ERRATUM AND ADDENDUM PROPERTY (FA) AND LATTICES IN $SU(2, 1)$

[Internet. J. Algebra Comput. Vol. 17, No. 7 (2007) 1335–1347]

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Received 10 September 2012

Accepted 25 September 2013

Published 5 November 2013

Communicated by J. Howie

Two proofs in the paper “Property (FA) and lattices in $SU(2, 1)$ ” contain errors and the statement of Theorem 1 requires correction. The purpose of this note is to correct these, which adds an extra assumption to Theorem 3. Fortunately, the corrections allow us to extend Theorem 3 to higher dimensions without this extra condition.

Keywords: Property (FA); discrete subgroups of Lie groups; complex hyperbolic geometry.

Mathematics Subject Classification: 20E08, 22E40; 20E06, 11F23

We first describe the organization of this addendum. Section 1 corrects the statement of [10, Theorem 1]. Section 2 corrects [10, Proposition 19], which applies to [10, Theorem 3 and Corollary 4]. This correction requires an extra hypothesis that, conjecturally, is unnecessary. Finally, Sec. 3 gives a corrected proof of [10, Proposition 20]. The corrected proof also implies Theorem 5 below, which generalizes [10, Theorem 3] to $SU(n, 1)$ for $n \geq 4$ (independent from the corrections to [10, Theorem 3]).

1. The Statement of Theorem 1

In [10, Theorem 1], it is shown that $PU(2, 1; \mathcal{O}_3)$ has Property (FA). This proof is correct. See [11] for a similar proof that $PU(2, 1; \mathbb{Z}[i])$ also has (FA). However, it is

then assumed that the lift of $\mathrm{PU}(2, 1; \mathcal{O}_3)$ to $\mathrm{SU}(2, 1)$ is $\mathrm{SU}(2, 1; \mathcal{O}_3)$. This is not the case. The correct statement of [10, Theorem 1] is the following theorem.

Theorem 1. *Let Γ be the lift of $\mathrm{PU}(2, 1; \mathcal{O}_3)$ to $\mathrm{SU}(2, 1)$. Then $\mathrm{PU}(2, 1; \mathcal{O}_3)$ and Γ have Property (FA). The groups $\mathrm{SU}(2, 1; \mathcal{O}_3)$ and $\mathrm{PSU}(2, 1; \mathcal{O}_3)$ do not.*

Proof. To show that $\mathrm{SU}(2, 1; \mathcal{O}_3)$ has (FA), [10] appeals to an exact sequence

$$1 \rightarrow \mathbb{Z}/3\mathbb{Z} \rightarrow \mathrm{SU}(2, 1; \mathcal{O}_3) \rightarrow \mathrm{PU}(2, 1; \mathcal{O}_3) \rightarrow 1. \quad (1)$$

However, the image $\mathrm{PSU}(2, 1; \mathcal{O}_3)$ of $\mathrm{SU}(2, 1; \mathcal{O}_3)$ in $\mathrm{PU}(2, 1)$ is a proper subgroup of $\mathrm{PU}(2, 1; \mathcal{O}_3)$. More precisely,

$$[\mathrm{PU}(2, 1; \mathcal{O}_3) : \mathrm{PSU}(2, 1; \mathcal{O}_3)] = 3 \quad (2)$$

because $\mathrm{PSU}(2, 1; \mathcal{O}_3)$ does not contain the image in $\mathrm{PU}(2, 1)$ of the element

$$n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{U}(2, 1; \mathcal{O}_3). \quad (3)$$

Therefore, the lift Γ of $\mathrm{PU}(2, 1; \mathcal{O}_3)$ to $\mathrm{SU}(2, 1)$ is generated by $\mathrm{SU}(2, 1; \mathcal{O}_3)$ and $\zeta_3^{-1/3} n$. This is a maximal lattice in $\mathrm{SU}(2, 1)$, and it has Property (FA) by the argument in [10].

