliuqp@163.com*

Lei Wang*

*Beijing International Center for Mathematical Research
 Peking University, Beijing 100871, P. R. China
 wanglei-2468@163.com*

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A Cayley graph Γ is called *arc-transitive* if its automorphism group $\text{Aut}\Gamma$ is transitive on the set of arcs in Γ . In this paper, we give a characterization of cubic arc-transitive Cayley graphs on a class of Frobenius groups.

Keywords: Arc-transitive graph; Frobenius group; Cayley graph.

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1. Introduction

Throughout the paper, graphs considered are simple, connected and undirected. For a graph Γ , denoted by $V\Gamma$, $E\Gamma$, $A\Gamma$, $\text{val}(\Gamma)$ and $\text{Aut}\Gamma$ the vertex set, edge set, arc set, valency and full automorphism group of Γ , respectively. An s -arc of Γ is a sequence of vertices $(u_0, u_1, \dots, u_s)$ such that u_i is adjacent to u_{i+1} and $u_{i-1} \neq u_{i+1}$ for all possible i . For a subgroup $G \leq \text{Aut}\Gamma$, Γ is said to be (G, s) -arc-transitive if G is transitive on the set of s -arcs in Γ . In particular, a 0-arc is called vertex, a 1-arc is called an arc for short, and a $(G, 1)$ -arc-transitive graph is called G -arc transitive graph. As we all know, a graph Γ is G -arc-transitive if and only if G is transitive on $V\Gamma$ and the vertex stabilizer G_v of $v \in V\Gamma$ in G is transitive on the neighborhood $\Gamma(v)$ of v .

A graph Γ is called a *Cayley graph* if there exists a group G and a subset $S \subset G \setminus \{1\}$ with $S = S^{-1} := \{g^{-1} \mid g \in S\}$ such that the vertices of Γ may be

*Corresponding author.

identified with the elements of G in such a way that x is adjacent to y if and only if $yx^{-1} \in S$. The Cayley graph Γ is denoted by $\text{Cay}(G, S)$. Throughout this paper, denote by $\mathbf{1}$ the vertex of $\text{Cay}(G, S)$ corresponding to the identity of G .

A graph Γ is a Cayley graph on G if and only if $\text{Aut}\Gamma$ contains a subgroup which is regular on vertices and isomorphic to G . It is well-known that a Cayley graph $\text{Cay}(G, S)$ is vertex-transitive. However, a Cayley graph is of course not necessarily arc-transitive. Thus much excellent work has dealt with arc-transitive Cayley graphs. In particular, there are many results about cubic Cayley graphs. For instance, see [11, 16, 17] for cubic symmetric Cayley graphs on finite nonabelian simple groups, which are normal except for A_{47} , see [12] for a characterization of connected cubic s -transitive Cayley graphs, and see [5] for a classification of the connected arc-transitive cubic Cayley graphs on $\text{PSL}(2, p)$ where $p \geq 5$ is a prime. The objective of this paper is to give a characterization of cubic arc-transitive Cayley graphs on a class of Frobenius groups.

A group G is said to be a Frobenius group if G has the form $G = W:H$ such that $xy \neq yx$ for any $x \in W \setminus \{1\}$ and $y \in H \setminus \{1\}$. In particular, G is called a *primitive Frobenius group* if H acts irreducibly on W .

Theorem 1.1. *Let $G = W:H \cong \mathbb{Z}_p^d:\mathbb{Z}_n$ be a primitive Frobenius group, where p is a prime, and d, n are positive integers. Assume that Γ is a connected cubic X -arc-transitive Cayley graph on G , where $G \leq X \leq \text{Aut}\Gamma$. Then one of the following statements holds:*

- (i) G is normal in X , and $X_1 = \mathbb{Z}_3$ or S_3 ;
- (ii) $G \cong D_6$, $\Gamma \cong K_{3,3}$, and $\text{Aut}\Gamma \cong S_3 \wr S_2$;
- (iii) $\Gamma_W = K_{3,3}$, $G \cong \mathbb{Z}_p:\mathbb{Z}_6$ and $X = (W \times L):H$, where $L \cong \mathbb{Z}_3$, $\mathbf{Z}(X) = 1$, and Γ is constructed as in Construction 3.1;
- (iv) $\Gamma_W = K_{3,3}$, $G \cong \mathbb{Z}_p^2:\mathbb{Z}_6$ and $X = ((W \times L):H):M$, where $p \equiv -1 \pmod{3}$, $L \cong \mathbb{Z}_3$, $M \cong \mathbb{Z}_2$ or 1 , $\mathbf{Z}(X) = 1$, and Γ is constructed as in Construction 3.3;
- (v) Γ is isomorphic to one of the graphs listed in Table 1, and $(G, \text{Aut}\Gamma, (\text{Aut}\Gamma)_1)$ is as follows.

