NON-DIFFERENTIABLE EXACT SOLUTIONS FOR THE NONLINEAR ODES DEFINED ON FRACTAL SETS

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Abstract

In the present paper, a family of the special functions via the celebrated Mittag-Leffler function defined on the Cantor sets is investigated. The nonlinear local fractional ODEs (NLFODEs) are presented by following the rules of local fractional derivative (LFD). The exact solutions for these problems are also discussed with the aid of the non-differentiable charts on Cantor sets. The obtained results are important for describing the characteristics of the fractal special functions.

Keywords: Nonlinear ODEs; Local Fractional Derivative; Mittag-Leffler Function; Cantor Sets.

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1. INTRODUCTION

Fractional ordinary differential equations (FODEs)^{1–3} have been successfully used to model the complexity in mathematics, physics and societies, such as the fractional evolution,⁴ control,⁵ circuits,^{6–9} relaxation^{10–14} and population dynamics.¹⁵ Finding the solutions for these mentioned models, many technologies were proposed in Ref. 16. For example, the Adomian decomposition technology (ADT)¹⁷ and its extended version¹⁸ were proposed to solve the approximate solutions for the FODEs. The spectral element method (SEM)¹⁹ and the finite difference method (FDM)²⁰ and the linear multiple step method (LMSM)²¹ were discussed to handle the numerical solutions for the FODEs. The technologies involving the differential transform (DT)²² and the fractional operational calculus (FOC)²³ technologies were reported in order to find the analytical and exact solutions for FODEs, respectively.

Recently, fractional calculus (FC) was considered to solve a class of the fractal problems in mathematical physics,^{24–28} mechanics,^{29–31} heat,³² biology³³ and others.^{34–37} There is an alternative operator (called local FC) to model the local FODEs in fractal electric circuits,³⁸ free damped vibrations,³⁹ shallow water surfaces⁴⁰ and populations.^{41–43} The fractal partial differential equations (FPDEs) in mathematical physics were also discussed in Refs. 44–48. The structure solutions for the nonlinear local fractional ordinary differential equations (NLFODEs) have not been sufficiently investigated. Motivated especially by the above idea, our aim in the present article is to structure the NLFODEs by means of a family of the special functions via the Mittag-Leffler function defined on the Cantor sets.

The structure of the paper is designed as follows. In Sec. 2, the basic definitions of the local fractional derivative (LFD) and special functions defined on Cantor sets are introduced. In Sec. 3, we present the NLFODEs with the use of the LFDs of the special functions defined on the Cantor sets. Finally, we give the conclusion in Sec. 4.

2. PRELIMINARIES, DEFINITIONS AND FRACTAL SPECIAL FUNCTIONS

Definition 1. The LFD of $\Pi_\tau(\mu)$ of fractal order τ ($0 < \tau < 1$) at the point $\mu = \mu_0$ is defined

by^{24,38–41,44–48}

$$\begin{aligned} D^{(\tau)}\Pi_\tau(\mu_0) &= \frac{d^\tau \Pi_\tau(\mu_0)}{d\mu^\tau} \\ &= \lim_{\mu \rightarrow \mu_0} \frac{\Delta^\tau(\Pi_\tau(\mu) - \Pi_\tau(\mu_0))}{(\mu - \mu_0)^\tau}, \end{aligned} \quad (1)$$

where

$$\begin{aligned} \Delta^\tau(\Pi_\tau(\mu) - \Pi_\tau(\mu_0)) &\cong \Gamma(1 + \tau)\Delta[\Pi_\tau(\mu) - \Pi_\tau(\mu_0)]. \end{aligned} \quad (2)$$

Definition 2. The LFD of $\Pi_\tau(\mu)$ of fractal order $\kappa\tau$ ($0 < \tau < 1, \kappa \in \mathbb{N}$) at the point $\mu = \mu_0$ is given as follows (see Refs. 24 and 43):

$$D^{(\kappa\tau)}\Pi_\tau(\mu_0) = \underbrace{\frac{d^\tau}{d\mu^\tau} \cdots \frac{\partial^\tau}{\partial \mu^\tau}}_{\kappa \text{ times}} \Pi_\tau(\mu_0). \quad (3)$$

If j^τ is a fractal imaginary unit and $\kappa \in \mathbb{N}_0$, then the fractal special functions defined on fractal sets^{24,38–41,43–48} are listed in Table 1, $\mathbb{N}_0$ being (as usual) the set of nonnegative integers.

If ρ is a constant, then the LFDs of the fractal special functions defined on fractal sets^{24,38–41,44–48} are listed in Table 2.

3. NONLINEAR LOCAL FRACTIONAL ODES

In this section, we apply the results of the LFDs of the special functions defined on Cantor sets in order to structure the NLFODEs.

Defining the following special functions on Cantor sets:

$$\Phi_\tau(\mu) = \varphi_1 \sin_\tau(\varphi_2 \mu^\tau) \quad (4)$$

and

$$\Phi_\tau(\mu) = \varphi_1 \cos_\tau(\varphi_2 \mu^\tau), \quad (5)$$

where φ_1 and φ_2 are two parameters, we find from Table 2 that

$$D^{(\tau)}\varphi_1 \sin_\tau(\varphi_2 \mu^\tau) = \varphi_1 \varphi_2 \cos_\tau(\varphi_2 \mu^\tau) \quad (6)$$

and

$$D^{(\tau)}\varphi_1 \cos_\tau(\varphi_2 \mu^\tau) = -\varphi_1 \varphi_2 \sin_\tau(\varphi_2 \mu^\tau), \quad (7)$$

so that we get the following NLFODE:

$$[D^{(\tau)}\Phi_\tau(\mu)]^2 = \varphi_2^2(\varphi_1^2 - \Phi_\tau^2(\mu)). \quad (8)$$

When $\varphi_1 = 1$ and $\varphi_2 = 1$, from Eq. (8), we get the NLFODE as follows:

$$[D^{(\tau)}\Phi_\tau(\mu)]^2 = 1 - \Phi_\tau^2(\mu), \quad (9)$$

Table 1 The Expressions of the Fractal Special Functions.

