Infinite number of conservation laws and Darboux transformations for a 6-field integrable lattice system

Fangcheng Fan

*School of Mathematics, Jilin University,
Changchun 130012, P. R. China
fanfc16@mails.jlu.edu.cn*

Shaoyun Shi

*School of Mathematics, Jilin University,
Changchun 130012, P. R. China
State Key Laboratory of Automotive Simulation and Control,
Jilin University, Changchun 130012, P. R. China
shisy@jlu.edu.cn*

Zhiguo Xu*

*School of Mathematics, Jilin University,
Changchun 130012, P. R. China
xuzg2014@jlu.edu.cn*

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In this paper, we study a 6-field integrable lattice system, which, in some special cases, can be reduced to the self-dual network equation, the discrete second-order nonlinear Schrödinger equation and the relativistic Volterra lattice equation. With the help of the Lax pair, we construct infinitely many conservation laws and a new Darboux transformation for system. Exact solutions resulting from the obtained Darboux transformation are presented by using a given seed solution. Further, we generate the soliton solutions and plot the figures of one-soliton solutions with properly parameters.

Keywords: A 6-field integrable lattice system; conservation law; Darboux transformation; exact solution.

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*Corresponding author.

1. Introduction

Dating back to the work of Fermi *et al.* in the 1950s,¹ nonlinear integrable lattice equations have been the focus of many nonlinear studies. Particularly, many physically interesting phenomena can be modeled with nonlinear integrable lattice equations, including particle vibrations in lattices, currents in electrical networks, pulses in biological chains, etc. The study of integrable systems of lattice versions, therefore, has aroused increasing interest in recent years. A number of physically important nonlinear integrable lattice equations have been obtained and systematically discussed, for instance the Ablowitz–Ladik lattice,² the Belov–Chaltikian lattice,³ the Blaszak–Marciniak lattice,⁴ and so on. There have been some studies on higher-order symmetries, Hamiltonian structures and Casoratian solutions for lattice system.^{5,6} The Volterra lattice and the modified Volterra lattice equation have recently been solved by Darboux transformation.^{7,8} Also, some active researches are interested in the kind of exact solutions to integrable equations called lumps.^{9–11}

In order to model the nonlinear Frenkel-like excitations on a triangular lattice ribbon, Vakhnenko introduced the following 6-field integrable lattice system¹²:

$$\begin{cases} p_{n,t} = p_n q_n x_{n-1} - p_n q_{n+1} x_n - r_n x_{n-1} + r_n x_n - q_n y_n + q_{n+1} y_n, \\ q_{n,t} = q_n r_{n-1} y_n - q_n q_{n+1} x_n + q_{n+1} z_n - p_n q_n - r_{n-1} + r_n, \\ r_{n,t} = r_n r_{n-1} y_n - q_{n+1} r_n x_n - p_n r_{n-1} + r_n z_n - q_n + q_{n+1}, \\ x_{n,t} = q_n x_n x_{n-1} - r_n x_n y_{n+1} - x_{n-1} z_n + p_n x_n - y_n + y_{n+1}, \\ y_{n,t} = q_n x_{n-1} y_n - r_n y_n y_{n+1} + p_n y_{n+1} - y_n z_n - x_{n-1} + x_n, \\ z_{n,t} = r_{n-1} y_n z_n - r_n y_{n+1} z_n - r_{n-1} x_n + r_n x_n - q_n y_n + q_n y_{n+1}, \end{cases} \quad (1)$$

where $t \in \mathbb{R}$ is a continuous variable, $n \in \mathbb{N}$ is a discrete variable. Moreover, in the framework of generalized recursive approach, the author constructed the most significant local conservation laws explicitly, and presented the Hamiltonian formulation and Poisson structure for (1). He also derived the Darboux transformation by using the Darboux-dressing method, from which the multicomponent one-soliton solution was generated. In fact, by taking an appropriate choice of discrete variables, Vakhnenko found that (1) can reduce to an integrable nonlinear Schrödinger system on a triangular lattice ribbon,¹³ which had been investigated in Ref. 14. In addition, (1) also permits at least two variants of purely real-valued reductions that yield the systems of semidiscrete KdV type understood in a sense used by Tsuchida, Ujino and Wadati.¹⁵

Interestingly, we should point out that (1) can reduce to some famous equations in some special cases. For example, when

$$\begin{aligned} p_n &= I_n V_n, & q_n &= \pm I_n, & r_n &= V_n, \\ x_n &= \pm V_n, & y_n &= I_n, & z_n &= I_n V_n, \end{aligned}$$

(1) comes to the following nonlinear self-dual network equation^{2,16}:

$$\begin{cases} I_{n,t} = (1 \pm I_n^2)(\pm V_{n-1} \mp V_n), \\ V_{n,t} = (1 \pm V_n^2)(\pm I_n \mp I_{n+1}), \end{cases} \quad (2)$$

which describes propagation of electrical signals in electric circuit. Rich mathematical structures related to (2) have been obtained. For example, the exact solution was found by inverse scattering in Ref. 2, the kink-shaped solitary solution was presented by ADM-padé technique in Ref. 17, and the N -fold Darboux transformation, infinite conservation laws were constructed in Ref. 18. If we set

$$p_n = R_n S_n, \quad q_n = \mp R_n^*, \quad r_n = S_n, \quad x_n = \mp S_n^*, \quad y_n = R_n, \quad z_n = R_n^* S_n^*,$$

and $\bar{t} = -it$, the system (1) is regarded as two coupled discrete nonlinear Schrödinger equations

$$\begin{cases} R_{n,\bar{t}} = i(1 \pm R_n R_n^*)(\mp S_n^* \pm S_{n-1}^*), \\ S_{n,\bar{t}} = i(1 \pm S_n S_n^*)(\mp R_{n+1}^* \pm R_n^*), \end{cases} \quad (3)$$

which is a special reduction of the four-field Ablowitz–Ladik equation,² and infinite conservation laws, N -fold Darboux transformation, N -soliton solutions of (3) were obtained in Ref. 19. Let

$$\begin{aligned} p_n &= 2w_{n+1} + 2v_n + 1, & q_n &= -(2v_n + 1), & r_n &= -(2w_{n+1} + 1), \\ x_n &= -(2w_{n+1} + 1), & y_n &= -(2v_n + 1), & z_n &= 2w_{n+1} + 2v_n + 1, \end{aligned}$$

the system (1) becomes the relativistic Volterra lattice²⁰

$$\begin{cases} u_{n,t} = u_n(v_n - v_{n-1} + u_n v_n - u_{n-1} v_{n-1}), \\ v_{n,t} = v_n(u_{n+1} - u_n + u_{n+1} v_{n+1} - u_n v_n). \end{cases} \quad (4)$$

Mathematical structures related to (4) such as Lax pairs,²¹ Hamiltonian structure, master symmetries, infinitely many conservation laws,²² bilinear structure,²³ and Darboux transformation²⁴ have been closely studied.