Remarks on Theorem 1.1. (a) The Cayley graph Γ in part (i) is called a *X-normal Cayley graph*. Furthermore, if $X = \text{Aut}\Gamma$, then Γ is called a *normal Cayley graph*, introduced in [15].

- (b) H acts irreducibly on W if and only if n does not divide $p^m - 1$ where $m < d$. We call such n a *primitive divisor* of $p^d - 1$, refer to [4, Proposition 2.3].

Table 1. Some exceptions.

Γ	Example 3.5	Example 3.6	Example 3.5
G	D_{14}	$\mathbb{Z}_{11}:\mathbb{Z}_{10}$	$\mathbb{Z}_{23}:\mathbb{Z}_{22}$
$\text{Aut}\Gamma$	$\text{PGL}(2, 7)$	$\text{PGL}(2, 11)$	$\text{PGL}(2, 23)$
$(\text{Aut}\Gamma)_1$	S_4	D_{12}	S_4

After this introductory section, some preliminary results are given in Sec. 2 and a few examples appearing in Theorem 1.1 are given in Sec. 3. Then we complete the proof of Theorem 1.1 in Sec. 4.

2. Preliminary Results

In this section, we quote some preliminary results, which will be used in the subsequent sections.

Let $\Gamma = (V, E)$ be a G -vertex-transitive graph. Then, for $\alpha \in V$, the stabilizer G_α is a core-free subgroup in G , that is, $\bigcap_{g \in G} G_\alpha^g = 1$. Set $H = G_\alpha$ and $D = \{x \mid \alpha^x \in \Gamma(\alpha)\}$, where $\Gamma(\alpha)$ is the set of neighbors of α in Γ . Then D is a union of several double cosets HxH . Moreover, Γ is isomorphic to the coset graph $\text{Cos}(G, H, D)$ defined over $\{Hx \mid x \in G\}$ with edge set $\{\{Hg_1, Hg_2\} \mid g_2g_1^{-1} \in D\}$.

The following statements for coset graphs are well known.

- (a) Γ is undirected if and only if $D = D^{-1} := \{x^{-1} \mid x \in D\}$.
- (b) Γ is connected if and only if $\langle H, D \rangle = G$.
- (c) Γ is G -arc-transitive if and only if $D = HgH$ for $g \in G$ with $g^2 \in H$; moreover, g can be chosen as a 2-element with $g \in N_G(H \cap H^g)$ and $g^2 \in H \cap H^g$.

For a Cayley graph $\Gamma = \text{Cay}(G, S)$. Let $\text{Aut}(G, S) = \{\alpha \in \text{Aut}(G) \mid S^\alpha = S\}$. Then we have the following basic result.

Lemma 2.1 ([7, Lemma 2.1]). *Let $\Gamma = \text{Cay}(G, S)$ be a Cayley graph. Then the normalizer $N_{\text{Aut}\Gamma}(G) = G:\text{Aut}(G, S)$.*

The vertex stabilizer for s -arc-transitive graphs of valency 3 is known.

Lemma 2.2 (Tutte). *Let Γ be a connected (G, s) -transitive cubic graph, where $s \geq 1$. Then $s \leq 5$ and the stabilizer G_u and $|G_u|$ satisfy the following table:*

s	1	2	3	4	5
G_u	$\mathbb{Z}_3$	S_3	D_{12}	S_4	$S_4 \times S_2$
$ G_u $	3	$2 \cdot 3$	$2^2 \cdot 3$	$2^3 \cdot 3$	$2^4 \cdot 3$

We end this section with two basic results for cubic arc-transitive graph.

