Kelvin wave and knot dynamics on entangled vortices

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In this paper, starting from Biot–Savart mechanics for entangled vortex-membranes, a new theory — knot physics — is developed to explore the underlying physics of quantum mechanics. Owing to the conservation conditions of the volume of knots on vortices in incompressible fluid, the shape of knots will never be changed and the corresponding Kelvin waves cannot evolve smoothly. Instead, the knot can only be split. The knot-pieces evolve following the equation of motion of Biot–Savart equation that becomes Schrödinger equation for probability waves of knots. The classical functions for Kelvin waves become wave-functions for knots. The effective theory of perturbative entangled vortex-membranes becomes a traditional model of relativistic quantum field theory — a massive Dirac model. As a result, this work would help researchers to understand the mystery in quantum mechanics.

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1. Introduction

One hundred years ago, Kelvin (Sir W. Thomson) studied the physical properties of vortex-lines that consist of the rotating motion of fluid around a common center-line. In an incompressible fluid, the vorticity of vortex-lines manifests itself in the circulation $\oint \mathbf{v} \cdot d\mathbf{l} = \kappa$, where κ is a constant. For classical hydrodynamic vortex-lines, Kelvin found a transverse and circularly polarized wave¹ (called Kelvin wave), in which the vortex-lines twist around their equilibrium position forming a helical

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structure.² For two vortex-lines, owing to the nonlocal interaction, the leapfrogging motion has been predicted in classical fluids from the works of Helmholtz and Kelvin.^{3–7} Owing to Kolmogorov-like turbulence,^{8,9} a variety of approaches have been used to study this phenomenon. In experiments, Kelvin waves have been observed in uniformly rotating ^4He superfluid (SF) (Ref. 10) and Bose–Einstein condensates.¹¹

In addition, Kelvin and Toit tried to develop an early atomic theory that is linked to the existence and dynamics of knotted vortex-rings in ether. However, they failed — the fundamental structure of atoms is irrelevant to knots. Today, the failure reason is clear. The elementary particles in our universe are not classical knots but quantum objects obeying quantum mechanics and Einstein’s relativity. Quantum mechanics (also known as quantum physics or quantum theory) is a fundamental branch of modern physics that was proposed from Planck’s solution in 1900 to the black-body radiation problem and Einstein’s 1905 paper which offered a quantum-based theory to explain the photoelectric effect. After several decades, quantum mechanics is established and becomes a successful theory that agrees very well with experiments and provides an accurate description of the dynamic behavior of microcosmic objects. In quantum mechanics, the energy is quantized and can only change by discrete amounts, i.e., $E = \hbar\omega$, where $\hbar$ is Planck constant. There are several fundamental principles in quantum mechanics: wave-particle duality, uncertainty principle, and superposition principle. The Schrödinger equation describes how wave-functions evolve, playing a role similar to Newton’s second law in classical mechanics. In quantum mechanics, when considering special relativity, the Schrödinger equation is replaced by the Dirac equation.

However, quantum mechanics is far from being well understood. There are a lot of unsolved mysteries in quantum mechanics including quantum entanglement problem and quantum measurement problem. Einstein said, “*Quantum mechanics is certainly imposing. But an inner voice tells me that it is not yet the real thing. The theory says a lot, but does not really bring us any closer to the secret of the ‘old one’.* I, at any rate, am convinced that He does not throw dice.” The exploration of the underlying physics of quantum mechanics is going on since its establishment.¹² There are a lot of attempts, such as De Broglie’s pivot-wave theory,⁷ the Bohmian mechanics,¹⁴ the many-world theory,¹⁵ the Nelsonian Mechanics¹⁶ and the idea of primary state diffusion.¹⁷ However, for all these interpretations of quantum mechanics, people focus on the issue of quantum motion but miss another important issue, *what is matter (or reality)?*

In this paper, we develop a new theory towards understanding quantum mechanics based on three-dimensional (3D) leapfrogging vortex-membranes in five-dimensional (5D) incompressible fluid. We call the new theory — *knot physics*. According to knot physics, it is the 3D quantum Dirac model that characterizes the knot dynamics of leapfrogging vortex-membranes. The knot physics gives a complete interpretation on quantum mechanics. Figure 1 is an illustration of the framework structure of knot physics.

