ERRATUM TO: TIGHT CONTACT STRUCTURES ON LENS SPACES

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The Proof of Lemma 3.6 in [1] is incorrect but all the main theorems in the paper are still correct. Lemma 3.6 can still be proven for the lens spaces $L(p, q)$ when $q = 1$ or $p - 1$. (We include the proof for these cases at the end of the paper since [2] used Lemma 3.6 when $q = p - 1$.) However, the lemma may be bypassed in the essential arguments in [1]. Recall that the prime use of Lemma 3.6 was to show that we could assume that the generalized projective plane D in $L(p, q)$ (*i.e.* the two skeleton) had at most p elliptic points in its characteristic foliation. This is still true:

Lemma 1. *We may (topologically) isotop D in $L(p, q)$ so that $e_+(D) \leq p$.*

Note that with this lemma in hand Theorem 4.3, Corollary 4.5 and Theorem 4.10 from [1] are true and their proofs are essentially unchanged.

Proof. We assume that we have isotoped D so that $e_+(D)$ is minimal. Now if $e_+(D) > p$ then we derive a contradiction.

Recall that by choosing a point x on the one skeleton C of $L(p, q)$ we break ∂D into p intervals $B_1, \dots, B_p$ using the p points $x_1, \dots, x_p$ on ∂D that map to x when D is glued to C . We say that an interval I on ∂D is longer than k if for any choice of x the interior of the interval I contains at least k of the x_i 's.

Note by [1, Remark 3.9] we know that if we look at an outermost hyperbolic singularity on D (*i.e.* one whose unstable manifolds separate off a disk Δ containing one elliptic point) then $\Delta \cap \partial D$ is longer than 1. Now consider a hyperbolic point h whose unstable manifolds separate off a disk Δ from D that contains two elliptic points and one hyperbolic point h' . We claim that $I = \Delta \cap \partial D$ is longer than 2. To see this let $I = I_c \cup I_m \cup I_a$ where I_m and the unstable manifolds of h' bound a disk on D . We know that I_m is longer than 1 so if I is not longer than 2 then both I_c and I_a are not longer than 1. Moreover, in this situation it is not hard to see that the intervals I_c and I_a are disjoint when mapped to C . Thus just as in [1, Fig. 9] we may construct an overtwisted disk by extending the unstable manifolds of h and h' across C .

The above argument generalizes to prove: if h is a hyperbolic point whose unstable manifolds separate off a disk Δ containing a linear graph in the characteristic foliation containing k elliptic points, then $\Delta \cap \partial D$ is longer than k . There is one complication in this case that is not seen above. If h' is the hyperbolic point in Δ whose unstable manifolds separate off a disk Δ' containing $k - 1$ elliptic points, then by induction on k we can assume that $\Delta' \cap \partial D$ is longer than $k - 1$. So if our claim is not true then $I_c \cup I_a = S \cap \partial D$ is not longer than 1 and I_c and I_a are disjoint when mapped to C , where $S = \overline{\Delta \setminus \Delta'}$. As above we would like to say that we can construct an overtwisted disk as in [1, Fig. 9]. Unfortunately one of the intervals, say I_a , might not limit to a single elliptic point when pushed across C (strangely enough this does not happen in the situation above), but if this happens the other interval will limit to a single elliptic point when pushed across C . So if K is the union of all leaves in D_ξ that intersect I_c and end at a fixed elliptic point, then we can find a disk B close to $S \cup K$ such that $B \cap C$ is a neighborhood of I_a on C and B contains a single hyperbolic point h whose unstable manifolds intersect C in ∂I_a . Note we might have $B \cap D \neq \emptyset$ but then we may (topologically) isotope D keeping C fixed so that it has the same number and type of singularities and is disjoint from B (away from C). Now as in the proof of [1, Lemma 3.8] we may use this disk B to decrease $e_+(D)$ by p (contradicting the minimality of $e_+(D)$). Combining this argument with the one in the proof of [1, Theorem 4.6] one may easily remove the word “linear” from the above statement.

