ERRATUM

Correction to:

D. Veneziano, “Large Deviations of Multifractal Measures,” *Fractals* Vol. 10, No. 1 (2002) 117–129

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An error was found in the refined limits for dressed multifractal measures [Eq. (15) of the paper] for the case $\gamma > \gamma^*$. The correct limits are

$$P[\varepsilon_n Z \geq m^{n\gamma}] \approx \begin{cases} E[Z^{c'(\gamma)}] \left(2\pi \frac{[c'(\gamma)]^2}{c''(\gamma)} n \ln(m) \right)^{-1/2} m^{-nc(\gamma)}, & \text{for } \gamma < \gamma^* \\ bm^{-n[c(\gamma^*) + q^*(\gamma - \gamma^*)]}, & \text{for } \gamma > \gamma^*. \end{cases} \quad (1)$$

The derivation in the Appendix to the paper is revised as follows.

We are interested in the behavior for $n \rightarrow \infty$ of the probability

$$\begin{aligned} P[\varepsilon_n Z \geq m^{n\gamma}] &= \frac{E}{Z} P[\varepsilon_n \geq m^{n\gamma}/Z] \\ &= \frac{E}{Z'} P[\varepsilon_n \geq m^{n(\gamma - Z'/n)}] \end{aligned} \quad (2)$$

where Z is the dressing factor of the cascade and $Z' = \log_m Z$. As stated in Eq. (A2) of the paper, this probability may be expressed as

$$\begin{aligned} P[\varepsilon_n Z \geq m^{n\gamma}] &\approx \int_{-\infty}^{\infty} \left(2\pi \frac{[c'(\gamma - \frac{z'}{n})]^2}{c''(\gamma - \frac{z'}{n})} n \ln(m) \right)^{-1/2} \\ &\quad \times m^{-nc(\gamma - \frac{z'}{n})} f_{Z'}(z') dz' \end{aligned} \quad (3)$$

where $\approx$ denotes equality up to a factor that approaches 1 as $n \rightarrow \infty$.

For $\gamma < \gamma^*$, the paper evaluated the integral after linearizing the exponent $c(\gamma - \frac{z'}{n})$ in Eq. (3) around $z'/n = 0$. This works because for large n , z'/n remains close to 0 over the range of z' values that contribute significantly to the integral.

For $\gamma = \gamma^*$ the above is not true, since after linearization $m^{-nc(\gamma^* - \frac{z'}{n})} f_{Z'}(z')$ is constant nonzero for large z' . For $\gamma > \gamma^*$ the situation is even more extreme, since after linearization the contribution to the integral by equal z' intervals in the upper tail of Z' increases with increasing z' . Hence, to evaluate the integral in Eq. (3) for $\gamma \geq \gamma^*$ one needs to use better approximations of the $c(\gamma)$ function.

The approach we follow here for $\gamma > \gamma^*$ is as follows:

- find the value of z'/n for which the integrand is maximum,
- replace $c(\gamma - \frac{z'}{n})$ by the quadratic Taylor series expansion around the maximum, and
- after showing that for large n the higher-order terms are negligible, evaluate the integral.

This approach works because for $\gamma > \gamma^*$ and n large the integrand is contributed almost exclusively by very large values of z' for which $f_{Z'}$ has known exponential form, and in addition the contributing values of z' are such that z'/n varies over a very narrow range.

For $\gamma = \gamma^*$, the interval of z' that contributes significantly to the integral is not in the extreme upper tail of Z' . For that case, we expect the formula for $\gamma > \gamma^*$ to provide an upper bound.

Below we use the following property of the distributions of Z and Z' : for some $b > 0$ and large z and z' ,

$$\begin{aligned} 1 - F_{Z'}(z') &= 1 - F_Z(mz') \approx bm^{-q^*z'} \\ \Rightarrow f_{Z'}(z') &\approx [bq^* \ln(m)]m^{-q^*z'}. \end{aligned} \quad (4)$$

Case $\gamma > \gamma^*$

As we shall show later, when $\gamma > \gamma^*$ and n is large the dominant contribution to the integral in Eq. (3) is from values of z' such that (a) z' is large and (b) z'/n varies over a narrow range. As a consequence of (a), one can use $f_{Z'}$ in Eq. (4) to write Eq. (3) as

$$\begin{aligned} P[\varepsilon_n Z \geq m^{n\gamma}] & \\ \approx bq^* \ln(m) \int_{-\infty}^{\infty} & \left(2\pi \frac{[c'(\gamma - \frac{z'}{n})]^2}{c''(\gamma - \frac{z'}{n})} n \ln(m) \right)^{-1/2} \\ \times m^{-n[c(\gamma - \frac{z'}{n}) + q^* \frac{z'}{n}]} & dz'. \end{aligned} \quad (5)$$

As a consequence of (b), one may regard $\left(2\pi \frac{[c'(\gamma - \frac{z'}{n})]^2}{c''(\gamma - \frac{z'}{n})} n \ln(m) \right)^{-1/2}$ in Eq. (5) as constant.

Then the integrand is maximum where $c(\gamma - \frac{z'}{n}) + q^* \frac{z'}{n}$ is minimum. To find the value of z'/n at which this minimum occurs, we define $\eta = \gamma - z'/n$ and find the value of η for which $c(\eta) - q^*\eta + q^*\gamma$ is minimum. Since γ is fixed, this is the same as the value of η for which $q^*\eta - c(\eta)$ is maximum. The solution to the latter maximum problem is $\eta = \gamma^*$; hence for large n , $z'/n = \gamma - \gamma^*$ maximizes the integrand.