Lemma 2.3. *Let G be a finite group. Let $\Gamma = \text{Cay}(G, S)$ be connected of valency 3. Assume that $\text{Aut}\Gamma$ has a subgroup X such that Γ is X -arc-transitive and $G \trianglelefteq X$. Then $X_1 = \mathbb{Z}_3$ or S_3 .*

Proof. Since Γ is connected, we have $G = \langle S \rangle$, and thus $\text{Aut}(G, S)$ acts faithfully on S . So $\text{Aut}(G, S) \lesssim S_3$. By Lemma 2.1, we have $X \leq N_{\text{Aut}\Gamma}(G) = G:\text{Aut}(G, S)$. Thus $X_1 \leq \text{Aut}(G, S) \lesssim S_3$. Since X_1 is transitive on S , it follows that $X_1 \cong \mathbb{Z}_3$ or S_3 . $\square$

Lemma 2.4. *Let Γ be a connected cubic X -arc-transitive graph, and let $N \triangleleft X$ such that X/N is insoluble. Then Γ is a normal cover of Γ_N .*

Proof. Let $u \in V\Gamma$. Let $B = u^N$ be an orbit of N acting on $V\Gamma$. By Lemma 2.2, X_u is a $\{2, 3\}$ -group, and hence X_u is soluble. Let K be the kernel of X acting on $V\Gamma_N$. Then K_u is soluble as $K_u \leq X_u$. By the Frattini argument, we have that $K = NK_u$. Note that $K/N \cong NK_u/N \cong K_u/(N \cap K_u)$, it follows that K/N is soluble. Since X/N is insoluble, $X/K \cong (X/N)/(K/N)$ is insoluble. Thus Γ is a normal cover of Γ_N . $\square$

3. Examples

In this section, we give some examples of graphs satisfying Theorem 1.1.

Construction 3.1. Let $G = W:H = \mathbb{Z}_p:\mathbb{Z}_6$ be a Frobenius group, where p is a prime. Let $X = (W \times L):H$, where $L \cong \mathbb{Z}_3$ and $\mathbf{Z}(X) = 1$. Write $H = \langle h \rangle$ and $L = \langle a \rangle$.

Let $N = \langle ah^2 \rangle$ and $g = wah^3$, where $1 \neq w \in W$. Set

$$\Gamma = \text{Cos}(X, N, NgN).$$

The next lemma shows that the graphs constructed here are as required.

Lemma 3.2. *Let Γ be a graph constructed in Construction 3.1. Then Γ is a connected cubic X -arc-transitive Cayley graph on G , and G is not normal in X . In particular, Γ is a cover of Γ_W , and $\Gamma_W \cong K_{3,3}$.*

Proof. By the definition, N is core-free in X , and hence $X \leq \text{Aut}\Gamma$. Now $X = GN$ and $G \cap N = 1$, and thus G acts regularly on the vertex set $[X:N]$. So Γ is a Cayley graph of G . Obviously, G is not normal in X .

Let $Y = \langle N, g \rangle$. Let $n = ah^2$. Since $\mathbf{Z}(X) = 1$, we have $a^h = a^{-1}$. Then $g^n = w^{h^2}a^{-1}h^3$, and $g^ng = w^{h^2}w^{-1}a^{-2}$. Let $x = w^{h^2}w^{-1}$. Since h acts fixed-point-free on W , it follows that $x \neq 1$. Clearly, $(g^ng)^3 = x^3 \neq 1$. Thus $x \in Y$, and $W \leq Y$. So $X = Y$. It is easy to show that $N \cap N^g = 1$. Therefore, Γ is a connected cubic X -arc-transitive Cayley graph. Since Γ is of valency 3, Γ is a cover of Γ_W . Clearly, Γ_W is a Cayley graph of G/W . By [1, Theorem 1.2], we obtain $\Gamma_W \cong K_{3,3}$. $\square$

Construction 3.3. Let $G = W:H = \mathbb{Z}_p^2:\mathbb{Z}_6$ be a Frobenius group, where $p \equiv -1 \pmod{3}$ is a prime. Let $X = (W \times L):H$, where $L \cong \mathbb{Z}_3$ and $\mathbf{Z}(X) = 1$. Write $H = \langle h \rangle$ and $L = \langle a \rangle$.

Let $N = \langle ah^2 \rangle$ and $g = wah^3$, where $1 \neq w \in W$. Set

$$\Gamma = \text{Cos}(X, N, NgN).$$

Arguing similarly as Lemma 3.2, we have the following lemma.

Lemma 3.4. *Let Γ be a graph constructed in Construction 3.3. Then Γ is a connected cubic X -arc-transitive Cayley graph on G , and G is not normal in X . In particular, Γ is a cover of Γ_W , and $\Gamma_W \cong K_{3,3}$.*

Applying [13, Examples 3.6 and 3.8], we have the following two examples.