Fractal Special Functions	Expressions
$T_\tau(\mu^\tau)$	$T_\tau(\eta^\tau) = \sum_{\kappa=0}^{\infty} \eta^{\kappa\tau} / \Gamma(1 + \kappa\tau)$
$\sin_\tau(\mu^\tau)$	$\sin_\tau(\mu^\tau) = \frac{T_\tau(j^\tau \mu^\tau) - T_\tau(-j^\tau \mu^\tau)}{2j^\tau}$
$\cos_\tau(\mu^\tau)$	$\cos_\tau(\mu^\tau) = \frac{T_\tau(j^\tau \mu^\tau) + T_\tau(-j^\tau \mu^\tau)}{2j^\tau}$
$\sin_\tau^2(\mu^\tau)$	$\sin_\tau^2(\mu^\tau) = 1 - \cos_\tau^2(\mu^\tau)$
$\cos_\tau^2(\mu^\tau)$	$\cos_\tau^2(\mu^\tau) = 1 - \sin_\tau^2(\mu^\tau)$
$\tan_\tau(\mu^\tau)$	$\tan_\tau(\mu^\tau) = \frac{T_\tau(j^\tau \mu^\tau) - T_\tau(-j^\tau \mu^\tau)}{j^\tau (T_\tau(j^\tau \mu^\tau) + T_\tau(-j^\tau \mu^\tau))}$
$\cot_\tau(\mu^\tau)$	$\cot_\tau(\mu^\tau) = \frac{j^\tau (T_\tau(j^\tau \mu^\tau) + T_\tau(-j^\tau \mu^\tau))}{T_\tau(j^\tau \mu^\tau) - T_\tau(-j^\tau \mu^\tau)}$
$\tan_\tau^2(\mu^\tau)$	$\tan_\tau^2(\mu^\tau) = \frac{\sin_\tau^2(\mu^\tau)}{\cos_\tau^2(\mu^\tau)}$
$\cot_\tau^2(\mu^\tau)$	$\cot_\tau^2(\mu^\tau) = \frac{\cos_\tau^2(\mu^\tau)}{\sin_\tau^2(\mu^\tau)}$
$\sec_\tau(\mu^\tau)$	$\sec_\tau(\mu^\tau) = \frac{1}{\cos_\tau(\mu^\tau)} = \frac{T_\tau(j^\tau \mu^\tau) + T_\tau(-j^\tau \mu^\tau)}{2j^\tau}$
$\csc_\tau(\mu^\tau)$	$\csc_\tau(\mu^\tau) = \frac{1}{\sin_\tau(\mu^\tau)} = \frac{T_\tau(j^\tau \mu^\tau) - T_\tau(-j^\tau \mu^\tau)}{2j^\tau}$
$\sec_\tau^2(\mu^\tau)$	$\sec_\tau^2(\mu^\tau) = 1 + \tan_\tau^2(\mu^\tau)$
$\csc_\tau^2(\mu^\tau)$	$\csc_\tau^2(\mu^\tau) = 1 + \cot_\tau^2(\mu^\tau)$
$\sinh_\tau(\mu^\tau)$	$\sinh_\tau(\mu^\tau) = \frac{T_\tau(\mu^\tau) - T_\tau(-\mu^\tau)}{2}$
$\cosh_\tau(\mu^\tau)$	$\cosh_\tau(\mu^\tau) = \frac{T_\tau(\mu^\tau) + T_\tau(-\mu^\tau)}{2}$
$\sinh_\tau^2(\mu^\tau)$	$\sinh_\tau^2(\mu^\tau) = \cosh_\tau^2(\mu^\tau) - 1$
$\cosh_\tau^2(\mu^\tau)$	$\cosh_\tau^2(\mu^\tau) = \sinh_\tau^2(\mu^\tau) + 1$
$\tanh_\tau(\mu^\tau)$	$\tanh_\tau(\mu^\tau) = \frac{\sinh_\tau(\mu^\tau)}{\cosh_\tau(\mu^\tau)}$
$\coth_\tau(\mu^\tau)$	$\coth_\tau(\mu^\tau) = \frac{\cosh_\tau(\mu^\tau)}{\sinh_\tau(\mu^\tau)}$
$\tanh_\tau^2(\mu^\tau)$	$\tanh_\tau^2(\mu^\tau) = \frac{\sinh_\tau^2(\mu^\tau)}{\cosh_\tau^2(\mu^\tau)}$
$\coth_\tau^2(\mu^\tau)$	$\coth_\tau^2(\mu^\tau) = \frac{\cosh_\tau^2(\mu^\tau)}{\sinh_\tau^2(\mu^\tau)}$
$\sec h_\tau(\mu^\tau)$	$\sec h_\tau(\mu^\tau) = \frac{1}{\cosh_\tau(\mu^\tau)} = \frac{T_\tau(\mu^\tau) + T_\tau(-\mu^\tau)}{2}$
$\csc h_\tau(\mu^\tau)$	$\csc h_\tau(\mu^\tau) = \frac{1}{\sinh_\tau(\mu^\tau)} = \frac{T_\tau(\mu^\tau) - T_\tau(-\mu^\tau)}{2}$
$\sec h_\tau^2(\mu^\tau)$	$\sec h_\tau^2(\mu^\tau) = 1 - \tanh_\tau^2(\mu^\tau)$
$\csc h_\tau^2(\mu^\tau)$	$\csc h_\tau^2(\mu^\tau) = \coth_\tau^2(\mu^\tau) - 1$

where the non-differentiable solution has the form and given by

$$\Phi_\tau(\mu) = \begin{cases} \varphi_1 \sin_\tau(\varphi_2 \mu^\tau), \\ \varphi_1 \cos_\tau(\varphi_2 \mu^\tau). \end{cases} \quad (10)$$