In this paper, we firstly establish infinitely many conservation laws formulaically for (1) by using the Lax pair. Secondly, we derive a new Darboux transformation via the method proposed and developed by Wu *et al.*^{25–27} Finally, by using the obtained Darboux transformation, we generate the exact solutions of the system. Meanwhile, the soliton solutions, and the figures of one-soliton solutions with properly parameters are presented.

The rest of the paper is organized as follows. In Sec. 2, by means of the Lax pair of (1), we derive infinitely many conservation laws. In Sec. 3, the Darboux transformation of (1) is constructed by means of the gauge transformation and Lax pair. In Sec. 4, we get the exact solutions through the obtained Darboux transformation, from which we generate the soliton solutions and plot the figures of the one-soliton solutions with properly parameters. Some remarks and summary will be presented in the last section.

2. Lax Pair and Infinitely Many Conservation Laws

The Lax pair corresponding to (1) can be given by the discrete spectral problem¹²

$$E\phi_n = U_n\phi_n, \quad U_n = \begin{pmatrix} \lambda^2 + p_n & \lambda q_n + \frac{1}{\lambda}r_n \\ \lambda x_n + \frac{1}{\lambda}y_n & z_n + \frac{1}{\lambda^2} \end{pmatrix} \quad (5)$$

and the auxiliary evolution problem

$$\phi_{n,t} = V_n\phi_n, \quad V_n = \begin{pmatrix} \lambda^2 - q_n x_{n-1} & \lambda q_n + \frac{1}{\lambda}r_{n-1} \\ \lambda x_{n-1} + \frac{1}{\lambda}y_n & \frac{1}{\lambda^2} - r_{n-1}y_n \end{pmatrix}, \quad (6)$$

where the shift operator E is defined by $Ef(n, t) = f(n+1, t) = f_{n+1}$, $\phi_n = (\phi_{1,n}, \phi_{2,n})^T$ is eigenfunction, λ is the spectral parameter and $\lambda_t = 0$, $p_n, q_n, r_n, x_n, y_n, z_n$ are potential functions. The compatibility condition of (5) and (6) leads to the discrete zero curvature equation²⁸

$$U_{n,t} = (EV_n)U_n - U_nV_n,$$

from which we arrive at (1). Based on these facts, we know that (1) is Lax integrable.

The existence of infinite number of conservation laws is the key integrability characteristic for an integrable equation. A systematic method for finding an infinite hierarchy of conservation laws for semi-discrete nonlinear integrable systems has a long history. In Refs. 15 and 29, by using the recursive approach, infinitely many conservation laws for the coupled KdV equations and coupled nonlinear Schrödinger equations were derived. Motivating by the above work, Zhu *et al.* gave a simple approach to derive conservation laws for some semi-discrete systems from Lax pair directly.^{22,30} Recently, Vakhnenko modified procedure of the recursive approach,^{31,32} which is applicable for any semi-discrete integrable system associated with the auxiliary square spectral and evolution matrices of an arbitrary order.

In fact, the hierarchy of conservation laws for (1) had been studied by Vakhnenko in Ref. 12. Next, we will derive infinite number of conservation laws for (1) by using the method in Refs. 22 and 30, and give conservation laws formulaically.

Firstly, let $\Gamma_n = \frac{\phi_{2,n}}{\phi_{1,n}}$, then a direct calculation from (5) leads to the following Riccati equation

$$\begin{aligned} &\lambda^4\Gamma_{n+1} + \lambda^3q_n\Gamma_n\Gamma_{n+1} + \lambda^2(p_n\Gamma_{n+1} - z_n\Gamma_n) \\ &+ \lambda r_n\Gamma_n\Gamma_{n+1} - \Gamma_n - \lambda^3x_n - \lambda y_n = 0. \end{aligned} \quad (7)$$

Suppose that

$$\Gamma_n = \sum_{j=1}^{\infty} \lambda^{-j} f_n^{(j)} \quad (8)$$

is a solution of (7), where $f_n^{(j)}$ are the functions to be determined. Substituting (8) into (7), and comparing the coefficients of the same power of λ , we obtain

$$\begin{aligned} f_n^{(1)} &= x_{n-1}, \\ f_n^{(3)} &= -q_{n-1}x_{n-2}x_{n-1} - p_{n-1}x_{n-1} + x_{n-2}z_{n-1} + y_{n-1}, \\ f_n^{(2j+1)} &= -q_{n-1}(f_{n-1}^{(1)}f_n^{(2j-1)} + f_{n-1}^{(3)}f_n^{(2j-3)} + \dots + f_{n-1}^{(2j-1)}f_n^{(1)}) \\ &\quad + r_{n-1}(f_{n-1}^{(1)}f_n^{(2j-3)} + f_{n-1}^{(3)}f_n^{(2j-5)} + \dots + f_{n-1}^{(2j-3)}f_n^{(1)}) \\ &\quad - p_{n-1}f_n^{(2j-1)} + z_{n-1}f_{n-1}^{(2j-1)} + f_{n-1}^{(2j-3)}, \quad j = 2, 3, \dots, \\ f_n^{(2k)} &= 0, \quad k = 1, 2, \dots \end{aligned}$$

A direct computation from (5) and (6) leads to

$$\begin{aligned} \frac{\phi_{1,n+1}}{\phi_{1,n}} &= \lambda^2 + p_n + \left(\lambda q_n + \frac{1}{\lambda} r_n \right) \Gamma_n, \\ \frac{\phi_{1,n,t}}{\phi_{1,n}} &= \lambda^2 - q_n x_{n-1} + \left(\lambda q_n + \frac{1}{\lambda} r_{n-1} \right) \Gamma_n. \end{aligned}$$

Note that

$$\left(\ln \frac{\phi_{1,n+1}}{\phi_{1,n}} \right)_t = (E - 1) \frac{\phi_{1,n,t}}{\phi_{1,n}},$$

we get

$$\left[\ln \left(1 + \frac{1}{\lambda^2} \Psi_n \right) \right]_t = (E - 1) \Phi_n, \quad (9)$$

where

$$\begin{aligned} \Psi_n &= p_n + \lambda q_n \Gamma_n + \frac{1}{\lambda} r_n \Gamma_n \triangleq \sum_{j=0}^{\infty} v_j \lambda^{-2j}, \\ \Phi_n &= -q_n x_{n-1} + \lambda q_n \Gamma_n + \frac{1}{\lambda} r_{n-1} \Gamma_n \triangleq \sum_{k=1}^{\infty} w_k \lambda^{-2k}, \end{aligned}$$

with

$$\begin{aligned} v_0 &= p_n + q_n f_n^{(1)}, \quad v_j = r_n f_n^{(2j-1)} + q_n f_n^{(2j+1)}, \quad j = 0, 1, \dots, \\ w_1 &= -q_n x_{n-1} + q_n f_n^{(1)}, \quad w_k = r_{n-1} f_n^{(2j-1)} + q_n f_n^{(2j+1)}, \quad k = 1, 2, \dots \end{aligned}$$