Example 3.5. Let $T = \text{PSL}(2, p)$, and let $X = \text{PGL}(2, p)$, where $p = 7$ or 23 . Take a subgroup $H \cong S_4$ of T , and let $K \cong D_8$ be a Sylow 2-subgroup of H . Then $N_G(K) = D_{16}$ and let $g \in N_G(K) \setminus K$ be an involution such that $\langle H, g \rangle = X$. Let $G = D_{14}$ or $\mathbb{Z}_{23}:\mathbb{Z}_{22}$. Then the coset graph $\Gamma = \text{Cos}(X, H, HgH)$ is a Cayley graph on G with full automorphism group $\text{PGL}(2, p)$.

Example 3.6. Let $T = \text{PSL}(2, 11)$, and let $X = \text{PGL}(2, 11)$. Take a subgroup $H \cong D_{12}$ of T , and let $K \cong \mathbb{Z}_2$ be a Sylow 2-subgroup of H . Then $N_G(K) = S_4$. Let $g \in N_G(K) \setminus K$ be an involution such that $\langle H, g \rangle = X$. Then the coset graph $\Gamma_1 = \text{Cos}(X, H, HgH)$ is a Cayley graph on $\mathbb{Z}_{11}:\mathbb{Z}_{10}$ with full automorphism group $\text{PGL}(2, 11)$.

4. Proof of Theorem 1.1

In this section, we will prove Theorem 1.1 by a series of lemmas.

Let $G = W:H \cong \mathbb{Z}_p^d:\mathbb{Z}_n$ be a primitive Frobenius group, where p is a prime, and d, n are integers. Assume that $\Gamma = \text{Cay}(G, S)$ is a connected cubic X -arc-transitive Cayley graph on G , where $G \leq X \leq \text{Aut}\Gamma$. First of all, we study the case where the automorphism group X is soluble.

For $G = D_{2p}$, we have the following lemma, see [13, Theorem 1.1].

Lemma 4.1. *Let G be the dihedral group of order $2p$, and let Γ be a connected cubic X -arc-transitive Cayley graph on G . Set $G = \langle a, b \rangle$ with $o(a) = p$, $o(b) = 2$, and $a^b = a^{-1}$. Assume that X is soluble. Then $\Gamma = \text{Cay}(G, S)$ for one of the following subsets S of G .*

- (i) $p = 3$ and $S = \{ab, a^2b, b\}$; in this case, $\Gamma \cong K_{3,3}$;
- (ii) $p > 3$ and $S = \{ab, a^kb, b\}$, where $k^2 - k + 1 \equiv 0 \pmod{p}$; in this case, $\text{Aut}\Gamma \cong D_{2p}:\mathbb{Z}_3$.

Lemma 4.2. *Let $B = S_4$. Then $B \not\leq \text{GL}(2, p)$, where p is a prime.*

Proof. If $p = 2$, then the statement follows. In the following, assume that p is odd.

Let $H = \text{GL}(2, p)$, and $L = \text{SL}(2, p)$. Suppose that $B \leq H$. By [8, Theorem 8.3, p. 42], L has a unique involution. Thus $B \cap L = 1$. It follows that $B/(B \cap L) \lesssim H/L \lesssim \mathbb{Z}_{p-1}$, which is a contradiction. Thus $B \not\leq H$. $\square$

Let F be the Fitting subgroup of X , that is, F is the largest nilpotent normal subgroup of X . Then $F \neq 1$ and $C_X(F) \leq F$ as X is soluble, and $F = \langle \mathbf{O}_p(X) \mid p \mid |X| \rangle$.

For a group H and a prime p , by H_p we mean a Sylow p -subgroup of H .

Lemma 4.3. *If $G \not\cong D_{2p}$, then W is normal in X .*

Proof. We first show that $G \cap F \neq 1$. Suppose that $G \cap F = 1$. Since $X = GX_u$, we have $|F| \mid |X_u|48$, see Lemma 2.2, where $u \in V\Gamma$.

Assume that F is transitive on $V\Gamma$. Then $|G| \mid |F|$. Since H acts irreducibly on W , we conclude that $G \cong \mathbb{Z}_2^3:\mathbb{Z}_3$. However, there exists no cubic symmetric graphs of order 12, see [6, Theorem 6.2]. Similarly, we also exclude the case where $\Gamma_F = K_2$.

Thus Γ is a normal cover of Γ_F . Then $|F|$ divides $|G|$. Since $C_G(F) \leq F$, we conclude that $C_G(F) = 1$. This implies that F is noncyclic. Thus $2 \mid |F|$.