Similarly, by taking the following special functions defined on Cantor sets:

$$\Phi_\tau(\mu) = \varphi_1 \sinh_\tau(\varphi_2 \mu^\tau) \quad (11)$$

$$\Phi_\tau(\mu) = \varphi_1 \cosh_\tau(\varphi_2 \mu^\tau), \quad (12)$$

we have

$$D^{(\tau)} \varphi_1 \sinh_\tau(\varphi_2 \mu^\tau) = \varphi_1 \varphi_2 \cosh_\tau(\varphi_2 \mu^\tau) \quad (13)$$

and

$$D^{(\tau)} \varphi_1 \cosh_\tau(\varphi_2 \mu^\tau) = \varphi_1 \varphi_2 \sinh_\tau(\varphi_2 \mu^\tau), \quad (14)$$

Table 2 The LFDs of the Fractal Special Functions Defined on Fractal Sets.

Fractal Special Functions	LFDs
$T_\tau(\rho\mu^\tau)$	$D^{(\tau)}T_\tau(\rho\mu^\tau) = \rho T_\tau(\rho\mu^\tau)$
$\sin_\tau(\rho\mu^\tau)$	$D^{(\tau)}\sin_\tau(\rho\mu^\tau) = \rho \cos_\tau(\rho\mu^\tau)$
$\cos_\tau(\rho\mu^\tau)$	$D^{(\tau)}\cos_\tau(\rho\mu^\tau) = -\rho \sin_\tau(\rho\mu^\tau)$
$\tan_\tau(\rho\mu^\tau)$	$D^{(\tau)}\tan_\tau(\rho\mu^\tau) = \rho(1 + \tan_\tau^2(\rho\mu^\tau))$
$\cot_\tau(\rho\mu^\tau)$	$D^{(\tau)}\cot_\tau(\rho\mu^\tau) = -\rho(1 + \cot_\tau^2(\rho\mu^\tau))$
$\sec_\tau(\rho\mu^\tau)$	$D^{(\tau)}\sec_\tau(\rho\mu^\tau) = \rho \sec_\tau(\rho\mu^\tau) \tan_\tau(\rho\mu^\tau)$
$\csc_\tau(\rho\mu^\tau)$	$D^{(\tau)}\csc_\tau(\rho\mu^\tau) = -\rho \csc_\tau(\rho\mu^\tau) \cot_\tau(\rho\mu^\tau)$
$\sinh_\tau(\rho\mu^\tau)$	$D^{(\tau)}\sinh_\tau(\rho\mu^\tau) = \rho \cosh_\tau(\rho\mu^\tau)$
$\cosh_\tau(\rho\mu^\tau)$	$D^{(\tau)}\cosh_\tau(\rho\mu^\tau) = \rho \sinh_\tau(\rho\mu^\tau)$
$\tanh_\tau(\rho\mu^\tau)$	$D^{(\tau)}\tanh_\tau(\rho\mu^\tau) = \rho(1 - \tanh_\tau^2(\rho\mu^\tau))$
$\coth_\tau(\rho\mu^\tau)$	$D^{(\tau)}\coth_\tau(\rho\mu^\tau) = -\rho(1 - \coth_\tau^2(\rho\mu^\tau))$
$\sec h_\tau(\rho\mu^\tau)$	$D^{(\tau)}\sec h_\tau(\rho\mu^\tau) = -\rho \sec h_\tau(\rho\mu^\tau) \tanh_\tau(\rho\mu^\tau)$
$\csc h_\tau(\rho\mu^\tau)$	$D^{(\tau)}\csc h_\tau(\rho\mu^\tau) = -\rho \csc h_\tau(\rho\mu^\tau) \coth_\tau(\rho\mu^\tau)$

so that we present the form of the NLFODE as follows:

$$[D^{(\tau)}\Phi_\tau(\mu)]^2 = \varphi_2^2(\Phi_\tau^2(\mu) - \varphi_1^2). \quad (15)$$

Thus, we easily structure from Eqs. (8) and (15), the following NLFODE:

$$[D^{(\tau)}\Phi_\tau(\mu)]^2 = \nu\varphi_2^2(\Phi_\tau^2(\mu) - \varphi_1^2), \quad (16)$$

where the non-differentiable solutions can be written as follows:

$$\Phi_\tau(\mu) = \begin{cases} \varphi_1 \sin_\tau(\varphi_2\mu^\tau), & (\nu = -1), \\ \varphi_1 \cos_\tau(\varphi_2\mu^\tau), & (\nu = -1), \\ \varphi_1 \sinh_\tau(\varphi_2\mu^\tau), & (\nu = 1), \\ \varphi_1 \cosh_\tau(\varphi_2\mu^\tau), & (\nu = 1). \end{cases} \quad (17)$$