Now substituting Ψ_n and Φ_n into (9), and making a comparison of the same powers of λ on both sides, we obtain the following infinitely many conservation laws for (1):

$$\rho_{n,t}^{(j)} = (E - 1) J_n^{(j)}, \quad (10)$$

where

$$\begin{aligned}\rho_n^{(1)} &= v_0, & J_n^{(1)} &= w_1, \\ \rho_n^{(2)} &= v_1 - \frac{1}{2}v_0^2, & J_n^{(2)} &= w_2, \\ \rho_n^{(j)} &= v_{j-1} - \frac{1}{2} \sum_{l_1+l_2=j-2} v_{l_1}v_{l_2} + \cdots + (-1)^{j-2}v_0^{j-2}v_1 + \frac{(-1)^{j-1}}{j}v_0^j, \\ J_n^{(j)} &= w_j, & j &= 3, 4, \dots\end{aligned}$$

Secondly, let $\tilde{\Gamma}_n = \frac{\phi_{1,n}}{\phi_{2,n}}$, then a direct calculation from (5) leads to the following Riccati equation:

$$\begin{aligned}\tilde{\Gamma}_{n+1} + \lambda y_n \tilde{\Gamma}_n \tilde{\Gamma}_{n+1} + \lambda^2(z_n \tilde{\Gamma}_{n+1} - p_n \tilde{\Gamma}_n) \\ + \lambda^3 x_n \tilde{\Gamma}_n \tilde{\Gamma}_{n+1} + \lambda^4 \tilde{\Gamma}_n - \lambda^3 q_n - \lambda r_n = 0.\end{aligned}\quad (11)$$

Suppose that

$$\tilde{\Gamma}_n = \sum_{j=1}^{\infty} \lambda^j \tilde{f}_n^{(j)} \quad (12)$$

is a solution of (11), where $\tilde{f}_n^{(j)}$ are the functions to be determined. Substituting (12) into (11), and making a comparison of the powers of λ on both sides of equation (11), we obtain

$$\begin{aligned}\tilde{f}_n^{(1)} &= r_{n-1}, \\ \tilde{f}_n^{(3)} &= -y_{n-1}r_{n-2}r_{n-1} - z_{n-1}r_{n-1} + r_{n-2}p_{n-1} + q_{n-1}, \\ \tilde{f}_n^{(2j+1)} &= -y_{n-1}(\tilde{f}_{n-1}^{(1)}\tilde{f}_n^{(2j-1)} + \tilde{f}_{n-1}^{(3)}\tilde{f}_n^{(2j-3)} + \cdots + \tilde{f}_{n-1}^{(2j-1)}\tilde{f}_n^{(1)}) \\ &\quad - x_{n-1}(\tilde{f}_{n-1}^{(1)}\tilde{f}_n^{(2j-3)} + \tilde{f}_{n-1}^{(3)}\tilde{f}_n^{(2j-5)} + \cdots + \tilde{f}_{n-1}^{(2j-3)}\tilde{f}_n^{(1)}) \\ &\quad - z_{n-1}\tilde{f}_n^{(2j-1)} + p_{n-1}\tilde{f}_{n-1}^{(2j-1)} - \tilde{f}_{n-1}^{(2j-3)}, \quad j = 2, 3, \dots, \\ \tilde{f}_n^{(2k)} &= 0, \quad k = 1, 2, \dots\end{aligned}$$

In this case, we obtain the following infinitely many conservation laws for (1):

$$\tilde{\rho}_{n,t}^{(j)} = (E - 1)\tilde{J}_n^{(j)}, \quad (13)$$

where

$$\begin{aligned}\tilde{\rho}_n^{(1)} &= \tilde{v}_0, & \tilde{J}_n^{(1)} &= \tilde{w}_1, \\ \tilde{\rho}_n^{(2)} &= \tilde{v}_1 - \frac{1}{2}\tilde{v}_0^2, & \tilde{J}_n^{(2)} &= \tilde{w}_2, \\ \tilde{\rho}_n^{(j)} &= \tilde{v}_{j-1} - \frac{1}{2} \sum_{l_1+l_2=j-2} \tilde{v}_{l_1}\tilde{v}_{l_2} + \cdots + (-1)^{j-2}\tilde{v}_0^{j-2}\tilde{v}_1 + \frac{(-1)^{j-1}}{j}\tilde{v}_0^j, \\ \tilde{J}_n^{(j)} &= \tilde{w}_j, & j &= 3, 4, \dots,\end{aligned}$$

with

$$\begin{aligned}\tilde{v}_0 &= z_n + y_n \tilde{f}_n^{(1)}, \quad \tilde{v}_j = x_n \tilde{f}_n^{(2j-1)} + y_n \tilde{f}_n^{(2j+1)}, \quad j = 0, 1, \dots, \\ \tilde{w}_1 &= x_{n-1} \tilde{f}_n^{(1)} + y_n \tilde{f}_n^{(3)}, \quad \tilde{w}_k = x_{n-1} \tilde{f}_n^{(2k-1)} + y_n \tilde{f}_n^{(2k+1)}, \quad k = 1, 2, \dots\end{aligned}$$

Remark 2.1. We point out that referring onto the already known general formulas for the collection of generating equations and the set of Riccati equation presented in Ref. 12, the hierarchy of infinite number of conservation laws for system (1) can also be obtained.

Remark 2.2. The infinite number of conservation laws (10) and (13) for system (1) satisfy the symmetry between the field variables $p_n \rightleftharpoons z_n$, $q_n \rightleftharpoons y_n$, $r_n \rightleftharpoons x_n$.

3. Darboux Transformation

In this section, we proceed to establish the Darboux transformation for (1). In fact, Darboux transformation is a special gauge transformation of the spectral problem and transforms the spectral problem into another spectral problem of the same type.^{33–35} For this, we introduce following gauge transformation:

$$\tilde{\phi}_n = T_n \phi_n,$$

which can transform the spectral problems (5) and (6) into

$$E \tilde{\phi}_n = \tilde{U}_n \tilde{\phi}_n, \quad \tilde{\phi}_{n,t} = \tilde{V}_n \tilde{\phi}_n,$$

where

$$\tilde{U}_n = T_{n+1} U_n T_n^{-1}, \quad \tilde{V}_n = (T_{nt} + T_n V_n) T_n^{-1},$$

and the matrixes $\tilde{U}_n$, $\tilde{V}_n$ have the same structures as U_n and V_n , respectively. We shall construct Darboux transformation as follows.