Suppose that $p = 2$. By the previous paragraph, we have $n \geq 5$. Assume that $\Phi(F) = 1$. Since $|F|$ divides both 48 and $|G|$, we obtain $F_2 \cong \mathbb{Z}_2^k$ and $F_{2'} \cong \mathbb{Z}_3$ or 1, where $k \leq \gcd(4, d)$. Note that $C_X(F) \leq F$, it follows that $G \leq \text{GL}(k, 2)$. By Atlas [3], there exists no k and G satisfying the above relation. Thus $\Phi(F) \neq 1$. Let $\overline{F} = F/\Phi(F)$ and $\overline{H} = H\Phi(F)/\Phi(F)$. Then $\overline{H} \cong H$. If $C_{\overline{H}}(\overline{F}) \neq 1$, then $C_H(F) \neq 1$, see [8, Theorem 1.4, p. 174]. Thus $C_{\overline{H}}(\overline{F}) = 1$. Clearly, $|\overline{F}_2| \leq 8$. By Atlas [3], $\overline{F}_2 \cong \mathbb{Z}_2^3$ and $\overline{H} \cong \mathbb{Z}_7$. For this case $d = 3$, and $|F_2| = 16$, which is a contradiction.

Suppose that p is odd. Assume $\Phi(F) = 1$. Then $F_2 \cong \mathbb{Z}_2^k$, where $k \leq 4$ and $2^k \mid n$. As $C_X(F) \leq F$, we have $G \leq \text{GL}(k, 2)$. It follows that $W \leq \text{GL}(k, 2)$, and $\mathbb{Z}_{2^k} \leq \text{GL}(k, 2)$. By Atlas [3], there exists no k and W satisfying the above relation. Thus $\Phi(F) \neq 1$. Since $F_3 \leq \mathbb{Z}_3$, we conclude that $\Phi(F) \leq F_2$. Let $\overline{W} = W\Phi(F)/\Phi(F)$ and $\overline{F}_2 = F_2/\Phi(F)$. If $C_{\overline{W}}(\overline{F}_2) \neq 1$, then $C_W(F_2) \neq 1$, refer to [8, Theorem 1.4, p. 174]. Thus $C_{\overline{W}}(\overline{F}_2) = 1$. Clearly, $\overline{F}_2 \cong \mathbb{Z}_2^3$. By Atlas [3], this is impossible.

To sum up, $F \cap G \neq 1$. Since W is minimal in G , we conclude that $W \leq F$. Recall that $G \not\cong D_{2p}$, it follows that Γ is a cover of Γ_{F_p} , and thus $W = F_p$. That is to say, W is normal in X . This completes the proof. $\square$

Lemma 4.4. *With the hypothesis of Lemma 4.3, then one of the following statements holds:*

- (a) G is normal in X , and $X_u = \mathbb{Z}_3$ or S_3 , where $u \in V\Gamma$;
- (b) $\Gamma_W = K_{3,3}$, $G \cong \mathbb{Z}_p:\mathbb{Z}_6$, and $X = (W \times L):H$, where $L \cong \mathbb{Z}_3$, and $\mathbf{Z}(X) = 1$;
- (c) $\Gamma_W = K_{3,3}$, $G \cong \mathbb{Z}_p^2:\mathbb{Z}_6$, and $X = ((W \times L):H):M$, where $p \equiv -1 \pmod{3}$, $L \cong \mathbb{Z}_3$, $M \cong \mathbb{Z}_2$ or 1, and $\mathbf{Z}(X) = 1$.

Proof. By Lemma 4.3, W is normal in X . Since $G \not\cong D_{2p}$, we have $n > 2$. Then Γ is a normal cover of Γ_W . By [1, Theorem 1.2], either $G/W \trianglelefteq X/W$ or $\Gamma_W = K_4$ or $K_{3,3}$. In the former case, $G \trianglelefteq X$. By Lemma 2.3, $X_u = \mathbb{Z}_3$ or S_3 , where $u \in V\Gamma$. If $\Gamma_W = K_4$, then $G/W \cong \mathbb{Z}_4$. Note that Γ is X -arc-transitive, we conclude that $X \cong \mathbb{Z}_p^2:S_4$. Since $H \cong \mathbb{Z}_4$, we have $X/W \cong S_4$ and $X/W \leq \text{GL}(2, p)$, a contradiction, see Lemma 4.2.