In a similar manner, we consider the following special functions defined on Cantor sets:

$$\Phi_\tau(\mu) = \varphi_1 \tan_\tau(\varphi_1\mu^\tau) \quad (18)$$

and

$$\Phi_\tau(\mu) = \varphi_1 \cot_\tau(\varphi_1\mu^\tau). \quad (19)$$

In view of Eqs. (18) and (19), we have

$$D^{(\tau)}\varphi_1 \tan_\tau(\varphi_1\mu^\tau) = \varphi_1\varphi_2(1 + \tan_\tau^2(\varphi_2\mu^\tau)) \quad (20)$$

and

$$D^{(\tau)}\varphi_1 \cot_\tau(\varphi_2\mu^\tau) = -\varphi_1\varphi_2(1 + \cot_\tau^2(\varphi_2\mu^\tau)) \quad (21)$$

so that

$$D^{(\tau)}\Phi_\tau(\mu) = \pm\varphi_2 \left(\varphi_1 + \frac{1}{\varphi_1}\Phi_\tau^2(\mu) \right). \quad (22)$$

Thus, we directly obtain the following NLFODE:

$$D^{(\tau)}\Phi_\tau(\mu) = \nu\varphi_2 \left(\varphi_1 + \frac{1}{\varphi_1}\Phi_\tau^2(\mu) \right), \quad (23)$$

where ν is a parameter and the non-differentiable solutions can be given as follows:

$$\Phi_\tau(\mu) = \begin{cases} \varphi_1 \tan_\tau(\varphi_2\mu^\tau), & (\nu = 1), \\ \varphi_1 \cot_\tau(\varphi_2\mu^\tau), & (\nu = -1). \end{cases} \quad (24)$$

In a similar manner, we can structure the following NLFODE:

$$D^{(\tau)}\Phi_\tau(\mu) = \varphi_2 \left(\varphi_1 - \frac{1}{\varphi_1}\Phi_\tau^2(\mu) \right), \quad (25)$$

where the non-differentiable solution is represented by

$$\Phi_\tau(\mu) = \begin{cases} \varphi_1 \tanh_\tau(\varphi_2\mu^\tau), \\ \varphi_1 \coth_\tau(\varphi_2\mu^\tau). \end{cases} \quad (26)$$

Making use of Eqs. (23) and (25), we can derive the following NLFODE:

$$D^{(\tau)}\Phi_\tau(\mu) = \pm\varphi_2 \left(\varphi_1 + \frac{\nu}{\varphi_1}\Phi_\tau^2(\mu) \right), \quad (27)$$

where the non-differentiable solutions are given by

$$\Phi_\tau(\mu) = \begin{cases} \varphi_1 \sinh_\tau(\varphi_2\mu^\tau), & (\nu = 1), \\ \varphi_1 \cosh_\tau(\varphi_2\mu^\tau), & (\nu = 1), \\ \varphi_1 \tanh_\tau(\varphi_2\mu^\tau), & (\nu = -1), \\ \varphi_1 \coth_\tau(\varphi_2\mu^\tau), & (\nu = -1). \end{cases} \quad (28)$$

Let us define the following special functions on Cantor sets:

$$\Phi_\tau(\mu) = \varphi_1 \sec_\tau(\varphi_2\mu^\tau), \quad (29)$$

$$\Phi_\tau(\mu) = \varphi_1 \csc_\tau(\varphi_2\mu^\tau), \quad (30)$$

$$\Phi_\tau(\mu) = \varphi_1 \sec h_\tau(\varphi_2\mu^\tau) \quad (31)$$

and

$$\Phi_\tau(\mu) = \varphi_1 \csc h_\tau(\varphi_2 \mu^\tau). \quad (32)$$

From Eqs. (29)–(32), we can establish the following formulas:

$$\begin{aligned} D^{(\tau)} \varphi_1 \sec_\tau(\varphi_2 \mu^\tau) \\ = \varphi_1 \varphi_2 \sec_\tau(\varphi_2 \mu^\tau) \tan_\tau(\varphi_2 \mu^\tau), \end{aligned} \quad (33)$$

$$\begin{aligned} D^{(\tau)} \varphi_1 \csc_\tau(\varphi_2 \mu^\tau) \\ = \varphi_1 \varphi_2 \csc_\tau(\varphi_2 \mu^\tau) \cot_\tau(\varphi_2 \mu^\tau), \end{aligned} \quad (34)$$

$$\begin{aligned} D^{(\tau)} \varphi_1 \sec h_\tau(\varphi_2 \mu^\tau) \\ = -\varphi_1 \varphi_2 \sec h_\tau(\varphi_2 \mu^\tau) \tanh_\tau(\varphi_2 \mu^\tau) \end{aligned} \quad (35)$$

and

$$\begin{aligned} D^{(\tau)} \varphi_1 \csc h_\tau(\varphi_2 \mu^\tau) \\ = -\varphi_1 \varphi_2 \csc h_\tau(\varphi_2 \mu^\tau) \coth_\tau(\varphi_2 \mu^\tau). \end{aligned} \quad (36)$$

From Eqs. (33)–(36), we have

$$\begin{aligned} [D^{(\tau)} \varphi_1 \sec_\tau(\varphi_2 \mu^\tau)]^2 \\ = \varphi_2^2 \sec_\tau^2(\varphi_2 \mu^\tau) (\varphi_1^2 \sec_\tau^2(\varphi_2 \mu^\tau) - \varphi_1^2), \end{aligned} \quad (37)$$

$$\begin{aligned} [D^{(\tau)} \varphi_1 \csc_\tau(\varphi_2 \mu^\tau)]^2 \\ = \varphi_2^2 \csc_\tau^2(\varphi_2 \mu^\tau) (\varphi_1^2 \cot_\tau^2(\varphi_2 \mu^\tau) - \varphi_1^2), \end{aligned} \quad (38)$$