Assume that $(p_n, q_n, r_n, x_n, y_n, z_n)$ is a given solution to (1), and $\varphi_n(\lambda) = (\varphi_{1,n}(\lambda), \varphi_{2,n}(\lambda))^T$, $\psi_n(\lambda) = (\psi_{1,n}(\lambda), \psi_{2,n}(\lambda))^T$ are two linear independent solutions of (5) and (6). Define

$$\sigma_{i,n} = \frac{\varphi_{2,n}(\lambda_i) + \gamma_i \psi_{2,n}(\lambda_i)}{\varphi_{1,n}(\lambda_i) + \gamma_i \psi_{1,n}(\lambda_i)}, \quad i = 1, 2 \quad (14)$$

and

$$\begin{aligned}a_n &= \frac{\lambda_1 \lambda_2 (\lambda_1 \sigma_{2,n} - \lambda_2 \sigma_{1,n})}{\lambda_1 \sigma_{1,n} - \lambda_2 \sigma_{2,n}}, \quad b_n = \frac{\lambda_2^2 - \lambda_1^2}{\lambda_1 \sigma_{1,n} - \lambda_2 \sigma_{2,n}}, \\ c_n &= \frac{\sigma_{1,n} \sigma_{2,n} (\lambda_2^2 - \lambda_1^2)}{\lambda_1 \lambda_2 (\lambda_1 \sigma_{1,n} - \lambda_2 \sigma_{2,n})}, \quad d_n = \frac{\lambda_1 \sigma_{2,n} - \lambda_2 \sigma_{1,n}}{\lambda_1 \lambda_2 (\lambda_1 \sigma_{1,n} - \lambda_2 \sigma_{2,n})},\end{aligned} \quad (15)$$

where parameters λ_i and γ_i ($\lambda_1 \neq \lambda_2 \neq 0$, $\gamma_1 \neq \gamma_2$) are suitably chosen such that all the denominators in (14) and (15) are not zero. From (5) and (14), we have

$$\sigma_{i,n+1} = \frac{\mu_{i,n}}{\nu_{i,n}}, \quad i = 1, 2, \quad (16)$$

where

$$\mu_{i,n} = \lambda_i x_n + \frac{y_n}{\lambda_i} + \left(z_n + \frac{1}{\lambda_i^2} \right) \sigma_{i,n}, \quad \nu_{i,n} = \lambda_i^2 + p_n + \left(\lambda_i q_n + \frac{r_n}{\lambda_i} \right) \sigma_{i,n}.$$

Shifting (15) by operator E , we get

$$\begin{aligned} a_{n+1} &= \frac{\lambda_1 \lambda_2 (\lambda_1 \sigma_{2,n+1} - \lambda_2 \sigma_{1,n+1})}{\lambda_1 \sigma_{1,n+1} - \lambda_2 \sigma_{2,n+1}}, & b_{n+1} &= \frac{\lambda_2^2 - \lambda_1^2}{\lambda_1 \sigma_{1,n+1} - \lambda_2 \sigma_{2,n+1}}, \\ c_{n+1} &= \frac{\sigma_{1,n+1} \sigma_{2,n+1} (\lambda_2^2 - \lambda_1^2)}{\lambda_1 \lambda_2 (\lambda_1 \sigma_{1,n+1} - \lambda_2 \sigma_{2,n+1})}, & d_{n+1} &= \frac{\lambda_1 \sigma_{2,n+1} - \lambda_2 \sigma_{1,n+1}}{\lambda_1 \lambda_2 (\lambda_1 \sigma_{1,n+1} - \lambda_2 \sigma_{2,n+1})}. \end{aligned} \quad (17)$$

It is obvious that the matrix T_n is a key step for constructing Darboux transformation, a proper T_n will ensure the correctness of Darboux transformation. Making use of (15), we take T_n as

$$T_n = \begin{pmatrix} \lambda^2 + a_n & \lambda b_n \\ \lambda c_n & 1 + \lambda^2 d_n \end{pmatrix}. \quad (18)$$

Through calculations, we can verify from (15) that $\pm \lambda_1$ and $\pm \lambda_2$ are all the roots of the determinant of T_n . Therefore, we have

$$\det T_n = d_n (\lambda + \lambda_1) (\lambda - \lambda_1) (\lambda + \lambda_2) (\lambda - \lambda_2). \quad (19)$$

In order to construct the Darboux transformation successfully, we need to ensure that $\tilde{U}_n$ and $\tilde{V}_n$ have the same forms as U_n and V_n , respectively. We first show that the structure of U_n is the same as $\tilde{U}_n$. Let

$$F_n(\lambda) = T_{n+1} U_n T_n^* = \begin{pmatrix} F_n^1(\lambda) & F_n^2(\lambda) \\ F_n^3(\lambda) & F_n^4(\lambda) \end{pmatrix},$$

where T_n^* denotes the adjoint matrix of T_n , $F_n^j(\lambda)$ ($j = 1, 2, 3, 4$) are polynomials in λ , and the details of $F_n^j(\lambda)$ are omitted. From (14) and (15), we have

$$\lambda_i^2 + a_n + \lambda_i \sigma_{i,n} b_n = 0, \quad \lambda_i c_n + \sigma_{i,n} + \lambda_i^2 \sigma_{i,n} d_n = 0, \quad i = 1, 2. \quad (20)$$

By using (15) and (20), it can be verified that $\pm \lambda_1$ and $\pm \lambda_2$ are the roots of $F_n^j(\lambda)$ ($j = 1, 2, 3, 4$). Take into account the order of $F_n^j(\lambda)$ and (19), we conclude that there exists a matrix $P_n(\lambda)$ such that

$$T_{n+1} U_n T_n^* = \det T_n \cdot P_n(\lambda), \quad (21)$$

where

$$P_n(\lambda) = \begin{pmatrix} \lambda^2 P_n^1 + P_n^2 & \lambda P_n^3 + \frac{P_n^4}{\lambda} \\ \lambda P_n^5 + \frac{P_n^6}{\lambda} & P_n^7 + \frac{P_n^8}{\lambda^2} \end{pmatrix},$$

with $P_n^k (k = 1, 2, \dots, 8)$ are independent of λ . To determine P_n^k , we rewrite (21) as

$$T_{n+1}U_n = P_nT_n. \quad (22)$$

Equating the coefficients of λ^j ($j = 4, 3, \dots, -2$) in (22) we have

$$\begin{aligned} P_n^1 &= 1, \quad P_n^2 = p_n + b_{n+1}x_n + a_{n+1} - a_n - \frac{c_n(q_n - b_n)}{d_n}, \quad P_n^3 = \frac{q_n - b_n}{d_n}, \\ P_n^4 &= a_{n+1}r_n + b_{n+1}, \quad P_n^5 = d_{n+1}x_n + c_{n+1}, \quad P_n^6 = \frac{y_n - c_n}{a_n}, \\ P_n^7 &= \frac{d_{n+1}z_n + c_{n+1}q_n - b_nc_{n+1} - b_nd_{n+1}x_n}{d_n}, \quad P_n^8 = 1. \end{aligned}$$