Fig. 1. (Color online) The framework structure of knot physics.

The paper is organized as below. In Sec. 2, we review Biot–Savart mechanics for 3D vortex-membrane in a 5D incompressible fluid. In Sec. 3, the Kelvin waves of 3D vortex-membrane are studied. In this section, the Biot–Savart mechanics for helical vortex-membranes is mapped onto “pseudo-quantum mechanics” for a particle, by which we apply to the calculation of the evolution of a deformed helical vortex-membrane. In Sec. 4, we develop the Biot–Savart mechanics for entangled vortex-membranes. In addition, we introduce tensor representation to describe the Kelvin waves. In Sec. 5, we develop an emergent quantum mechanics to describe the knot dynamics on knot-crystal. In particular, a 3D massive Dirac model is obtained to characterize the knot dynamics on the entangled vortex-membranes. We also address the measurement theory and quantum entanglement of emergent quantum mechanics. The emergent quantum mechanics help us understand the mysteries of quantum mechanics. Finally, the conclusions are drawn in Sec. 6.

2. Biot–Savart Mechanics for Vortices

2.1. The Euler equation on vorticity

In this paper, we would study the vortex in an inviscid incompressible fluid. In condensed matter physics, we may consider superfluid (SF) as an inviscid incompressible fluid. For a divergence-free inviscid incompressible fluid ($\nabla \cdot \mathbf{v} = 0$), the fluid motion is described by the classical Euler equation:

$$\partial_t \mathbf{v} + \nabla_v \mathbf{v} = -\frac{1}{\rho} \nabla p, \quad (1)$$

Fig. 2. (Color online) (a) An illustration of a flat 1D vortex-line in 3D space; (b) An illustration of a curved 1D vortex-line in 3D space. The local orthogonal coordinate system on vortex-line described by the local vectors $\mathbf{s}'$, $\mathbf{s}''$, $\mathbf{s}' \times \mathbf{s}''$ is shown.

where $\mathbf{v}$ is the fluid velocity field, ρ is the uniform density and p is the pressure function. $\nabla_v \mathbf{v}$ is Riemannian covariant derivative of the field $\mathbf{v}$ in the direction of itself. On the other hand, to characterize vorticity, the Helmholtz form of the Euler equation is written as

$$\partial_t \boldsymbol{\Omega} + L_v \boldsymbol{\Omega} = 0, \quad (2)$$

where L_v is the Lie derivative along the velocity field $\mathbf{v}$ and $\boldsymbol{\Omega} = \nabla \times \mathbf{v}$ is the vorticity field.

Vortices are extended objects with singular vorticity in an inviscid incompressible fluid that can be regarded as a closed oriented embedded subvortex-membrane with Marsden–Weinstein (MW) symplectic structure. For example, for two-dimensional (2D) inviscid incompressible fluid, we have zero-dimensional point-vortices; for 3D case, we have one-dimensional (1D) vortex-lines; for 4D case, we have 2D vortex-surfaces; for 5D case, we have 3D vortex-membranes. See the illustration of vortex-line in 3D space in Fig. 2. In this paper, we focus on 3D vortex-membranes in the 5D inviscid incompressible fluid.

2.2. Vortices in inviscid incompressible fluid

2.2.1. Point-vortices in 2D inviscid incompressible fluid

Firstly, we review the dynamics of point-vortices within the framework of Marsden–Weinstein symplectic structure.

For N point-vortices in a 2D inviscid incompressible fluid, we define scalar vorticity field

$$\Omega = \sum_{j=1}^N \kappa_j \delta(\mathbf{z} - \mathbf{z}_j) \quad (1 \leq j \leq N), \quad (3)$$

where the complex value $\mathbf{z}_j = \xi_j + i\eta_j$ denotes the position of the j th point-vortex and κ_j denotes the constant vorticity of j th point-vortex. So we have a topological

condition

$$\oint_C \Omega dl = \sum_{j=1}^N \kappa_j \quad (4)$$

with loop integral along a closed path C .