$$\begin{aligned} [D^{(\tau)} \varphi_1 \sec h_\tau(\varphi_2 \mu^\tau)]^2 \\ = \varphi_2^2 \sec^2 h_\tau(\varphi_2 \mu^\tau) [\varphi_1^2 - \varphi_1^2 \sec^2 h_\tau(\varphi_2 \mu^\tau)] \end{aligned} \quad (39)$$

and

$$\begin{aligned} [D^{(\tau)} \varphi_1 \csc h_\tau(\varphi_2 \mu^\tau)]^2 \\ = \varphi_2^2 \csc^2 h_\tau(\varphi_2 \mu^\tau) [\varphi_1^2 \csc^2 h_\tau(\varphi_2 \mu^\tau) + \varphi_1^2], \end{aligned} \quad (40)$$

so that

$$\begin{aligned} [D^{(\tau)} \Phi_\tau(\mu)]^2 \\ = \frac{\nu_1 \varphi_2^2}{\varphi_1^2} \Phi_\tau^2(\mu) [\Phi_\tau^2(\mu) + \nu_2 \varphi_1^2], \end{aligned} \quad (41)$$

where the non-differentiable solutions are given as follows:

$$\begin{aligned} \Phi_\tau(\mu) \\ = \begin{cases} \varphi_1 \sec_\tau(\varphi_2 \mu^\tau), & (\nu_1 = 1; \nu_2 = -1), \\ \varphi_1 \csc_\tau(\varphi_2 \mu^\tau), & (\nu_1 = 1; \nu_2 = -1), \\ \varphi_1 \sec h_\tau(\varphi_2 \mu^\tau), & (\nu = -1; \nu_2 = -1), \\ \varphi_1 \csc h_\tau(\varphi_2 \mu^\tau), & (\nu = 1; \nu_2 = 1). \end{cases} \end{aligned} \quad (42)$$

From Table 1, we set up the following special function defined on Cantor sets:

$$\Phi_\tau(\mu) = \frac{\varphi_3}{[\varphi_1 - \varphi_2 T_\tau(\rho \mu^\tau)]^{\frac{2}{\kappa}}}, \quad (43)$$

where ρ , φ_1 , φ_2 and φ_3 are parameters.

From Eq. (43), we easily have

$$T_\tau(\rho \mu^\tau) = \frac{\varphi_1 - \frac{2}{\kappa} \sqrt{\frac{\varphi_3}{\Phi_\tau(\mu)}}}{\varphi_2} \quad (44)$$

and

$$\frac{2}{\kappa} \sqrt{\frac{\Phi_\tau(\mu)}{\varphi_3}} = \frac{1}{\varphi_1 - \varphi_2 T_\tau(\rho \mu^\tau)}. \quad (45)$$

For finding the LFD of Eq. (43), we present

$$D^{(\tau)} \Phi_\tau(\mu) = \frac{\frac{2\varphi_3\varphi_2\rho}{\kappa} T_\tau(\rho \mu^\tau)}{(\varphi_1 - \varphi_2 T_\tau(\rho \mu^\tau))^{\frac{2}{\kappa}+1}}. \quad (46)$$

Upon substituting Eqs. (44) and (45) into Eq. (46), we have

$$D^{(\tau)} \Phi_\tau(\mu) = \frac{2\rho\Phi_\tau(\mu)}{\kappa} \left(\varphi_1 \left(\frac{\Phi_\tau(\mu)}{\varphi_3} \right)^{\frac{\kappa}{2}} - 1 \right). \quad (47)$$

When $\kappa = 1$, we find from Eq. (47) that

$$D^{(\tau)} \Phi_\tau(\mu) = \frac{2\rho\varphi_1}{\kappa\varphi_3} \Phi_\tau^2(\mu) - \frac{2\rho}{\kappa} \Phi_\tau(\mu) \quad (48)$$

together with the non-differentiable solution given by

$$\Phi_\tau(\mu) = \frac{\varphi_3}{\varphi_1 - \varphi_2 T_\tau(\rho \mu^\tau)}. \quad (49)$$

Similarly, we propose the following special function defined on Cantor sets:

$$\Phi_\tau(\mu) = \frac{\varphi_3}{[\varphi_1 - \varphi_2 T_\tau(-\rho \mu^\tau)]^{\frac{2}{\kappa}}}, \quad (50)$$

which leads to

$$T_\tau(-\rho \mu^\tau) = \frac{\varphi_1 - \frac{2}{\kappa} \sqrt{\frac{\varphi_3}{\Phi_\tau(\mu)}}}{\varphi_2} \quad (51)$$

and

$$\frac{2}{\kappa} \sqrt{\frac{\Phi_\tau(\mu)}{\varphi_3}} = \frac{1}{\varphi_1 - \varphi_2 T_\tau(-\rho \mu^\tau)}. \quad (52)$$

In order to find the LFD of Eq. (49), we have

$$D^{(\tau)} \Phi_\tau(\mu) = \frac{-\frac{2\varphi_3\varphi_2\rho}{\kappa} T_\tau(-\rho \mu^\tau)}{(\varphi_1 - \varphi_2 T_\tau(-\rho \mu^\tau))^{\frac{2}{\kappa}+1}}, \quad (53)$$

which leads to

$$D^{(\tau)}\Phi_{\tau}(\mu) = \frac{2\rho\Phi_{\tau}(\mu)}{\kappa} \left(1 - \varphi_1 \left(\frac{\Phi_{\tau}(\mu)}{\varphi_3} \right)^{\frac{\kappa}{2}} \right), \quad (54)$$

where ρ , φ_1 , φ_2 and φ_3 are parameters.