We now know that $e_+(D) \leq p$ since the total “length” of ∂D is p . □

Theorem 4.6 (and hence Corollary 4.7 and Theorems 4.8 and 4.9) in [1] follow from the above proof as follows: Note that under the hypothesis of Theorem 4.6 there must be p elliptic and $p - 1$ hyperbolic singularities in D_ξ . If we take an outermost hyperbolic point h then its unstable manifolds separate D into two disks Δ_0 and Δ_1 and by the remark at the end of the above proof $I_0 = \Delta_0 \cap \partial D$ is longer than 1 while $I_1 = \Delta_1 \cap \partial D$ is longer than $p - 1$. Now take a point x on C that is in the intersection of C with the unstable manifolds of h . The interior of I_0 contains at least 1 of the points x_i while the interior of I_1 contains at least $p - 1$ of the points x_i . That means that at least p of the x_i ’s are accounted for on the interiors of the respective intervals but one of the x_i ’s is on the boundary of both intervals. This contradicts the fact that there are only p , x_i ’s. Thus there can be no tight contact structure satisfying the hypothesis of Theorem 4.6.

We have now demonstrated that the main theorems in [1] are correct; but, in order to repair a gap in [2] caused by the incorrect proof of [1, Lemma 3.6] we show that the lemma is indeed correct in the cases relevant to [2].

Proof of [1, Lemma 3.6] for $q=1$ or $p-1$. We will show how to isotope D to a disk D' with transverse boundary in ∂V_1 , whose graph of singularities relates to D ’s as shown in [1, Fig. 2]. Since the graph of singularities in D_ξ must be a tree, a sequence of such moves will clearly yield the conclusion of the lemma.

Assume that part of the graph of singularities in D_ξ is as shown on the left hand side of [1, Fig. 2] and let h be the hyperbolic singularity whose stable separatrix we wish to move. The unstable separatrices of h cut D into two pieces: one, Δ , containing only one elliptic singularity e and one, $D \setminus \Delta$, containing all the other singularities. Let U be a (closed) neighborhood, in V_1 , of $\partial V_1 \cup \Delta$ for which $U \cap D$ contains only the singularities h and e . We may assume that U is diffeomorphic to $(\partial V_1) \times [0, 1]$ and that the characteristic foliations on both boundary components of ∂U are non singular. Moreover we need $D \cap U$ to have transverse boundary. Here is where we must assume that $q = 1$ or $p - 1$ since $(\partial U) \setminus (\partial V_1)$ will naturally have four singularities that need to be canceled. We would like to do this cancellation in the complement of $D \cap U$ so as to keep it transverse. By taking V_0 to be a sufficiently small neighborhood of its core C when $q = 1$ or $p - 1$ we may achieve this (as the reader may easily verify by looking at the monodromy on ∂D induced by the characteristic foliation of ∂V_1). Now let $A = D \cap U$, $D' = \overline{D \setminus A}$, $c = \partial D'$ and x be the intersection of c with the stable separatrix of h that we wish to move. Note there is a region $R \subset c$ such that if D'_ξ is glued to A_ξ via a diffeomorphism $\psi : c \rightarrow c$ that takes x into R the resulting singular foliation is as seen on the right hand side of [1, Fig. 2]. We now show how to “realize” such a diffeomorphism by isotoping A .

Let T' be the boundary component of U that lies in the interior of V_1 . Since the characteristic foliation of T' is non singular and contains no leaves parallel to c (since ξ is tight) we may use T'_ξ , thought of as a flow, to define a Poincaré return map $\phi : c \rightarrow c$. We can assume that ϕ has irrational rotation number since by isotoping T' in the neighborhood of a meridional curve (away from D) we will change ϕ . During this isotopy the rotation numbers for the corresponding ϕ 's will change and thus at some point be irrational.

If we cut T' along c we get annulus A' . Gluing one boundary component of A' to $c \subset A$ and the other to $c \subset D'$ and rounding corners we have a new meridional disk D_1 whose characteristic foliation is D'_ξ glued to A_ξ via ϕ . (Note that using Makar-Limanov's corner rounding method [3] one can round both corners above without altering the topological type of the characteristic foliation.) It is useful to think of D_1 as obtained from D by pushing part of the interior of D once around V_1 .

If ϕ does not take x into R then we do not have the desired characteristic foliation. But since ϕ has irrational rotation number the orbit of x under ϕ is dense in c . Thus there is some power, say n , of ϕ that will take x into R . If we take n disjoint copies of T' then we can do the above procedure using all n copies of T' to obtain a disk D_n . Of course the characteristic foliation on D_n will not be exactly D'_ξ glued to A_ξ via ϕ^n since the Poincaré return maps on the copies of T' are not exactly ϕ . But if we take the copies of T' to be sufficiently close to T' then the gluing map will be close enough to ϕ^n to still take x into R . Thus D_n will be the desired new meridional disk whose characteristic foliation is related to D 's as seen in [1, Fig. 2]. $\square$

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References

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