Corrigendum

The $\forall \exists$ theory of Peano Σ_1 sentences

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We would like to correct an unfortunate error in the statement and applications of [1, Fact 1.1(a)]. Here is the corrected version of that fact.

Fact 1.1(a) bis. *Consider a function $\varphi : \mathcal{J}(\ell) \rightarrow L$, where ℓ and L are nontrivial bounded distributive lattices, ℓ finite. Then φ extends to a bounded homomorphism $\tilde{\varphi} : \ell \rightarrow L$ if and only if*

- (i) $\bigvee_{a \in \mathcal{J}(\ell)} \varphi(a) = 1$,
- (ii) $a \leq b \Rightarrow \varphi(a) \leq \varphi(b)$ for all $a, b \in \mathcal{J}(\ell)$,
- (iii) for all $a, b \in \mathcal{J}(\ell)$ one has

$$\varphi(a) \wedge \varphi(b) \leq \bigvee_{\mathcal{J}(\ell) \ni c \leq a \wedge b} \varphi(c).$$

The extension $\tilde{\varphi}$ is unique.

Here $\mathcal{J}(\ell)$ is the set of nonzero $\vee$ -irreducible elements of ℓ . A *bounded* lattice possesses 0 and 1. *Bounded* homomorphisms preserve those.

In [1], Fact 1.1(a) appeared without clause (iii). With that clause absent, the statement admits a simple counterexample: Consider the five-element distributive lattice ℓ together with the function $\varphi : \mathcal{J}(\ell) \rightarrow \ell$,

where the elements of $\mathcal{J}(\ell) = \{a, b_0, b_1\}$ are marked by triangles, $\varphi(a) = a$, and $\varphi(b_i) = 1$. Then φ satisfies clauses (i) and (ii), but φ cannot be extended to any endomorphism $\tilde{\varphi}$ of ℓ , let alone a bounded one, for, if it could, then

$$a = \tilde{\varphi}(a) = \tilde{\varphi}(b_0 \wedge b_1) = \tilde{\varphi}(b_0) \wedge \tilde{\varphi}(b_1) = 1.$$

Clause (b) of [1, Fact 1.1] remains unaffected.

There are two applications of [1, Fact 1.1(a)] — one in Construction 4.1, p. 267, and one in Construction 5.1, p. 270. Let us explain why the extra clause (iii) of Fact 1.1(a) bis holds in both cases.

The role of the function φ is played by functions $\sigma : \mathcal{J}(m) \rightarrow \Sigma_1/T$, where $m = \ell$ in Construction 4.1 and $m = k$ in Construction 5.1 are nontrivial finite distributive lattices. For each $a \in \mathcal{J}(m)$, the Σ_1 sentence σ_a is defined by

$$\sigma_a \equiv \exists x \, h(x) \leq a,$$

where $h : \omega \rightarrow \mathcal{J}(m)$ are certain elementary functions ($h = f$ in Construction 4.1 and $h = g$ in Construction 5.1), and the ordering relation in $h(x) \leq a$ is the partial order on $\mathcal{J}(m)$ inherited from m . Furthermore, the functions h are T -provably decreasing with respect to the ordering on $\mathcal{J}(m)$, i.e.

$$T \vdash \forall x \forall y \geq x \, h(y) \leq h(x).$$

This is Lemma 4.2(a) for $h = f$ and condition (G1(n)) in Construction 5.1 for $h = g$. Therefore,

$$\begin{aligned} T \vdash \sigma_a \wedge \sigma_b &\leftrightarrow \exists x \, (h(x) \leq a \wedge h(x) \leq b) \\ &\leftrightarrow \bigvee_{\mathcal{J}(m) \ni c \leq a \wedge b} \sigma_c. \end{aligned}$$

Thus clause (iii) of Fact 1.1(a) bis holds, and σ extends to a homomorphism $\tilde{\sigma} : m \rightarrow \Sigma_1/T$ in both instances.

In [1], Fact 1.1 went without proof. We now include a detailed proof of Fact 1.1(a) bis even though it is quite elementary.