Let

$$\begin{aligned} \tilde{p}_n &= p_n + b_{n+1}x_n + a_{n+1} - a_n - \frac{c_n(q_n - b_n)}{d_n}, \quad \tilde{q}_n = \frac{q_n - b_n}{d_n}, \\ \tilde{r}_n &= a_{n+1}r_n + b_{n+1}, \quad \tilde{x}_n = d_{n+1}x_n + c_{n+1}, \\ \tilde{y}_n &= \frac{y_n - c_n}{a_n}, \quad \tilde{z}_n = \frac{d_{n+1}z_n + c_{n+1}q_n - b_nc_{n+1} - b_nd_{n+1}x_n}{d_n}, \end{aligned} \quad (23)$$

then

$$P_n^2 = \tilde{p}_n, \quad P_n^3 = \tilde{q}_n, \quad P_n^4 = \tilde{r}_n, \quad P_n^5 = \tilde{x}_n, \quad P_n^6 = \tilde{y}_n, \quad P_n^7 = \tilde{z}_n,$$

which means

$$\tilde{U}_n = T_{n+1}U_nT_n^{-1} = P_n(\lambda) = \begin{pmatrix} \lambda^2 + \tilde{p}_n & \lambda\tilde{q}_n + \frac{1}{\lambda}\tilde{r}_n \\ \lambda\tilde{x}_n + \frac{1}{\lambda}\tilde{y}_n & \tilde{z}_n + \frac{1}{\lambda^2} \end{pmatrix}.$$

Thus, we show that the matrix $\tilde{U}_n$ has the same form as U_n .

Proceeding in a similar manner, we can show that $\tilde{V}_n$ has the same structure with V_n . Let

$$G_n(\lambda) = (T_{n,t} + T_nV_n)T_n^* = \begin{pmatrix} G_n^1(\lambda) & G_n^2(\lambda) \\ G_n^3(\lambda) & G_n^4(\lambda) \end{pmatrix},$$

where T_n^* denotes the adjoint matrix of T_n , and the details of $G_n^j(\lambda)$ ($j = 1, 2, 3, 4$) are omitted. By virtue of (6) and (14), we obtain

$$\begin{aligned} \sigma_{i,n,t} &= \lambda_i x_{n-1} + \frac{y_n}{\lambda_i} + \left(\frac{1}{\lambda_i^2} - r_{n-1}y_n - \lambda_i^2 + q_n x_{n-1} \right) \sigma_{i,n} \\ &\quad - \left(\lambda_i q_n + \frac{r_{n-1}}{\lambda_i} \right) \sigma_{i,n}^2, \quad i = 1, 2. \end{aligned} \quad (24)$$

Deriving on both sides of (20), we have

$$\begin{aligned} a_{n,t} &= -\lambda_i b_{n,t} \sigma_{i,n} - \lambda_i b_n \sigma_{i,n,t}, \\ c_{n,t} &= -\frac{1}{\lambda_i} \sigma_{i,n,t} - \lambda_i d_{n,t} \sigma_{i,n} - \lambda d_n \sigma_{i,n,t}, \quad i = 1, 2. \end{aligned} \quad (25)$$

With the help of (20), (24) and (25), it can be verified that $\pm\lambda_1, \pm\lambda_2$ are the roots of $G_n^j(\lambda)$. Hence, there exists a matrix $Q_n(\lambda)$ such that

$$(T_{n,t} + T_n V_n) T_n^* = \det T_n \cdot Q_n(\lambda), \quad (26)$$

with

$$Q_n(\lambda) = \begin{pmatrix} Q_n^1 \lambda^2 + Q_n^2 & Q_n^3 \lambda + \frac{1}{\lambda} Q_n^4 \\ Q_n^5 \lambda + \frac{1}{\lambda} Q_n^6 & Q_n^7 + \frac{1}{\lambda^2} Q_n^8 \end{pmatrix},$$

where $Q_n^k (k = 1, 2, \dots, 8)$ are undetermined functions independent of λ . In order to obtain Q_n^k , we rewrite (26) as

$$T_{n,t} + T_n V_n = Q_n T_n. \quad (27)$$

Comparing the coefficients of $\lambda^j (j = 4, 3, \dots, -2)$ in (27), we have

$$\begin{aligned} Q_n^1 &= 1, & Q_n^2 &= b_n x_{n-1} - q_n x_{n-1} - \frac{c_n(q_n - b_n)}{d_n}, \\ Q_n^3 &= \frac{q_n - b_n}{d_n}, & Q_n^4 &= b_n + a_n r_{n-1}, \\ Q_n^5 &= c_n + d_n x_{n-1}, & Q_n^6 &= \frac{y_n - c_n}{a_n}, \\ Q_n^7 &= c_n r_{n-1} - r_{n-1} y_n - \frac{b_n(y_n - c_n)}{a_n}, & Q_n^8 &= 1. \end{aligned} \quad (28)$$

By (23) and (28), we can verify

$$\begin{aligned} Q_n^2 &= -\tilde{q}_n \tilde{x}_{n-1}, & Q_n^3 &= \tilde{q}_n, & Q_n^4 &= \tilde{r}_{n-1}, \\ Q_n^5 &= \tilde{x}_{n-1}, & Q_n^6 &= \tilde{y}_n, & Q_n^7 &= -\tilde{r}_{n-1} \tilde{y}_n, \end{aligned}$$

so

$$\tilde{V}_n = (T_{n,t} + T_n V_n) T_n^{-1} = \begin{pmatrix} \lambda^2 - \tilde{q}_n \tilde{x}_{n-1} & \lambda \tilde{q}_n + \frac{1}{\lambda} \tilde{r}_{n-1} \\ \lambda \tilde{x}_{n-1} + \frac{1}{\lambda} \tilde{y}_n & \frac{1}{\lambda^2} - \tilde{r}_{n-1} \tilde{y}_n \end{pmatrix}.$$

Therefore, the matrix $\tilde{V}_n$ has the same form as V_n .

Based on the above facts, we can now conclude the following theorem.