In fact, $\mathrm{SU}(2, 1; \mathcal{O}_3)$ and $\mathrm{PSU}(2, 1; \mathcal{O}_3)$ do not have Property (FA), as we now prove. An explicit homomorphism of $\mathrm{PSU}(2, 1; \mathcal{O}_3)$ onto the infinite dihedral group D_∞ can be constructed as follows.

Using the presentation of Falbel–Parker (see [4, Proposition 5.11]) for $\mathrm{PU}(2, 1; \mathcal{O}_3)$ with generators J , R_1 and GAP [5], there is a unique normal index three subgroup of $\mathrm{PU}(2, 1; \mathcal{O}_3)$, which is necessarily $\mathrm{PSU}(2, 1; \mathcal{O}_3)$. Then $\mathrm{PSU}(2, 1; \mathcal{O}_3)$ has generators $a = J^2 R_1^{-1}$, $b = J R_1$, $c = R_1^2 J^{-1}$ and relations

$$\begin{aligned} & a^4, \\ & b^4, \\ & (bc)^2, \\ & bacacb^{-1}a^{-2}, \\ & c^{-1}a^2b^2ac^{-1}a^{-1}, \\ & bca^2c^{-1}b^{-1}a^2, \\ & a^{-1}b^{-1}a^{-2}bc^{-1}a^{-1}c^{-1}. \end{aligned} \quad (4)$$

Presenting D_∞ as

$$\langle x, y : x^2, y^2 \rangle, \quad (5)$$

the following condition determines a surjective representation $\rho : \mathrm{PSU}(2, 1; \mathcal{O}_3) \rightarrow D_\infty$:

$$\rho(a) = x, \quad \rho(b) = y, \quad \rho(c) = 1. \quad (6)$$

Therefore $\mathrm{PSU}(2, 1; \mathcal{O}_3)$, and thus $\mathrm{SU}(2, 1; \mathcal{O}_3)$, does not have Property (FA). $\square$

2. Lattices of Second Type

The proof of [10, Proposition 19] is not correct because Eq. (22) does not imply Eq. (23). The purpose of this section is to give a corrected statement and proof. At the end of the section, we discuss their relation to a conjectural picture of torsion in the cohomology of arithmetic groups. We replace [10, Proposition 19] with the following proposition.

Proposition 2. *Let $\Gamma < \mathrm{SU}(2, 1)$ be a congruence arithmetic lattice of second type, and suppose that there is a torsion-free congruence subgroup $\Gamma' < \Gamma$ such that $H^1(\Gamma'; \mathbb{F}_2)$ is trivial. Then Γ does not admit a homomorphism onto the infinite dihedral group.*

Proof. As in [10], we can assume that Γ is torsion free. Let $\rho : \Gamma \rightarrow D_\infty$ be a nontrivial representation. This produces a nontrivial element of $H^1(\Gamma; \mathbb{F}_2)$, which determines a nontrivial element of $H^1(M; \mathbb{F}_2)$, where $M = \mathbf{H}_{\mathbb{C}}^2/\Gamma$ and $\mathbb{F}_2$ is the field of two elements. (Since Γ is torsion free, M is a manifold and thus a $K(\Gamma, 1)$.)

Let $M' \rightarrow M$ be a congruence covering with fundamental group $\Gamma' < \Gamma$. Then, as explained in [10], $\rho|_{\Gamma'}$ has infinite dihedral image in $\rho(D_\infty)$. In particular, we see that $H^1(M'; \mathbb{F}_2)$ is nontrivial for every congruence covering of M . This proves the proposition. $\square$

This leads to the following corrected statement of [10, Theorem 3 and Corollary 4], which we combine here as a single statement.