According to Kirchhoff's theorem, the Euler equation for point-vortices is

$$\kappa_j \dot{\xi}_j = \frac{\partial H_{\text{Kirchhoff}}}{\partial \eta_j}, \quad \kappa_j \dot{\eta}_j = -\frac{\partial H_{\text{Kirchhoff}}}{\partial \xi_j} \quad (5)$$

or

$$i\kappa_i \frac{dz_i}{dt} = \frac{\delta H_{\text{Kirchhoff}}}{\delta z_i^*}, \quad (6)$$

where $H_{\text{Kirchhoff}}$ is the Kirchhoff Hamiltonian given by

$$H_{\text{Kirchhoff}} = -\frac{1}{2\pi} \sum_{j < k}^N \kappa_j \kappa_k \ln |z_j - z_k|. \quad (7)$$

The Kirchhoff Hamiltonian is really potential energy that characterizes the long range interaction between two point-vortices: for two vortices with vorticities of same signs, we have repulsive interaction; for two vortices with vorticities of opposite signs, we have attractive interaction. In addition, the above Euler equation can be changed in terms of Poisson bracket $\{, \} = \sum_{j=1}^N \frac{1}{\kappa_j} \left(\frac{\partial}{\partial \xi_j} \frac{\partial}{\partial \eta_j} - \frac{\partial}{\partial \eta_j} \frac{\partial}{\partial \xi_j} \right)$ to be Refs. 18 and 19

$$\partial_t \Omega = \{H_{\text{Kirchhoff}}, \Omega\}. \quad (8)$$

2.2.2. Vortex-lines in 3D inviscid incompressible fluid

Secondly, we review the dynamics of vortex-lines in 3D inviscid incompressible fluid.

(a) Biot-Savart equation

Vortex-lines are 1D topological objects in 3D inviscid incompressible fluid with $\nabla \cdot \mathbf{v} \equiv 0$. For a fluid with a vortex-line, a rotationless superfluid component flow $\nabla \times \mathbf{v} = 0$ is violated on 1D singularities $\mathbf{s}(\zeta, t)$, which depends on the variables — arc length ζ and the time t . Away from the singularities, the velocity increases to infinity so that the circulation of the fluid velocity remains constant,

$$\oint \mathbf{v} \cdot d\mathbf{l} = \kappa, \quad (9)$$

where $\mathbf{v} = \dot{\mathbf{r}}$. As a result, the vortex filament could be described by the Biot-Savart equation

$$\dot{\mathbf{r}} = \frac{\kappa}{4\pi} \int \frac{d\mathbf{s} \times (\mathbf{r} - \mathbf{s})}{|\mathbf{r} - \mathbf{s}|^3}, \quad (10)$$

where κ is the circulation and $\mathbf{s}$ is the vector that denotes the position of vortex filament. For a 3D superfluid, $\kappa = h/m$ is the discreteness of the circulation owing to its quantum nature. h is Planck constant and m is particle mass of SF.

(b) *Impulse and angular impulse*

In fluid with a vortex-line, the conventional momentum of the fluid motion cannot be well defined. Instead, the hydrodynamic impulse (the Lamb impulse) plays the role of the effective momentum that denotes the total mechanical impulse of the nonconservative body force applied to a limited fluid volume to generate instantaneously from rest the given motion of the whole of the fluid at time t . In general, the (effective) momentum for a vortex-line from Lamb impulse density is defined by

$$\begin{aligned}\mathbf{P}_{\text{Lamb}} &= \rho_0 \int \mathbf{v} dV_{3D} \\ &\Rightarrow \frac{1}{2} \rho_0 \int (\mathbf{s} \times \boldsymbol{\Omega}) dV_{3D},\end{aligned}\quad (11)$$

where dV_{3D} is an infinitesimal volume in 3D fluid and ρ_0 is the superfluid mass density. For a vortex-line with a global velocity $\mathbf{U}$, we have

$$\delta H - \mathbf{U} \cdot \delta \mathbf{P}_{\text{Lamb}} = 0. \quad (12)$$

As a result, the Lamb impulse becomes the right physical quantity describing the momentum of the vortex filament.