Theorem 3.1. $(\phi_n, p_n, q_n, r_n, x_n, y_n, z_n) \rightarrow (\tilde{\phi}_n, \tilde{p}_n, \tilde{q}_n, \tilde{r}_n, \tilde{x}_n, \tilde{y}_n, \tilde{z}_n)$ is a Darboux transformation of (1). Furthermore, if $(p_n, q_n, r_n, x_n, y_n, z_n)$ are the solutions

of (1), then $(\tilde{p}_n, \tilde{q}_n, \tilde{r}_n, \tilde{x}_n, \tilde{y}_n, \tilde{z}_n)$ are the new solutions under the transformation

$$\begin{aligned}\tilde{p}_n &= p_n + b_{n+1}x_n + a_{n+1} - a_n - \frac{c_n(q_n - b_n)}{d_n}, \\ \tilde{q}_n &= \frac{q_n - b_n}{d_n}, \quad \tilde{r}_n = a_{n+1}r_n + b_{n+1}, \\ \tilde{x}_n &= d_{n+1}x_n + c_{n+1}, \quad \tilde{y}_n = \frac{y_n - c_n}{a_n}, \\ \tilde{z}_n &= \frac{d_{n+1}z_n + c_{n+1}q_n - b_nc_{n+1} - b_nd_{n+1}x_n}{d_n}.\end{aligned}\tag{29}$$

Remark 3.1. We point out that Vakhnenko also constructed the Darboux transformation of system (1) through the following Darboux matrix^{12,14}:

$$S(n|\lambda) = \begin{pmatrix} \lambda^2 + u_{11}(n) & t_{12}(n)\lambda + u_{12}(n)\lambda^{-1} \\ t_{21}(n)\lambda + u_{21}(n)\lambda^{-1} & t_{22}(n) + \lambda^{-2} \end{pmatrix},$$

which satisfies the interchange $\lambda \rightleftharpoons \frac{1}{\lambda}$ of spectral parameters and symmetries between the field variables $p_n \rightleftharpoons z_n, q_n \rightleftharpoons y_n, r_n \rightleftharpoons x_n$. In Sec. 3, we establish explicit Darboux transformation of system (1) by using of the Darboux matrix (18) with four Darboux functions. Considering this interchange and symmetries, the Darboux matrix we consider is not unique and there exist alternative form with the preferable use of parameter $\frac{1}{\lambda}$.

4. Exact Solutions

The aim of this section is to derive some explicit exact solutions for (1) by applying the Darboux transformation obtained in Sec. 3. Obviously, (1) has a seed solution $p_n = 1, q_n = 0, r_n = 1, x_n = 0, y_n = 1, z_n = 0$, and the Lax pair related to the seed solution is

$$E\phi_n = \begin{pmatrix} \lambda^2 + 1 & \frac{1}{\lambda} \\ \frac{1}{\lambda} & \frac{1}{\lambda^2} \end{pmatrix} \phi_n, \quad \phi_{n,t} = \begin{pmatrix} \lambda^2 & \frac{1}{\lambda} \\ \frac{1}{\lambda} & \frac{1}{\lambda^2} - 1 \end{pmatrix} \phi_n.\tag{30}$$

We can obtain two linearly independent solutions of (30) as

$$\begin{aligned}\begin{pmatrix} \varphi_{1,n}(\lambda) \\ \varphi_{2,n}(\lambda) \end{pmatrix} &= \alpha^n e^{\tau t} \begin{pmatrix} 1 \\ -\lambda^3 - \lambda + \lambda\alpha \end{pmatrix}, \\ \begin{pmatrix} \psi_{1,n}(\lambda) \\ \psi_{2,n}(\lambda) \end{pmatrix} &= \beta^n e^{\rho t} \begin{pmatrix} 1 \\ -\lambda^3 - \lambda + \lambda\beta \end{pmatrix},\end{aligned}$$

where

$$\begin{aligned}\alpha &= \alpha(\lambda) = \frac{\lambda^2 + \frac{1}{\lambda^2} + 1 + \sqrt{\lambda^4 + \frac{1}{\lambda^4} + 2\lambda^2 + \frac{2}{\lambda^2} - 1}}{2}, \\ \tau &= \tau(\lambda) = \frac{\lambda^2 + \frac{1}{\lambda^2} - 1 + \sqrt{\lambda^4 + \frac{1}{\lambda^4} + 2\lambda^2 + \frac{2}{\lambda^2} - 1}}{2}, \\ \beta &= \beta(\lambda) = \frac{\lambda^2 + \frac{1}{\lambda^2} + 1 - \sqrt{\lambda^4 + \frac{1}{\lambda^4} + 2\lambda^2 + \frac{2}{\lambda^2} - 1}}{2}, \\ \rho &= \rho(\lambda) = \frac{\lambda^2 + \frac{1}{\lambda^2} - 1 - \sqrt{\lambda^4 + \frac{1}{\lambda^4} + 2\lambda^2 + \frac{2}{\lambda^2} - 1}}{2}.\end{aligned}$$

It follows from (14) and (16) that

$$\begin{aligned}\sigma_{i,n} &= \frac{[-\lambda_i^3 - \lambda_i + \lambda_i \alpha_i] \alpha_i^n e^{\tau_i t} + \gamma_i [-\lambda_i^3 - \lambda_i + \lambda_i \beta_i] \beta_i^n e^{\rho_i t}}{\alpha_i^n e^{\tau_i t} + \gamma_i \beta_i^n e^{\rho_i t}}, \\ \mu_{i,n} &= \frac{1}{\lambda_i} + \frac{1}{\lambda_i^2} \sigma_{i,n}, \quad \nu_{i,n} = \lambda_i^2 + 1 + \frac{1}{\lambda_i} \sigma_{i,n},\end{aligned}$$

where

$$\alpha_i = \alpha(\lambda_i), \quad \beta_i = \beta(\lambda_i), \quad \tau_i = \tau(\lambda_i), \quad \rho_i = \rho(\lambda_i), \quad i = 1, 2.$$