Taylor series expansion of $c(\gamma - \frac{z'}{n}) + q^* \frac{z'}{n}$ around $z'/n = \gamma - \gamma^*$ gives

$$\begin{aligned} c(\gamma - \frac{z'}{n}) + q^* \frac{z'}{n} &= [c(\gamma^*) + q^*(\gamma - \gamma^*)] \\ &+ \frac{1}{2} c''(\gamma^*) \left[\frac{z'}{n} - (\gamma - \gamma^*) \right]^2 \\ &+ R \left(\frac{z'}{n} - (\gamma - \gamma^*) \right) \end{aligned} \quad (6)$$

where R is a residual term. Inserting Eq. (6) into Eq. (5), one sees that, as $n \rightarrow \infty$, the integral in Eq. (5) is dominantly contributed by values of z'/n infinitesimally close to $(\gamma - \gamma^*)$. For these values, R in Eq. (6) is negligible. Neglecting this term, one obtains

$$\begin{aligned} P[\varepsilon_n Z \geq m^{n\gamma}] &\approx bq^* \ln(m) \left(2\pi \frac{[c'(\gamma^*)]^2}{c''(\gamma^*)} n \ln(m) \right)^{-1/2} \\ &\times m^{-n[c(\gamma^*) + q^*(\gamma - \gamma^*)]} \\ &\times \int_{-\infty}^{\infty} e^{-\frac{\ln(m)}{2n} c''(\gamma^*) [z' - n(\gamma - \gamma^*)]^2} dz'. \end{aligned} \quad (7)$$

Finally, using $c'(\gamma^*) = q^*$ and the fact that the integral in Eq. (7) equals $\left(\frac{2\pi n}{\ln(m)c''(\gamma^*)} \right)^{1/2}$ gives

$$P[\varepsilon_n Z \geq m^{n\gamma}] \approx bm^{-n[c(\gamma^*) + q^*(\gamma - \gamma^*)]} \quad (8)$$

where we recall that $b = \lim_{z \rightarrow \infty} \frac{P[Z \geq z]}{z^{-q^*}}$. This concludes the proof of the refined dressed limits in Eq. (1).

Case $\gamma = \gamma^*$

When $\gamma = \gamma^*$, the only condition of the previous analysis that is violated is $f_{Z'}(z') \approx [bq^* \ln(m)]m^{-q^*z'}$ for the values of z' that contribute significantly to the integral in Eq. (3). In fact, the values with significant distribution are now in a (wide) range around $z' = 0$, as opposed to being very large. Since we do not have an analytical expression for $f_{Z'}(z')$ in this central region of the distribution, we cannot obtain results analogous to Eq. (8). However we conjecture that in many cases $[bq^* \ln(m)]m^{-q^*z'}$ should be an upper bound for $f_{Z'}(z')$ and hence $bm^{-nc(\gamma^*)}$ from Eq. (8) should be an upper bound for $P[\varepsilon_n Z \geq m^{n\gamma^*}]$.

Comments on the Refined Limits

The refined limits in Eqs. (1a) and (1b) can be written in a different form that emphasizes their similarity and eases their interpretation. For $\gamma < \gamma^*$, one may use the refined limit for the bare density ε_n and the equality $c'_d(\gamma) = c'(\gamma)$ to write the right hand side of Eq. (1a) as $P[\varepsilon_n \geq m^{n\gamma}]E[Zc'_d(\gamma)]$. For $\gamma > \gamma^*$, one may use $P[Z \geq m^{n\gamma}] \approx bm^{-n\gamma q^*}$ and $q^* = c'_d(\gamma)$ to write the probability in Eq. (1b) as

$$\begin{aligned} b(m^{n\gamma})^{-q^*} m^{n[q^*\gamma^* - c(\gamma^*)]} &= P[Z \geq m^{n\gamma}]m^{nK(q^*)} \\ &= P[Z \geq m^{n\gamma}]E[\varepsilon_n^{q^*}] \\ &= P[Z \geq m^{n\gamma}]E[\varepsilon_n^{c'_d(\gamma)}]. \end{aligned} \quad (9)$$

Therefore, an alternative form of Eq. (1) is

$$P[\varepsilon_n Z \geq m^{n\gamma}] \approx \begin{cases} P[\varepsilon_n \geq m^{n\gamma}] E[Z^{c'_d(\gamma)}], & \text{(a)} \\ P[Z \geq m^{n\gamma}] E[\varepsilon_n^{c'_d(\gamma)}], & \text{(b)} \end{cases} \quad (10)$$

Equation (10) shows that the dressed exceedance probability $P[\varepsilon_n Z \geq m^{n\gamma}]$ is dominated by $P[\varepsilon_n \geq m^{n\gamma}]$ when $\gamma < \gamma^*$ and by $P[Z \geq m^{n\gamma}]$ when $\gamma > \gamma^*$. The non-dominant factor (Z for $\gamma < \gamma^*$ and ε_n for $\gamma > \gamma^*$) enhances these probabilities through a prefactor equal to its moment of order $c'_d(\gamma)$.