When $\kappa = \varphi_1 = 1$ and $\varphi_3 = 1$, we find from Eq. (54) that (see Ref. 41)

$$D^{(\tau)}\Phi_{\tau}(\mu) = 2\rho\Phi_{\tau}(\mu)(1 - \Phi_{\tau}(\mu)), \quad (55)$$

where the non-differentiable solution becomes (see Ref. 41)

$$\Phi_{\tau}(\mu) = \frac{1}{1 - \varphi_2 T_{\tau}(-\rho\mu^{\tau})} \quad (56)$$

with the parameters ρ and φ_2 .

Let us now consider the following special functions defined on Cantor sets:

$$\Phi_{\tau}(\mu) = \varphi_1 \sin_{\tau}^2(\varphi_2\mu^{\tau}) \quad (57)$$

and

$$\Phi_{\tau}(\mu) = \varphi_1 \cos_{\tau}^2(\varphi_2\mu^{\tau}). \quad (58)$$

By finding the LFDs of Eqs. (57) and (58), we have

$$\begin{aligned} D^{(\tau)}\varphi_1 \sin_{\tau}^2(\varphi_2\mu^{\tau}) \\ = 2\varphi_1\varphi_2 \sin_{\tau}(\varphi_2\mu^{\tau}) \cos_{\tau}(\varphi_2\mu^{\tau}) \end{aligned} \quad (59)$$

and

$$\begin{aligned} D^{(\tau)}\varphi_1 \cos_{\tau}^2(\varphi_2\mu^{\tau}) \\ = -2\varphi_1\varphi_2 \cos_{\tau}(\varphi_2\mu^{\tau}) \sin_{\tau}(\varphi_2\mu^{\tau}), \end{aligned} \quad (60)$$

which yield

$$\begin{aligned} [D^{(\tau)}\varphi_1 \sin_{\tau}^2(\varphi_2\mu^{\tau})]^2 \\ = (2\varphi_1\varphi_2)^2 \sin_{\tau}^2(\varphi_2\mu^{\tau}) \cos_{\tau}^2(\varphi_2\mu^{\tau}) \end{aligned} \quad (61)$$

and

$$\begin{aligned} [D^{(\tau)}\varphi_1 \cos_{\tau}^2(\varphi_2\mu^{\tau})]^2 \\ = (2\varphi_1\varphi_2)^2 \cos_{\tau}^2(\varphi_2\mu^{\tau}) \sin_{\tau}^2(\varphi_2\mu^{\tau}). \end{aligned} \quad (62)$$

From Eqs. (61) and (62), we get

$$\begin{aligned} [D^{(\tau)}\varphi_1 \sin_{\tau}^2(\varphi_2\mu^{\tau})]^2 \\ = (2\varphi_1\varphi_2)^2 \sin_{\tau}^2(\varphi_2\mu^{\tau})(1 - \sin_{\tau}^2(\varphi_2\mu^{\tau})) \end{aligned} \quad (63)$$

and

$$\begin{aligned} [D^{(\tau)}\varphi_1 \cos_{\tau}^2(\varphi_2\mu^{\tau})]^2 \\ = (2\varphi_1\varphi_2)^2 \cos_{\tau}^2(\varphi_2\mu^{\tau})(1 - \cos_{\tau}^2(\varphi_2\mu^{\tau})), \end{aligned} \quad (64)$$

which lead us to the following NLFODE:

$$[D^{(\tau)}\Phi_{\tau}(\mu)]^2 = (2\varphi_1\varphi_2)^2 \frac{\Phi_{\tau}(\mu)}{\varphi_1} \left(1 - \frac{\Phi_{\tau}(\mu)}{\varphi_1} \right). \quad (65)$$

Thus, from Eq. (65), we easily obtain the following NLFODE:

$$[D^{(\tau)}\Phi_{\tau}(\mu)]^2 = 4\varphi_2^2(\varphi_1\Phi_{\tau}(\mu) - \Phi_{\tau}^2(\mu)), \quad (66)$$

where the non-differentiable solutions are presented as follows:

$$\Phi_{\tau}(\mu) = \begin{cases} \varphi_1 \sin_{\tau}^2(\varphi_2\mu^{\tau}), \\ \varphi_1 \cos_{\tau}^2(\varphi_2\mu^{\tau}). \end{cases} \quad (67)$$

By a similar process, we present the special functions defined on Cantor sets as follows:

$$\Phi_{\tau}(\mu) = \varphi_1 \sinh_{\tau}^2(\varphi_2\mu^{\tau}) \quad (68)$$

and

$$\Phi_{\tau}(\mu) = \varphi_1 \cosh_{\tau}^2(\varphi_2\mu^{\tau}), \quad (69)$$

which lead us to the following formulas:

$$\begin{aligned} D^{(\tau)}\varphi_1 \sinh_{\tau}^2(\varphi_2\mu^{\tau}) \\ = 2\varphi_1\varphi_2 \sinh_{\tau}(\varphi_2\mu^{\tau}) \cosh_{\tau}(\varphi_2\mu^{\tau}) \end{aligned} \quad (70)$$

and

$$\begin{aligned} D^{(\tau)}\varphi_1 \cosh_{\tau}^2(\varphi_2\mu^{\tau}) \\ = -2\varphi_1\varphi_2 \cosh_{\tau}(\varphi_2\mu^{\tau}) \sinh_{\tau}(\varphi_2\mu^{\tau}), \end{aligned} \quad (71)$$

respectively. Therefore, we find from Eqs. (70) and (71) that

$$\begin{aligned} [D^{(\tau)}\varphi_1 \sinh_{\tau}^2(\varphi_2\mu^{\tau})]^2 \\ = (2\varphi_1\varphi_2)^2 \sinh_{\tau}^2(\varphi_2\mu^{\tau})(1 + \sinh_{\tau}^2(\varphi_2\mu^{\tau})) \end{aligned} \quad (72)$$