Proof of Fact 1.1(a) bis. (Only if) Suppose $\tilde{\varphi} : \ell \rightarrow L$ is a bounded homomorphism extending φ . Then

$$1 = \tilde{\varphi}(1) = \tilde{\varphi}\left(\bigvee \mathcal{J}(\ell)\right) = \bigvee_{a \in \mathcal{J}(\ell)} \tilde{\varphi}(a) = \bigvee_{a \in \mathcal{J}(\ell)} \varphi(a),$$

for any nonzero element of the finite lattice ℓ , including 1, is a join of $\vee$ -irreducibles, establishing clause (i). Clause (ii) is immediate, while for (iii) we consider $a, b \in \mathcal{J}(\ell)$

and note that

$$\varphi(a) \wedge \varphi(b) = \tilde{\varphi}(a \wedge b) = \tilde{\varphi} \left(\bigvee_{\mathcal{J}(\ell) \ni c \leq a \wedge b} c \right) = \bigvee_{\mathcal{J}(\ell) \ni c \leq a \wedge b} \tilde{\varphi}(c) = \bigvee_{\mathcal{J}(\ell) \ni c \leq a \wedge b} \varphi(c).$$

(If) Define $\tilde{\varphi}(x) = \bigvee_{\mathcal{J}(\ell) \ni c \leq x} \varphi(c)$. For $a \in \mathcal{J}(\ell)$ we have

$$\tilde{\varphi}(a) = \bigvee_{\mathcal{J}(\ell) \ni c \leq a} \varphi(c) = \varphi(a)$$

by clause (ii), so $\tilde{\varphi}$ extends φ . Further,

$$\tilde{\varphi}(x \vee y) = \bigvee_{\mathcal{J}(\ell) \ni c \leq x \vee y} \varphi(c) = \bigvee_{\mathcal{J}(\ell) \ni c \leq x} \varphi(c) \vee \bigvee_{\mathcal{J}(\ell) \ni c \leq y} \varphi(c) = \tilde{\varphi}(x) \vee \tilde{\varphi}(y)$$

because $c \leq x \vee y$ implies $c \leq x$ or $c \leq y$ for $\vee$ -irreducible c . As for meets, we first observe that

$$\tilde{\varphi}(x \wedge y) = \bigvee_{\mathcal{J}(\ell) \ni c \leq x \wedge y} \varphi(c) \leq \bigvee_{\mathcal{J}(\ell) \ni c \leq x} \varphi(c) \wedge \bigvee_{\mathcal{J}(\ell) \ni c \leq y} \varphi(c) = \tilde{\varphi}(x) \wedge \tilde{\varphi}(y).$$

Next, consider $a, b \in \mathcal{J}(\ell)$ such that $a \leq x$ and $b \leq y$. By clause (iii),

$$\varphi(a) \wedge \varphi(b) \leq \bigvee_{\mathcal{J}(\ell) \ni c \leq a \wedge b} \varphi(c) \leq \bigvee_{\mathcal{J}(\ell) \ni c \leq x \wedge y} \varphi(c) = \tilde{\varphi}(x \wedge y).$$

Hence,

$$\tilde{\varphi}(x) \wedge \tilde{\varphi}(y) = \bigvee_{\mathcal{J}(\ell) \ni a \leq x} \varphi(a) \wedge \bigvee_{\mathcal{J}(\ell) \ni b \leq y} \varphi(b) = \bigvee_{\substack{\mathcal{J}(\ell) \ni a \leq x \\ \mathcal{J}(\ell) \ni b \leq y}} (\varphi(a) \wedge \varphi(b)) \leq \tilde{\varphi}(x \wedge y).$$

Thus $\tilde{\varphi}$ is a homomorphism. It is bounded because $\tilde{\varphi}(0) = \bigvee \emptyset = 0$ and $\tilde{\varphi}(1) = \bigvee_{a \in \mathcal{J}(\ell)} \varphi(a) = 1$ in view of clause (i).

Finally, if $\bar{\varphi}$ is any bounded homomorphism extending φ , then

$$\bar{\varphi}(x) = \bar{\varphi} \left(\bigvee_{\mathcal{J}(\ell) \ni c \leq x} c \right) = \bigvee_{\mathcal{J}(\ell) \ni c \leq x} \bar{\varphi}(c) = \bigvee_{\mathcal{J}(\ell) \ni c \leq x} \varphi(c) = \tilde{\varphi}(x).$$

This takes care of uniqueness. $\square$

I am most grateful to the referee who pointed out a gap in Example 4.1 of the initially submitted version of [2]. The reasons for the occurrence of that gap were eventually traced down to the deficiency of [1, Fact 1.1(a)].

References

- [1] P. Lindström, V. Yu. Shavrukov, The $\forall\exists$ theory of Peano Σ_1 sentences, *J. Math. Logic* **8** (2008) 251–280.
- [2] V. Yu. Shavrukov, Dense chains of Σ_n sentences with strong conservativity properties, in *Liber Amicorum Alberti: A Tribute to Albert Visser*, eds. J. van Eijck, R. Iemhoff and J. J. Joosten (College Publications, 2016), pp. 209–222.