Now assume that $\Gamma_W = K_{3,3}$. Then $G/W \cong \mathbb{Z}_6$. Since G is a Frobenius group, p is coprime to 6. Note that Γ is X -arc-transitive, we obtain that 3 divides $|X_u|$ by Lemma 2.2. Thus $X_3 \cong \mathbb{Z}_3^2$. By Schur–Zassenhaus’s Theorem, $X = W:(X_3:X_2)$, where $X_3:X_2 \lesssim S_3 \wr S_2$. If $X_2 \cong D_8$, then X_3 is a unique minimal normal subgroup of $X_3:X_2 \cong S_3 \wr S_2$. Since $H \cong \mathbb{Z}_6$, we conclude that $C_{X_3}(W) = 1$. It follows

that $W \cong \mathbb{Z}_p^2$, and $X_3:X_2 \lesssim \text{GL}(2, p)$. Recall that H acts irreducibly on W , we have $p \equiv -1 \pmod{3}$. By [9, Theorem 7.3, p. 187], $C_{\text{GL}(2, p)}(H_3) \cong \mathbb{Z}_{p^2-1}$. However $C_X(H_3) \geq X_3$, a contradiction. Thus $X_2 = \mathbb{Z}_2^\ell$, where $\ell = 1$ or 2 . By the above argument, we also obtain that $C_{X_3}(W) \cong \mathbb{Z}_3$. Recall that $H \cong \mathbb{Z}_6$, and H acts irreducibly on W , it follows that $W \cong \mathbb{Z}_p$ or $\mathbb{Z}_p^2$. For the former, $p \equiv 1 \pmod{3}$, and for the latter, $p \equiv -1 \pmod{3}$.

Assume that $G \cong \mathbb{Z}_p:\mathbb{Z}_6$. Then $X_2 \cong \mathbb{Z}_2$, and so $X = (W \times L):H$, where $L \cong \mathbb{Z}_3$. Write $\Gamma = \text{Cos}(X, X_u, X_u g X_u)$, where $X_u \cong \mathbb{Z}_3$, $u \in V\Gamma$ and $o(g) = 2$. Let $H = \langle h \rangle$. By Sylow's Theorem, we may assume $Y_u = \langle lh^2 \rangle$, where $l \in L$. If h centralizes L , then $g = wh^3$, where $w \in W$. It is easily shown that $\langle Y_u, g \rangle = W:\langle l^2 h \rangle < X$. So Γ is not connected. Thus $\mathbf{Z}(X) = 1$.

Assume that $G \cong \mathbb{Z}_p^2:\mathbb{Z}_6$. Then $X_2 \cong \mathbb{Z}_2$ or $\mathbb{Z}_2^2$. Thus $X = ((W \times L):H):M$, where $L \cong \mathbb{Z}_3$ and $M \cong \mathbb{Z}_2$ or 1 . Arguing as above, L does not centralize H . Since S_3^2 has no abelian subgroup of order 12 , it follows that M does not centralize H_3 . So $\mathbf{Z}(X) = 1$.

This completes the proof of Lemma 4.4. $\square$

In the remaining section, we study the case where the automorphism group X is insoluble. Denote by R the *radical* of X , that is, R is the largest soluble normal subgroup of X . We will deal with the case where $R = 1$.

Lemma 4.5. *If $R = 1$, then, for $u \in V\Gamma$, $(G, \text{Aut}\Gamma, (\text{Aut}\Gamma)_u)$ is one of the triples listed in Table 1.*

Proof. Let N be a minimal normal subgroup of X . Since $R = 1$, we have $N = T_1 \times \cdots \times T_\ell \cong T^\ell$, where $T_i \cong T$ is nonabelian simple. By [10], T is one of the following:

$\text{PSL}(4, 2)$, $\text{PSU}(3, 8)$, M_{11} , $\text{PSp}(4, 3)$, $\text{PSL}(3, q) (q < 9)$, $\text{PSL}(2, q) (q > 3)$.

Let $W_i = T_i \cap W$, where $1 \leq i \leq \ell$. Let $L = N \cap H$. Since G is a Frobenius group, it follows that L is a diagonal subgroup of N . Write $L = \langle h_1 \cdots h_\ell \rangle$, where $\langle h_i \rangle \cong L$. Let $H_i = \langle h_i \rangle$ and $G_i = W_i:H_i$, where $1 \leq i \leq \ell$. Then G_i is a Frobenius group. Let $m = \frac{|T_1|}{|G_1|} \prod_{i=2}^\ell \frac{|T_i|}{|W_i|}$. Since $N \cap G = W:L$, and $\frac{|N|}{|N \cap G|}$ divides $|X_u|$, it follows that $m = \frac{|N|}{|N \cap G|}$, and $m \mid |X_u|$.