By using Theorem 3.1, the explicit exact solutions of system (1) are given by

$$\left\{ \begin{aligned} \tilde{p}_n &= -\frac{(\lambda_1 \lambda_2 (\lambda_1 \sigma_{1,n} - \lambda_2 \sigma_{2,n}) + \sigma_{1,n} \sigma_{2,n} (\lambda_1^2 - \lambda_2^2))}{(\lambda_1^3 \lambda_2 - \lambda_1 \lambda_2^3 + \lambda_2 \sigma_{1,n} - \lambda_1 \sigma_{2,n})}, \\ \tilde{q}_n &= \frac{\lambda_1 \lambda_2 (\lambda_1^2 - \lambda_2^2)}{\lambda_1 \sigma_{2,n} - \lambda_2 \sigma_{1,n}}, \\ \tilde{r}_n &= -\frac{\lambda_1^2 \lambda_2^2 (\lambda_1^3 \lambda_2 - \lambda_1 \lambda_2^3 + \lambda_2 \sigma_{1,n} - \lambda_1 \sigma_{2,n})}{\lambda_1 \lambda_2^3 - \lambda_1^3 \lambda_2 + \lambda_2^3 \sigma_{1,n} - \lambda_1^3 \sigma_{2,n}}, \\ \tilde{x}_n &= -\frac{(\lambda_1^2 - \lambda_2^2) (\lambda_1 \lambda_2 + \lambda_1 \sigma_{2,n} + \lambda_2 \sigma_{1,n} + \sigma_{1,n} \sigma_{2,n})}{\lambda_1^2 \lambda_2^2 (\lambda_1 \lambda_2^3 - \lambda_1^3 \lambda_2 + \lambda_2^3 \sigma_{1,n} - \lambda_1^3 \sigma_{2,n})}, \\ \tilde{y}_n &= \frac{\lambda_1 \lambda_2 (\lambda_1 \sigma_{1,n} - \lambda_2 \sigma_{2,n}) + \sigma_{1,n} \sigma_{2,n} (\lambda_1^2 - \lambda_2^2)}{\lambda_1^2 \lambda_2^2 (\lambda_1 \sigma_{2,n} - \lambda_2 \sigma_{1,n})}, \\ \tilde{z}_n &= -\frac{(\lambda_1^2 - \lambda_2^2)^2 (\lambda_1 \lambda_2 + \lambda_1 \sigma_{2,n} + \lambda_2 \sigma_{1,n} + \sigma_{1,n} \sigma_{2,n})}{\lambda_1 \lambda_2 (\lambda_1 \sigma_{2,n} - \lambda_2 \sigma_{1,n}) (\lambda_1 \lambda_2^3 - \lambda_1^3 \lambda_2 + \lambda_2^3 \sigma_{1,n} - \lambda_1^3 \sigma_{2,n})}. \end{aligned} \right. \quad (31)$$

A direct computation from (31) leads to the following relations

$$\tilde{p}_n = \tilde{r}_n \tilde{y}_n, \quad \tilde{z}_n = \tilde{q}_n \tilde{x}_n, \quad (32)$$

which means the solutions $\tilde{p}_n$ and $\tilde{z}_n$ are dependent of the solutions $\tilde{q}_n, \tilde{r}_n, \tilde{x}_n, \tilde{y}_n$.

As we known, the soliton solutions with strictly defined free physical parameters are very meaningful and valuable in the physical applications. In what follows, we proceed to establish the soliton solutions of system (1) by means of (31). When we choose $\lambda_2 = \frac{1}{\lambda_1}(\lambda_1 \neq \pm i, \pm 1)$, we find that $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, $\tau_1 = \tau_2$, $\rho_1 = \rho_2$, thus we can rewrite $\sigma_{1,n}$ and $\sigma_{2,n}$ as

$$\begin{aligned}\sigma_{1,n} &= \frac{(-\lambda_1^3 - \lambda_1 + \lambda_1 \alpha_1) + \gamma_1(-\lambda_1^3 - \lambda_1 + \lambda_1 \beta_1)e^{kn+wt}}{1 + \gamma_1 e^{kn+wt}}, \\ \sigma_{2,n} &= \frac{(-\lambda_1^{-3} - \lambda_1^{-1} + \lambda_1^{-1} \alpha_1) + \gamma_2(-\lambda_1^{-3} - \lambda_1^{-1} + \lambda_1^{-1} \beta_1)e^{kn+wt}}{1 + \gamma_2 e^{kn+wt}},\end{aligned}\quad (33)$$

with $k = \ln \frac{\beta_1}{\alpha_1}$, $w = \rho_1 - \tau_1$. From (31) and (33), we obtain

$$\begin{aligned}\tilde{q}_n &= \frac{\tilde{\Theta}_1 + e^{-(kn+wt)} + \gamma_1 \gamma_2 e^{kn+wt}}{\Theta_1 + e^{-(kn+wt)} + \gamma_1 \gamma_2 e^{kn+wt}}, \\ \tilde{r}_n &= \frac{\tilde{\Theta}_2}{\Theta_2 - \alpha_1(\lambda_1^2 - \lambda_1^{-2})e^{-(kn+wt)} - \gamma_1 \gamma_2 \beta_1(\lambda_1^2 - \lambda_1^{-2})e^{kn+wt}}, \\ \tilde{x}_n &= \frac{\tilde{\Theta}_3 + \alpha_1 e^{-(kn+wt)} + \gamma_1 \gamma_2 \beta_1 e^{kn+wt}}{\Theta_3 + \alpha_1 e^{-(kn+wt)} + \gamma_1 \gamma_2 \beta_1 e^{kn+wt}}, \\ \tilde{y}_n &= \frac{\tilde{\Theta}_4}{\Theta_4 + (\lambda_1^2 - \lambda_1^{-2})e^{-(kn+wt)} + \gamma_1 \gamma_2 (\lambda_1^2 - \lambda_1^{-2})e^{kn+wt}},\end{aligned}\quad (34)$$

where

$$\begin{aligned}\Theta_1 &= \gamma_1 + \gamma_2 - \frac{(\gamma_1 - \gamma_2)(\alpha_1 - \beta_1)}{\lambda_1^2 - \lambda_1^{-2}}, \\ \tilde{\Theta}_1 &= \gamma_1 + \gamma_2, \\ \Theta_2 &= \gamma_1 \beta_1 \lambda_1^{-2} - \gamma_2 \beta_1 \lambda_1^2 + \gamma_2 \alpha_1 \lambda_1^{-2} - \gamma_1 \alpha_1 \lambda_1^2, \\ \tilde{\Theta}_2 &= (\gamma_1 - \gamma_2)(\alpha_1 - \beta_1), \\ \Theta_3 &= \frac{\gamma_1 \beta_1 \lambda_1^{-2} - \gamma_2 \beta_1 \lambda_1^2 + \gamma_2 \alpha_1 \lambda_1^{-2} - \gamma_1 \alpha_1 \lambda_1^2}{\lambda_1^2 - \lambda_1^{-2}}, \\ \tilde{\Theta}_3 &= 2(\gamma_1 + \gamma_2) - \gamma_1 \beta_1 \lambda_1^{-2} - \gamma_2 \beta_1 \lambda_1^2 - \gamma_2 \alpha_1 \lambda_1^{-2} - \gamma_1 \alpha_1 \lambda_1^2, \\ \Theta_4 &= (\gamma_1 - \gamma_2)(\alpha_1 - \beta_1) + (\gamma_1 + \gamma_2)(\lambda_1^2 - \lambda_1^{-2}), \\ \tilde{\Theta}_4 &= -(\lambda_1^2 - \lambda_1^{-2})(\gamma_1(\lambda_1^{-2} \beta_1 + \lambda_1^2 \alpha_1) + \gamma_2(\lambda_1^2 \beta_1 + \lambda_1^{-2} \alpha_1)) \\ &\quad + (\lambda_1^2 - \lambda_1^{-2})((\gamma_1 + \gamma_2)(\alpha_1 - 1)(\beta_1 - 1)) \\ &\quad + \gamma_2(\lambda_1^2 \alpha_1 - \lambda_1^{-2} \beta_1) + \gamma_1(\lambda_1^2 \beta_1 - \lambda_1^{-2} \alpha_1),\end{aligned}$$