Theorem 3. *Let $\Gamma < \mathrm{SU}(2, 1)$ be a congruence arithmetic lattice of second type, and suppose that Γ contains a torsion-free congruence subgroup Γ' such that $H^1(\Gamma'; \mathbb{F}_2)$ is trivial. Then Γ has Property (FA).*

We now briefly describe how the full statement of [10, Theorem 3] follows from a conjectural picture of the cohomology of complex hyperbolic manifolds coming from congruence arithmetic lattices of second type. The above argument implies that any $\rho : \Gamma \rightarrow D_\infty$ determines an element order two in $H^1(M'; \mathbb{Z})$ for all congruence coverings M' of M . Let M_∞ be the inverse limit over all congruence subgroups Γ_K of the corresponding quotients of complex hyperbolic space. Its cohomology is the inverse limit of the cohomology of the Γ_K (see [9]). Thus ρ determines a nontrivial

element of $H^1(M_\infty; \mathbb{Z})$, which is necessarily of finite order. Indeed, an element of infinite order in $H^1(M_\infty; \mathbb{Z})$ determines a nontrivial element of $H^1(M_\infty; \mathbb{Q})$, which contradicts [8, Theorem 15.3.1]. Recent works of Boyer [1] and Emerton–Gee [3] suggest that $H^1(M_\infty; \mathbb{Z})$ should be torsion free for all primes ℓ . See in particular the discussion on p. 4 of [3]. This conjecture, along with Proposition 2, implies [10, Proposition 19], and thus proves [10, Theorem 3 and Corollary 4].

3. Fake Projective Planes and Lattices of Second Type in Higher Dimensions

We now give a correct proof of [10, Proposition 20]. The proof there uses Klingler’s archimedean superrigidity theorem for fundamental groups of fake projective planes [6]. For this theorem to apply, the representation must have Zariski-dense image, and the representation given in [10] does not. Fortunately, the proof is quite elementary. The key observation is the following lemma (cf. [6, Lemma 2.14]).

Lemma 4. *Let Γ be the fundamental group of a compact Kähler manifold M for which $h^{1,0}(M) = 0$ and $H^2(M; \mathbb{C})$ is generated by the Kähler class. Then there is no representation of Γ onto the infinite dihedral group D_∞ .*

Proof. Given $\rho : \Gamma \rightarrow D_\infty$, there exists $\Lambda < \Gamma$ of index two so that $\rho(\Lambda) \cong \mathbb{Z}$. Thus the corresponding double cover N of M supports a nontrivial holomorphic 1-form α . Let g be the Deck transformation for the covering.

Since $H^{1,0}(M) = \{0\}$, g acts on $H^{1,0}(N)$ by -1 . In particular, $\alpha \wedge \bar{\alpha}$ is nonzero and g -invariant, and thus projects to a nontrivial $(1, 1)$ -form on M . This projection is necessarily a nonzero multiple of the Kähler class on M . Since the square of the Kähler class under the cup product is nonzero, this implies, as in [6, Lemma 2.14], that

$$(\alpha \wedge \bar{\alpha}) \wedge (\alpha \wedge \bar{\alpha}) \neq 0. \tag{7}$$

This is a contradiction. □

Then [10, Proposition 20] follows immediately. This also leads to the following generalization of [10, Theorem 3] to higher dimensions. Note that this does not require the additional assumptions required in dimension 2.

Theorem 5. *Let $\Gamma < \mathrm{SU}(n, 1)$, $n \geq 4$, be a congruence arithmetic lattice constructed via a division algebra of degree $n + 1$ over a CM-field. Then Γ has Property (FA).*

See [2] for the appropriate generalizations of the definitions from [10] and Sec. 2. In that paper, Clozel proves that $h^{1,0}(\mathbf{H}_{\mathbb{C}}^2/\Gamma; \mathbb{C}) = 0$ and $H^2(\mathbf{H}_{\mathbb{C}}^2/\Gamma; \mathbb{C})$ is generated by the Kähler class, so the proof follows the exact same strategy of the proof for fake projective planes in [10].

Remark 6. For $n = 3$, [2] does not imply $H^{1,1}$ is generated by the Kähler class, though it does imply $h^{1,0} = 0$. Thus, if the Picard number is 1, then Theorem 5 still applies. According to [7, Theorem 1.4.5], this is indeed the case.

Acknowledgments

I thank Alan Reid, Dick Canary, and Ralf Spatzier for conversations related to this addendum, Frank Calegari for bringing [1] to my attention, and Pascal Boyer and Matthew Emerton for correspondence about their work. Partially supported by NSF RTG grant DMS 0602191.

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