Suppose that $T = \text{PSL}(4, 2)$. By Atlas [3], $G_i \lesssim A_7$. Clearly, 105 does not divide $|G_i|$. So m does not divide $|X_u|$, which is a contradiction. Arguing similarly as above, we exclude the cases where $T = \text{PSU}(3, 8)$, M_{11} , $\text{PSp}(4, 3)$, and $\text{PSL}(3, q) (q < 9)$.

Suppose that $T = \text{PSL}(2, q)$ where $q = r^e$. According to [14, Theorem 6.25], G_i is a subgroup of one of the following groups:

$$\mathbb{Z}_r^e:\mathbb{Z}_{\frac{(r^e-1)}{d}}, D_{\frac{2(r^e \pm 1)}{d}}, A_5 \quad \text{and} \quad \text{PGL}(2, r^f),$$

where $d = (2, r - 1)$, and $f \mid e$. In the following, we process our analysis by several cases.

Case 1: Suppose that $G_i \leq \mathbb{Z}_r^e : \mathbb{Z}_{\frac{(r^e-1)}{d}}$. If $\ell > 1$, then $(q+1)^2(q-1)$ divides 48 since m divides $|X_u|$. It follows that $(q+1)^2$ divides 16, and thus $q \leq 3$, which is a contradiction. Thus $\ell = 1$, namely, $N = T$. For this case, $q+1$ divides 48. Calculation shows that $r^e = q = 5, 7, 11$ or 23. Since $W = \mathbb{Z}_r$, we conclude that $W = \mathbb{Z}_5, \mathbb{Z}_7, \mathbb{Z}_{11}$ or $\mathbb{Z}_{23}$.

Assume that $W = \mathbb{Z}_5$. Then $G = \mathbb{Z}_5 : \mathbb{Z}_2$ or $\mathbb{Z}_5 : \mathbb{Z}_4$. By [2, Table 1], there exists no cubic Cayley graphs of order 10, the former case does not occur. For the latter case $\text{Aut}(\text{Cay}(G, S))$ is soluble, this case is also excluded, see [18, Lemma 2.2].

Assume that $W = \mathbb{Z}_7, \mathbb{Z}_{11}$ or $\mathbb{Z}_{23}$. Then G is one of the following

$$D_{14}, \quad \mathbb{Z}_7 : \mathbb{Z}_6, \quad D_{22}, \quad \mathbb{Z}_{11} : \mathbb{Z}_{10}, \quad D_{46}, \quad \mathbb{Z}_{23} : \mathbb{Z}_{22}.$$

Thus $|G|$ is square-free. By [13, Theorem 1.1], the triple $(G, \text{Aut}\Gamma, (\text{Aut}\Gamma)_u)$ lies in Table 1.

Case 2: Suppose that $G_i \leq D_{\frac{2(r^e+1)}{d}}$. Arguing as above, $N = T$. If $G_i \leq D_{\frac{2(r^e-1)}{d}}$, then $q(q+1)/d$ divides 48, which is a contradiction since $q > 3$. Thus $G_i \leq D_{\frac{2(r^e+1)}{d}}$. For this case, $q(q-1)/d$ divides 48. Calculation shows $q = 4$, and then $W = \mathbb{Z}_5$. By Case 1, this case is also excluded.

Case 3: Suppose that $G_i \leq A_5$. Then $G_i = D_{10}$. Arguing as Case 1, we obtain that $\ell = 1$. Since $15 \mid q(q^2-1)$, it follows that $5 \mid q, 5 \mid q-1$ or $5 \mid q+1$.

Suppose that $5 \mid q$. Since $\frac{|X|}{|G_1|}$ divides $|X_u|$, we conclude that $q = 5$. By Case 1, this case does not occur. Similarly, we also exclude other cases.

Case 4: Suppose that $G_i \leq \text{PGL}(2, r^f)$. Let $\hat{r} = \frac{r^{e-f}(r^{2e}-1)}{r^{2f}-1}$. Then $\hat{r}$ divides 48. If $e-f = 1$, then $e = 2$ and $f = 1$ as $f \mid e$. However, $\hat{r}$ does not divide 48, a contradiction. Thus $e > f+1$, and so $r = 2$. Then $\frac{(r^{2e}-1)}{r^{2f}-1} = 3$. Clearly, this is a contradiction. $\square$

In what follows, we will show $R = 1$.