and $\tilde{p}_n$, $\tilde{z}_n$ can be obtained directly from (32). Based on (32) and (34), we know that $\tilde{p}_n$, $\tilde{q}_n$, $\tilde{r}_n$, $\tilde{x}_n$, $\tilde{y}_n$ and $\tilde{z}_n$ are soliton solutions with free parameters $\lambda_1, \gamma_1, \gamma_2$.

Fig. 1. (Color online) Plot of the soliton solutions $p_n, q_n, r_n, x_n, y_n, z_n$ with the parameters $\lambda_1 = 0.5, \gamma_1 = 1$ and $\gamma_2 = 1.5$.

Next, we choose suitable parameters λ_i and γ_i to illustrate the structures of soliton solutions. When $\lambda_1 = 0.5, \gamma_1 = 1, \gamma_2 = 1.5$, Fig. 1 shows the bell-shaped one-soliton structure, respectively.

We remark that if the resulting solution is taken as the new starting point, and make the Darboux transformation once again, then another new solutions of (1) are obtained. This process can be done continuously. Therefore, we can obtain a sequence of explicit exact solutions for (1). We can also get the exact solution or soliton solutions of the self-dual network equation, the discrete second-order nonlinear Schrödinger equation and the relativistic Volterra lattice equation under the transformation (29) using the similar method.

Remark 4.1. We point out that by using the Darboux transformation in Refs. 12 and 14, Vakhnenko also obtained explicit analytical multicomponent solutions with strictly defined free physical parameters: amplitude, velocity, energy, wideness of a soliton.

5. Conclusion

In this paper, we investigate a 6-field integrable lattice system (1) under the spectral problem (5). Infinitely many conservation laws and a new Darboux transformation for (1) are derived by means of the Lax pair. As an application, explicit exact solutions are generated, from which we obtain the soliton solutions. The figures of the one-soliton solutions with properly parameters are presented to illustrate the

soliton structures. In fact, with the help of Darboux transformation, the multi-soliton solutions of system (1) can be obtained successfully.

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References

1. E. Fermi *et al.*, *Studies of the nonlinear problems*, Technical report, Los Alamos Scientific Lab. (New Mexico, 1955).
2. M. J. Ablowitz and J. F. Ladik, *J. Math. Phys.* **16**, 598 (1975).
3. A. A. Belov and K. D. Chaltikian, *Phys. Lett. B* **309**, 268 (1993).
4. M. Blaszkak and K. Marciniak, *J. Math. Phys.* **35**, 4661 (1994).
5. W. X. Ma and B. Fuchssteiner, *J. Math. Phys.* **40**, 2400 (1999).
6. W. X. Ma and Y. C. You, *Chaos, Solitons Fractals* **22**, 395 (2004).
7. W. X. Ma *et al.*, *Anal. Math. Phys.* **8**, 427 (2018).
8. W. X. Ma *et al.*, *Mod. Phys. Lett. B* **32**, 1850409 (2018).
9. W. X. Ma *et al.*, *J. Geom. Phys.* **133**, 10 (2018).
10. W. X. Ma *et al.*, *East Asian J. Appl. Math.* **9**, 100218 (2018).
11. W. X. Ma, *Phys. Lett. A* **379**, 1975 (2015).
12. O. O. Vakhnenko, *J. Math. Phys.* **56**, 033505 (2015).
13. O. O. Vakhnenko, *J. Phys. A, Math. Gen.* **39**, 11013 (2006).
14. O. O. Vakhnenko, *J. Phys. Soc. Jpn.* **84**, 014003 (2015).
15. T. Tsuchida, H. Ujino and M. Wadati, *J. Math. Phys.* **39**, 4785 (1998).
16. R. Hirota, *J. Phys. Soc. Jpn.* **35**, 289 (1973).
17. P. Yang, Y. Chen and Z. B. Li, *Appl. Math. Comput.* **210**, 362 (2009).
18. X. Y. Wen, *J. Phys. Soc. Jpn.* **81**, 114006 (2012).
19. X. Y. Wen, D. S. Wang and X. H. Meng, *Rep. Math. Phys.* **72**, 349 (2013).
20. Y. B. Suris and O. Ragnisco, *Commun. Math. Phys.* **200**, 445 (1999).
21. Z. N. Zhu, H. C. Huang and W. M. Xue, *J. Phys. Soc. Jpn.* **68**, 771 (1999).
22. Z. N. Zhu *et al.*, *J. Phys. A, Math. Gen.* **35**, 5079 (2002).
23. K. Maruno and W. X. Ma, *J. Nonlinear Math. Phys.* **9**, 127 (2002).
24. H. Q. Zhao, *Appl. Math. Lett.* **38**, 79 (2014).
25. Y. T. Wu and X. G. Geng, *J. Phys. A* **31**, L677 (1998).
26. X. X. Xu, *Phys. Lett. A* **362**, 205 (2007).
27. E. G. Fan and H. H. Dai, *Phys. Lett. A* **372**, 4578 (2008).
28. G. Z. Tu, *J. Phys. A* **23**, 3903 (1990).
29. T. Tsuchida, H. Ujino and M. Wadati, *J. Phys. A, Math. Gen.* **32**, 2239 (1999).

30. D. J. Zhang and D. Y. Chen, *Chaos, Solitons & Fractals* **14**, 573 (2002).
31. O. O. Vakhnenko, *J. Nonlinear Math. Phys.* **18**, 401 (2011).
32. O. O. Vakhnenko, *J. Nonlinear Math. Phys.* **24**, 250 (2017).
33. H. W. A. Riaz *et al.*, *Commun. Nonlinear Sci. Numer. Simul.* **48**, 387 (2017).
34. X. J. Zhao, R. Guo and H. Q. Hao, *Appl. Math. Lett.* **75**, 114 (2018).
35. N. Liu and X. Y. Wen, *Mod. Phys. Lett. B* **32**, 1850085 (2018).