By using the definition of a vortex-line^{20–24}

$$\boldsymbol{\Omega} = \kappa \delta_{\mathbf{s}}, \quad (13)$$

we have the (Lamb impulse) momentum

$$\mathbf{P}_{\text{Lamb}} = \frac{\rho_0 \kappa}{2} \int \mathbf{s} \times d\mathbf{s} \quad (14)$$

and the (Lamb impulse) angular momentum for a vortex-line

$$\begin{aligned}\mathbf{J}_{\text{Lamb}} &= \rho_0 \int (\mathbf{s} \times \mathbf{v}) dV_{3D} \\ &= \frac{1}{2} \rho_0 \int \mathbf{s} \times (\mathbf{s} \times \boldsymbol{\Omega}) dV_{3D}.\end{aligned}\quad (15)$$

It is obvious that $\mathbf{P}_{\text{Lamb}}$ and $\mathbf{J}_{\text{Lamb}}$ are conserved quantities, i.e.,

$$\frac{d\mathbf{P}_{\text{Lamb}}}{dt} \equiv 0 \quad (16)$$

and

$$\frac{d\mathbf{J}_{\text{Lamb}}}{dt} \equiv 0. \quad (17)$$

(c) *Geometric Biot–Savart mechanics for vortex-lines with 1D Marsden–Weinstein symplectic structure under localized induction approximation*

Localized induction approximation (LIA) of the vorticity motion is to keep the local terms in the vorticity Euler equation. The Biot–Savart mechanics under LIA for 1D vortex-lines is reduced to a special classical mechanics — geometric Biot–Savart

mechanics for geometric objects with 1D Marsden–Weinstein symplectic structure. The corresponding evolution becomes the vortex filament equation

$$\partial_t \mathbf{s} = \left(\frac{\kappa}{4\pi} \ln \epsilon \right) (\mathbf{s}' \times \mathbf{s}''), \quad (18)$$

where $\epsilon = \frac{\ell}{a_0}$, ℓ is the length of the order of the curvature radius (or inter-vortex distance when the considered vortex filament is a part of a vortex tangle) and a_0 denotes the vortex filament radius which is much smaller than any other characteristic size in the system.

For an arc-length parametrization, the tangent vectors $\mathbf{t} = \partial \mathbf{s} / \partial \zeta = \mathbf{s}'$ have unit length and the acceleration vectors are $\mathbf{s}'' = \partial \mathbf{t} / \partial \zeta = \varkappa \cdot \mathbf{n}$, i.e., $\partial_t \mathbf{s} = (\frac{\kappa}{4\pi} \ln \epsilon) (\mathbf{s}' \times \mathbf{s}'')$ becomes

$$\partial_t \mathbf{s} = \left(\frac{\kappa}{4\pi} \ln \epsilon \right) (\varkappa \cdot \mathbf{b}), \quad (19)$$

where $\varkappa$ and $\mathbf{b} = \mathbf{t} \times \mathbf{n}$ denote the curvature value and bi-normal unit vector of the curve $\mathbf{s}$ at the corresponding point, respectively. See the illustration in Fig. 2.

In addition, in terms of the Hamiltonian $H_{\text{length}}(\mathbf{s})$, the Biot–Savart equation can be written into

$$\dot{\mathbf{s}} = \hat{J} \left[\frac{\delta H_{\text{length}}(\mathbf{s})}{\delta \mathbf{s}} \right] \quad (20)$$

or

$$\dot{\xi} = \frac{\partial H_{\text{length}}(\xi, \eta)}{\partial \eta}, \quad \dot{\eta} = -\frac{\partial H_{\text{length}}(\xi, \eta)}{\partial \xi} \quad (21)$$

where $\hat{J}$ is an operator rotating the plane by $\pi/2$. The Hamiltonian $H_{\text{length}}(\mathbf{s})$ is defined by

$$H_{\text{length}}(\mathbf{s}) = \frac{\kappa \ln \epsilon}{4\pi} \cdot \text{length}(\mathbf{s}), \quad (22)$$

where $\text{length}(\mathbf{s}) = \int dx \sqrt{1 + |\frac{d\mathbf{z}}{dx}|^2}$ is the length of the vortex-line.

