Corrigendum

The minimum crossing number of essential tangles

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ABSTRACT

We correct some of the arguments proving that the tangles on Theorems 1.1 and 1.2 are essential.

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1. Introduction

In this note, we correct three arguments on the proofs of the tangles of Theorems 1.1 and 1.2 being essential. The results of the original paper remain unaltered, as stated below.

Theorem 1.1. *Let $\mathcal{T}$ be a n -string essential tangle. Then*

$$c(\mathcal{T}) \geq 2n + 1.$$

We have $c(\mathcal{T}) = 2n + 1$ if and only if $\mathcal{T}$ is equivalent to the tangle in Fig. 1.

Theorem 1.2. *Let $\mathcal{T}$ be a n -string essential tangle, with all strings unknotted. Then*

$$c(\mathcal{T}) \geq 2n + 2.$$

We have $c(\mathcal{T}) = 2n + 2$ if and only if $\mathcal{T}$ is equivalent to one of the tangles in Fig. 2.

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Fig. 1. The n -string essential tangle with the minimum crossing number.

Fig. 2. The all strings unknotted n -string essential tangles with the minimum crossing number.

2. The n -String Tangle of Theorem 1.1

In this section, we correct the proof that the n -string tangle of Theorem 1.1 is essential. The original argument, on p. 1450054-7, leads to a contradiction for a trivial link having linking number $n - 1$, however, the linking number of the argument can actually be $n - 1$ as well as other non-zero values. When correcting this argument, we opted for another path for the proof, which is summed up in Lemma 2.2. We first give a definition of prime tangle, inspired by Bleiler [1]:

Definition 2.1. An essential tangle (B, t) is said to be *prime* if:

- (1) Any 2-sphere embedded in B which meets t transversely in two points is parallel to ∂B or it bounds a 3-ball in B which meets t in a single unknotted arc.
- (2) Any properly embedded disc D which meets t transversely in a single point is such that ∂D bounds a disc in ∂B which also meets t in a single point.

Note that if (B, t) is a 1-string essential tangle, then it is prime if and only if the knot $t \cup \alpha$ is prime, where α is an arc on ∂B joining the endpoints of t .

Lemma 2.2. Let $\mathcal{T} = (B, s \cup s_n)$ be a n -string tangle, where $s = s_1 \cup \dots \cup s_{n-1}$ and (B, s) is a prime tangle. Suppose there is an arc α on ∂B such that $s_n \cup \alpha$

Fig. 3. The possible types of connected components δ (dashed) of $D_\alpha \cap D_\beta$ and the corresponding disks Δ_α and Δ_β (grayed).

bounds a disk D_α that meets s transversely in a single point x , decomposing s along x into a and b . Let c be an arc on D_α from x to ∂B . Then $\mathcal{T}$ is not essential if and only if s is ambient isotopic to $a \cup c$ or $b \cup c$ in $B - s_n$.

Proof. Since (B, s) is essential, then $\mathcal{T}$ is not essential if and only if there is a properly embedded disk in B separating s from s_n . It is clear that if s is isotopic to $a \cup c$ or $b \cup c$, then there is such a disk, therefore $\mathcal{T}$ is not essential.

Suppose now that there is a disk separating s from s_n . Since s_n is unknotted, there is a disk D_β with boundary $s_n \cup \beta$, where β is a simple arc in ∂B such that $D_\beta \cap s$ is empty.

In the following, denote the interior of X by X° and the number of connected components of X by $|X|$, and assume that all sub-manifolds are in general position. Suppose that $|D_\alpha^\circ \cap D_\beta^\circ|$ is minimal among all disks D_β as above. In Fig. 3 we have a schematic illustration for the possible types of connected components of $D_\alpha \cap D_\beta$ in D_α and D_β . We will show that $|D_\alpha^\circ \cap D_\beta^\circ| = 0$.

Suppose first that $D_\alpha^\circ \cap D_\beta^\circ$ contains some simple closed curve δ , that bounds a disk Δ_α in D_α and a disk Δ_β in D_β .

Assume δ to be innermost in D_α , that is $|\Delta_\alpha^\circ \cap D_\beta^\circ| = 0$. As the 2-sphere $\Delta_\alpha \cup \Delta_\beta$ is separating in B , the disk Δ_α cannot contain x . Hence, $\Delta_\alpha \cup \Delta_\beta$ is disjoint from s . By substituting D_β° by the disk $(D_\beta^\circ - \Delta_\beta) \cup \Delta_\alpha$ and perturbing it slightly, we obtain a contradiction with the minimality of $|D_\alpha^\circ \cap D_\beta^\circ|$. Therefore, $D_\alpha^\circ \cap D_\beta^\circ$ contains no simple closed curves.