and

$$\begin{aligned} [D^{(\tau)}\varphi_1 \cosh_{\tau}^2(\varphi_2\mu^{\tau})]^2 \\ = (2\varphi_1\varphi_2)^2 \cosh_{\tau}^2(\varphi_2\mu^{\tau})(\cosh_{\tau}^2(\varphi_2\mu^{\tau}) - 1), \end{aligned} \quad (73)$$

which deduce to

$$\begin{aligned} [D^{(\tau)}\Phi_{\tau}(\mu)]^2 \\ = (2\varphi_1\varphi_2)^2 \frac{\Phi_{\tau}(\mu)}{\varphi_1} \left(1 + \frac{\Phi_{\tau}(\mu)}{\varphi_1} \right) \end{aligned} \quad (74)$$

and

$$\begin{aligned} [D^{(\tau)}\Phi_{\tau}(\mu)]^2 \\ = (2\varphi_1\varphi_2)^2 \frac{\Phi_{\tau}(\mu)}{\varphi_1} \left(\frac{\Phi_{\tau}(\mu)}{\varphi_1} - 1 \right), \end{aligned} \quad (75)$$

respectively.

Next, from Eqs. (74) and (75), there exists the following NLFODE:

$$[D^{(\tau)}\Phi_{\tau}(\mu)]^2 = 4\varphi_2^2\Phi_{\tau}(\mu)(\Phi_{\tau}(\mu) + \nu\varphi_1^2), \quad (76)$$

where the non-differentiable solutions are given by

$$\Phi_{\tau}(\mu) = \begin{cases} \varphi_1 \sinh_{\tau}^2(\varphi_2\mu^{\tau}), & (\nu = 1), \\ \varphi_1 \cosh_{\tau}^2(\varphi_2\mu^{\tau}), & (\nu = -1). \end{cases} \quad (77)$$

If the special functions defined on Cantor sets are given as follows:

$$\Phi_{\tau}(\mu) = \varphi_1 \tan_{\tau}^2(\varphi_2\mu^{\tau}) \quad (78)$$

and

$$\Phi_{\tau}(\mu) = \varphi_1 \cot_{\tau}^2(\varphi_2\mu^{\tau}), \quad (79)$$

then we have

$$\begin{aligned} D^{(\tau)}\varphi_1 \tan_{\tau}^2(\varphi_2\mu^{\tau}) \\ = 2\varphi_1\varphi_2 \tan_{\tau}(\varphi_2\mu^{\tau})(1 + \tan_{\tau}^2(\varphi_2\mu^{\tau})) \end{aligned} \quad (80)$$

and

$$\begin{aligned} D^{(\tau)}\varphi_1 \cot_{\tau}^2(\varphi_2\mu^{\tau}) \\ = -2\varphi_1\varphi_2 \cot_{\tau}(\varphi_2\mu^{\tau})(1 + \cot_{\tau}^2(\varphi_2\mu^{\tau})), \end{aligned} \quad (81)$$

so that

$$\begin{aligned} [D^{(\tau)}\varphi_1 \tan_{\tau}^2(\varphi_2\mu^{\tau})]^2 \\ = (2\varphi_1\varphi_2)^2 \tan_{\tau}^2(\varphi_2\mu^{\tau})(1 + \tan_{\tau}^2(\varphi_2\mu^{\tau}))^2 \end{aligned} \quad (82)$$

and

$$\begin{aligned} [D^{(\tau)}\varphi_1 \cot_{\tau}^2(\varphi_2\mu^{\tau})]^2 \\ = (2\varphi_1\varphi_2)^2 \cot_{\tau}^2(\varphi_2\mu^{\tau})(1 + \cot_{\tau}^2(\varphi_2\mu^{\tau}))^2, \end{aligned} \quad (83)$$

which lead us to the following NLFODE:

$$\begin{aligned} [D^{(\tau)}\Phi_{\tau}(\mu)]^2 \\ = 4\varphi_1\varphi_2^2\Phi_{\tau}(\mu) \left(1 + \frac{1}{\varphi_1}\Phi_{\tau}(\mu)\right)^2. \end{aligned} \quad (84)$$

Therefore, we get the following NLFODE:

$$\begin{aligned} [D^{(\tau)}\Phi_{\tau}(\mu)]^2 \\ = 4\varphi_1\varphi_2^2\Phi_{\tau}(\mu) \left(1 + \frac{1}{\varphi_1}\Phi_{\tau}(\mu)\right)^2, \end{aligned} \quad (85)$$

where the non-differentiable solutions are given by:

$$\Phi_{\tau}(\mu) = \begin{cases} \varphi_1 \tan_{\tau}^2(\varphi_2\mu^{\tau}), \\ \varphi_1 \cot_{\tau}^2(\varphi_2\mu^{\tau}). \end{cases} \quad (86)$$