We introduce a complex description on the vortex-line, $z = \xi + i\eta = |z|e^{i\phi}$, where $|z|$ is the amplitude and ϕ is the angle in the complex plane. Under LIA²⁵ and a simple geometrical constraint $z' \ll 1$, in terms of the complex canonical coordinate $z(x)$, the Biot–Savart equation becomes⁸

$$i \frac{dz}{dt} = \frac{\delta H_{\text{length}}(z)}{\delta z^*}. \quad (23)$$

When the vortex-line is rotating $\frac{dz}{dt} \neq 0$, it becomes longer, $\Delta H_{\text{length}}(z) \neq 0$.

2.2.3. 3D vortex-membranes in 5D inviscid incompressible fluid

Thirdly, we develop the physics of 3D vortex-membranes in 5D inviscid incompressible fluid. The Biot–Savart mechanics under LIA for 3D vortex-membranes is reduced to 3D geometric Biot–Savart mechanics for geometric objects with 3D Marsden–Weinstein symplectic structure.

For 5D case, we have 3D vortex-membranes with MW symplectic structure.¹⁹ The 3D vortex-membrane is defined by a given singular vorticity

$$\mathbf{\Omega} = \kappa \delta_P, \quad (24)$$

where the singular δ -type vorticity denotes the submanifold P in 5D space, and κ is the constant circulation strength.

(a) *Generalized Biot–Savart equation in 5D fluid*

A generalized Biot–Savart equation in 5D fluid is given by (Ref. 19)

$$\mathbf{v}(q) = \kappa \int_{p \in P} \hat{J}(\text{Proj}_N \nabla_p G(q, p)) dV_{5D}, \quad (25)$$

where $\text{Proj}_N \mathbf{A}$ is the orthogonal projection of $\mathbf{A}$ to the fiber NpP of the normal bundle to P at $p \in P$, and the operator $\hat{J}$ is the positive rotation around p by $\pi/2$ in extra dimensional space NpP . $G(q, p) = \frac{\alpha}{|q-p|^{-3}}$ is the Green's function for the Laplace operator, i.e.,

$$\Delta_p G(q, p) = \delta_q(p), \quad (26)$$

the δ -function supported at q . dV_{5D} is an infinitesimal volume in 5D fluid and $\alpha = \frac{\Gamma(\frac{5}{2})}{6\pi^{\frac{5}{2}}}$, where $\Gamma(z)$ is the Gamma function.

(b) *Impulse and angular impulse*

We then consider the (effective) momentum for a vortex-membrane from Lamb impulse density

$$\mathbf{P}_{\text{Lamb}} = \frac{1}{2} \rho_0 \int (\mathbf{s} \times \mathbf{\Omega}) dV_{5D}. \quad (27)$$

Here, “ $\times$ ” denotes positive rotation around p after vector product. On the other hand, the (Lamb impulse) angular momentum for a vortex-membrane is defined by

$$\mathbf{J}_{\text{Lamb}} = \frac{1}{2} \rho_0 \int \mathbf{s} \times (\mathbf{s} \times \mathbf{\Omega}) dV_{5D}. \quad (28)$$

$\mathbf{P}_{\text{Lamb}}$ and $\mathbf{J}_{\text{Lamb}}$ are also conserved quantities, i.e., $\frac{d\mathbf{P}_{\text{Lamb}}}{dt} \equiv 0$ and $\frac{d\mathbf{J}_{\text{Lamb}}}{dt} \equiv 0$.