Now suppose that $D_\alpha^\circ \cap D_\beta^\circ$ contains an arc δ , that, along with an arc of ∂D_α , bounds a disk Δ_α in D_α and, along with an arc of ∂D_β , bounds a disk Δ_β in D_β . We will assume δ to be innermost in D_α , that is, $|\Delta_\alpha^\circ \cap D_\beta^\circ| = 0$, and that Δ_α does not contain x . Again, by substituting D_β° by the disk $(D_\beta^\circ - \Delta_\beta) \cup \Delta_\alpha$ and perturbing it slightly, we obtain a contradiction with the minimality of $|D_\alpha^\circ \cap D_\beta^\circ|$. Therefore, $D_\alpha^\circ \cap D_\beta^\circ$ contains no arcs, which completes the proof that $|D_\alpha^\circ \cap D_\beta^\circ| = 0$.

Since the properly embedded disk $D_\alpha \cup D_\beta$ meets the prime tangle s transversely in a single point, then $\alpha \cup \beta$ bounds a disk O in ∂B that meets s in a single point. Then the sphere $D_\alpha \cup D_\beta \cup O$ meets s transversely in two points. This sphere is either parallel to ∂B or it bounds a 3-ball in B which meets s in a single unknotted arc. In either case, a or b is a trivial arc in the corresponding ball separated by

Fig. 4. If $\mathcal{T}$ is not essential, then s_1 is unknotted.

$D_\alpha \cup D_\beta$. We can then isotope this trivial arc, inside the ball containing it, into c . That shows that s can be isotoped into $b \cup c$ or $a \cup c$, respectively. $\square$

Proof. (Proof that the n -string tangle of Theorem 1.1 is essential) Let s_1 be the string in $\mathcal{T}$ with the pattern of a trefoil, and s_2 be one of the remaining parallel arcs. Since the trefoil is a prime knot, then (B, s_1) is prime. By Lemma 2.2, $(B, s_1 \cup s_2)$ is essential. Otherwise, s_1 could be isotoped into one of the strings $a \cup c$ or $b \cup c$ on Fig. 4, and s_1 would be unknotted. As all strings but s_1 are parallel to s_2 , then $\mathcal{T}$ is essential. $\square$

3. The 2-String Tangle of Fig. 2(a)

In this section, we correct the proof that the 2-string tangle of Fig. 2(a) is essential. The original argument is on the proof of Theorem 1.2, on p. 1450054-10. In this argument, we choose to add a certain rational tangle to $(B, s_1 \cup s_2)$ obtaining the knot 6_2 from Rolfsen's list [3]. We then follow by assuming the tangle $(B, s_1 \cup s_2)$ is inessential, hence rational, as the two strings are unknotted. From here, we isotope s_2 to the complement of B and jump to the conclusion that the knot 6_2 is the connected sum of two unknots. The fallacy on this argument is on the assumption

Fig. 5. A decomposition of the knot 8_{16} in tangles.

that by an isotopy of s_2 to the complement of B , we obtain an unknotted arc. In general, the argument is not valid as it would imply that all 2-bridge knots are trivial, which is not true.

Proof. (Proof that the 2-string tangle of Fig. 2(a) is essential) The knot 8_{16} , as in *KnotInfo* [2], has a decomposition defined by the tangle $(B, s_1 \cup s_2)$ and a rational tangle, depicted in Fig. 5.

As observed before, if the tangle $(B, s_1 \cup s_2)$ is inessential, then it is rational. This contradicts the knot 8_{16} being 3-bridge, known from the classification of 2-bridge knots by Schubert [4]. (See *KnotInfo* [2].) $\square$

4. The n -String Tangle of Theorem 1.2

In this section, we correct the proof that the n -string tangles of Theorem 1.2, for $n \geq 3$, are essential. The original argument, on p. 1450054-11, leads to a reduction on the crossing number of the diagram, but this is not a contradiction as the crossing number is not assumed to be minimal. Instead, it is minimal as a consequence of the corresponding diagram being essential (the subject of the proof itself) and previous results on the original paper. To prove that these tangles are essential, we follow a similar argument to the one in Sec. 2.

Proof. (Proof that the n -string tangles of Theorem 1.2 are essential) We prove that the n -string tangles of Fig. 2(a) are essential; the proofs for the n -string tangles of Figs. 2(b) and 2(c) follow similarly. Let s_1 and s_2 be the strings as in the previous section and s_3 one of the remaining parallel unknotted arcs. As all strings but s_1 and s_2 are parallel to s_3 , it suffices to prove that the tangle defined by $s_1 \cup s_2 \cup s_3$ is essential. Since $(B, s_1 \cup s_2)$ is essential, and both s_1 and s_2 are unknotted, then $(B, s_1 \cup s_2)$ is prime. If $(B, s_1 \cup s_2 \cup s_3)$ is inessential, then, by Lemma 2.2, $(B, s_1 \cup s_2)$ is isotopic to one of the tangles of Fig. 6, which is a contradiction with $(B, s_1 \cup s_2)$ being essential. $\square$

Fig. 6. If $(B, s_1 \cup s_2 \cup s_3)$ is not essential, then $(B, s_1 \cup s_2)$ is not essential.

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