Lemma 4.6. *The radical $R = 1$.*

Proof. Suppose that $R \neq 1$. Let $L = R \cap G$. Suppose that $L \neq 1$. Since W is minimal in G , it follows that $W \leq L$. Since X/R is insoluble, Γ is a normal cover of Γ_R , see Lemma 2.4.

Let $\overline{G} = GR/R$. Then $\overline{G} \cong G/(G \cap R)$ is cyclic since $W \leq R$. Since Γ is a Cayley graph of G , Γ_R is a connected Cayley graph of $\overline{G}$. By [1, Theorem 1.2], we obtain either $\Gamma_R \cong K_4$ or $K_{3,3}$. In either case, $\text{Aut}\Gamma_R$ is soluble. Since $X/R \leq \text{Aut}\Gamma_R$, X/R is soluble. So is X , which is a contradiction. Thus $L = 1$.

Let $\overline{G} = GR/R$. Since $L = 1$, we have $\overline{G} \cong G$. Let $\overline{X} = X/R$. Then $\overline{X} = \overline{G} \overline{X}_u$. Arguing as the proof of Lemma 4.5 with $\overline{X} = \overline{G} \overline{X}_u$ in place of $X = GX_u$, $\overline{X}$ is almost simple, and $\overline{G} \cong D_{14}, \mathbb{Z}_{11} : \mathbb{Z}_{10}$ or $\mathbb{Z}_{23} : \mathbb{Z}_{22}$. By Table 1, X is almost simple, which is a contradiction. Thus $R = 1$. $\square$

The assertion of Theorem 1.1 follows from Lemmas 4.1 and 4.4–4.6.

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References

- [1] Y. G. Baik, Y. Q. Feng, H. S. Sim and M. Y. Xu, On the normality of Cayley graphs of abelian groups, *Algebra Colloq.* **5** (1998) 297–304.
- [2] Y. Cheng and J. Oxley, On weakly symmetric graphs of order twice a prime, *J. Combin. Theory Ser. B* **42** (1987) 196–211.
- [3] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of Finite Groups* (Cambridge Press, Oxford, 1985).
- [4] A. S. Detinko and D. L. Flannery, Nilpotent primitive linear groups over finite fields, *Comm. Algebra*. **33** (2005) 497–505.
- [5] S. F. Du and F. R. Wang, Arc-transitive cubic Cayley graphs on $\text{PSL}(2, p)$, *Sci. China A* **48** (2005) 1297–1308.
- [6] Y. Q. Feng and J. H. Kwak, Cubic symmetric graphs of order a small number times a prime or a prime square, *J. Combin. Theory Ser. B* **97** (2007) 627–646.
- [7] C. D. Godsil, On the full automorphism group of a graph, *Combinatorica* **10** (1981) 243–256.
- [8] D. Gorenstein, *Finite Groups* (Harper and Row, 1968).
- [9] B. Huppert, *Endliche Gruppen I* (Springer, Berlin, 1967).
- [10] L. S. Kazarin, Groups that can be represented as a product of two solvable subgroups, *Comm. Algebra* **14** (1986) 1001–1066.
- [11] C. H. Li, Isomorphisms of finite Cayley graphs, Ph.D. thesis, The University of Western Australia (1996).
- [12] J. J. Li and Z. P. Lu, Cubic s -arc transitive Cayley graphs, *Discrete Math.* **209**(20) (2009) 6014–6025.
- [13] C. H. Li, Z. P. Lu and G. X. Wang, Vertex-transitive cubic graphs of square-free order, *J. Graph Theory* **75** (2014) 1–19.
- [14] M. Suzuki, *Group Theory I* (Springer-Verlag, Berlin, 1982).
- [15] M. Y. Xu, Automorphism groups and isomorphisms of Cayley digraphs, *Discrete Math.* **182** (1998) 309–319.
- [16] S. J. Xu, X. G. Fang, J. Wang and M. Y. Xu, On cubic s -arc transitive Cayley graphs of finite simple groups, *European J. Combin.* **26** (2005) 133–143.
- [17] S. J. Xu, X. G. Fang, J. Wang and M. Y. Xu, 5-Arc transitive cubic Cayley graphs on finite simple groups, *European J. Combin.* **28** (2007) 1023–1036.
- [18] C. X. Zhou and Y. Q. Feng, Automorphism groups of connected cubic Cayley graphs of order $4p$, *Algebra Colloq.* **14**(2) (2007) 351–359.