Let us suppose that the special functions defined on Cantor sets can be expressed as follows:

$$\Phi_{\tau}(\mu) = \varphi_1 \tanh_{\tau}^2(\varphi_1\mu^{\tau}) \quad (87)$$

and

$$\Phi_{\tau}(\mu) = \varphi_1 \coth_{\tau}^2(\varphi_1\mu^{\tau}). \quad (88)$$

Then, we have

$$\begin{aligned} D^{(\tau)}\varphi_1 \tanh_{\tau}^2(\varphi_1\mu^{\tau}) \\ = 2\varphi_1\varphi_2 \tanh_{\tau}(\varphi_1\mu^{\tau})(1 + \tanh_{\tau}^2(\varphi_2\mu^{\tau})) \end{aligned} \quad (89)$$

and

$$\begin{aligned} D^{(\tau)}\varphi_1 \coth_{\tau}^2(\varphi_1\mu^{\tau}) \\ = -2\varphi_1\varphi_2 \coth_{\tau}(\varphi_1\mu^{\tau})(1 + \coth_{\tau}^2(\varphi_2\mu^{\tau})), \end{aligned} \quad (90)$$

so that

$$\begin{aligned} [D^{(\tau)}\varphi_1 \tanh_{\tau}^2(\varphi_1\mu^{\tau})]^2 \\ = (2\varphi_1\varphi_2)^2 \tanh_{\tau}^2(\varphi_1\mu^{\tau})(1 + \tanh_{\tau}^2(\varphi_2\mu^{\tau}))^2 \end{aligned} \quad (91)$$

and

$$\begin{aligned} [D^{(\tau)}\varphi_1 \coth_{\tau}^2(\varphi_1\mu^{\tau})]^2 \\ = (2\varphi_1\varphi_2)^2 \coth_{\tau}^2(\varphi_1\mu^{\tau})(1 + \coth_{\tau}^2(\varphi_2\mu^{\tau}))^2, \end{aligned} \quad (92)$$

which lead us to the following NLFODE:

$$\begin{aligned} [D^{(\tau)}\Phi_{\tau}(\mu)]^2 \\ = 4\varphi_1\varphi_2^2\Phi_{\tau}(\mu) \left(1 + \frac{1}{\varphi_1}\Phi_{\tau}(\mu)\right)^2. \end{aligned} \quad (93)$$

In case the special functions defined on Cantor sets can be written as follows:

$$\Phi_{\tau}(\mu) = \varphi_1 \sec_{\tau}^2(\varphi_2\mu^{\tau}) \quad (94)$$

and

$$\Phi_{\tau}(\mu) = \varphi_1 \csc_{\tau}^2(\varphi_2\mu^{\tau}), \quad (95)$$

then we have

$$\begin{aligned} D^{(\tau)}\varphi_1 \sec_{\tau}^2(\varphi_2\mu^{\tau}) \\ = 2\varphi_1\varphi_2 \sec_{\tau}^2(\varphi_2\mu^{\tau}) \tan_{\tau}(\varphi_2\mu^{\tau}) \end{aligned} \quad (96)$$

and

$$\begin{aligned} D^{(\tau)}\varphi_1 \csc_{\tau}^2(\varphi_2\mu^{\tau}) \\ = 2\varphi_1\varphi_2 \csc_{\tau}^2(\varphi_2\mu^{\tau}) \cot_{\tau}(\varphi_2\mu^{\tau}). \end{aligned} \quad (97)$$

Following Eqs. (96) and (97), we obtain

$$\begin{aligned} [D^{(\tau)}\varphi_1 \sec_{\tau}^2(\varphi_2\mu^{\tau})]^2 \\ = (2\varphi_1\varphi_2)^2 \sec_{\tau}^4(\varphi_2\mu^{\tau})(\sec_{\tau}^2(\varphi_2\mu^{\tau}) - 1) \end{aligned} \quad (98)$$

and

$$\begin{aligned} [D^{(\tau)}\varphi_1 \csc_{\tau}^2(\varphi_2\mu^{\tau})]^2 \\ = (2\varphi_1\varphi_2)^2 \csc_{\tau}^4(\varphi_2\mu^{\tau})(\cot_{\tau}^2(\varphi_2\mu^{\tau}) - 1), \end{aligned} \quad (99)$$

respectively.

From Eqs. (98) and (99), we get the following NLFODE:

$$[D^{(\tau)}\Phi_{\tau}(\mu)]^2 = 4\varphi_2^2\Phi_{\tau}^2(\mu)\left(\frac{\Phi_{\tau}(\mu)}{\varphi_1} - 1\right), \quad (100)$$

where the non-differentiable solution is determined by

$$\Phi_{\tau}(\mu) = \begin{cases} \varphi_1 \sec_{\tau}^2(\varphi_2\mu^{\tau}), \\ \varphi_1 \csc_{\tau}^2(\varphi_2\mu^{\tau}). \end{cases} \quad (101)$$

In the same manner, we establish the following special function defined on Cantor sets (see Ref. 40):

$$\Phi_{\tau}(\mu) = \varphi_1 \sec h_{\tau}^2(\varphi_2\mu^{\tau}), \quad (102)$$

which leads to

$$\begin{aligned} D^{(\tau)}\varphi_1 \sec h_{\tau}^2(\varphi_2\mu^{\tau}) \\ = -2\varphi_1\varphi_2 \sec h_{\tau}^2(\varphi_2\mu^{\tau}) \tan h_{\tau}(\varphi_2\mu^{\tau}). \end{aligned} \quad (103)$$

In view of Eq. (103), we have the following NLFODE (see Ref. 40):

$$[D^{(\tau)}\Phi_{\tau}(\mu)]^2 = 4\varphi_2^2\Phi_{\tau}^2(\mu)\left(1 - \frac{\Phi_{\tau}(\mu)}{\varphi_1}\right). \quad (104)$$

Equation (104) was used to find the traveling-wave solution for the fractal Korteweg–de Vries equation within LFD (see, for details, Ref. 40).

4. CONCLUSION

In our present work, the fractal special functions defined on Cantor sets were structured for the first time. With the use of the LFDs of the given special functions, we proposed the NLFODEs and their exact solutions of non-differentiable type. The results are applicable for designing the exact traveling-wave solutions for the nonlinear FPDEs in mathematical physics (see Ref. 40).

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