(c) *Geometric Biot–Savart mechanics for vortex-membranes with 3D Marsden–Weinstein symplectic structure under LIA*

Under LIA, the generalized Biot–Savart equation for a 3D vortex-piece is reduced to (Ref. 19)

$$\mathbf{v}(q) \simeq (\kappa \alpha \ln \epsilon) \cdot J(\mathbf{M}_C(p)), \quad (29)$$

where $\mathbf{M}_C(p)$ is the mean curvature vector to P at the point p . The mean curvature vector $\mathbf{M}_C(p) \in NpP$ is the mean value of the curvature vectors of geodesics in P passing through the point p when we average over the sphere S^4 of all possible unit tangent vectors in TpP for these geodesics. Thus, the generalized Biot–Savart equation under LIA is given by the skew-mean-curvature flow

$$\frac{\partial P(p)}{\partial t} = -(\kappa \alpha \ln \epsilon) J(\mathbf{M}_C(p)). \quad (30)$$

According to the fact that the mean curvature vector field is the gradient to the volume functional, the generalized Biot–Savart equation for a 3D vortex-piece under LIA can be described by Hamiltonian formula and the Hamiltonian on the vortex-membranes is just 3-volume

$$H_{\text{volume}}(P) = (\kappa\alpha \ln \epsilon) \cdot \text{volume}(P) \tag{31}$$

with $\text{volume}(P) = \int_P dV_P$. In terms of the Hamiltonian $H(P)$ for 3D vortex-membrane, the generalized Biot–Savart equation becomes

$$\frac{\partial P(p)}{\partial t} = (\kappa\alpha \ln \epsilon) \hat{J} \left[\frac{\delta H(P)}{\delta P} \right]. \tag{32}$$

The volume of the subvortex-membrane P becomes a conserved quantity that plays the role of Hamiltonian function of the corresponding dynamics. In addition, the generalized Biot–Savart equation becomes

$$\dot{\xi} = \frac{\partial H_{\text{volume}}(\xi, \eta)}{\partial \eta}, \quad \dot{\eta} = -\frac{\partial H_{\text{volume}}(\xi, \eta)}{\partial \xi}. \tag{33}$$

In complex representation, $z = \xi + i\eta$, Eq. (33) can also be written into (Ref. 19)

$$i \frac{dz}{dt} = \frac{\delta H_{\text{volume}}(P)}{\delta z^*}. \tag{34}$$

In particular, we emphasize that the evolution equation of the subvortex-membrane P is really mechanics of geometric objects — the Hamiltonian

	Classical mechanics	Biot-Savart mechanics
Physical objects	Mass-point (a tiny ball with mass m)	Vortex (point vortex in 2D, vortex-line in 3D, vortex-membrane in 4D/5D) under LIA ,
Motion	Classical shifting with given velocity	Skew-mean-curvature flow
Equation of motion	Hamiltonian equation: $\dot{q} = \frac{\delta H(p, q)}{\delta p}, \dot{p} = -\frac{\delta H(p, q)}{\delta q}$	Biot-Savart equation: $i \frac{dz_{A/B}}{dt} = \frac{\delta H(z_{A/B})}{\delta z_{A/B}^*}$
Hamiltonian	Kinetic energy T and potential energy V, $H = T + V$	Under LIA, the Hamiltonian is proportion to the size of the system: $H \propto \int dV_P$
Canonical variables	Position q and momentum p	Orthotropic coordinates for position in extra dimensional space, z, z^*
Momentum	Classical momentum $\vec{p}$	(Lamb impulse) momentum
Angular momentum	Classical angular momentum $\vec{J} = \vec{q} \times \vec{p}$	(Lamb impulse) angular momentum

Fig. 3. (Color online) Comparison between classical mechanics for mass-point and (geometric) Biot–Savart mechanics for vortices.

$H_{\text{volume}}(P)$ is 3-volume

$$\text{volume}(P) = \int_P dV_P \quad (35)$$

that is a geometric quantity and the dynamic variables $(\xi(t), \eta(t))$ or $z(t)$ denote the position in extra dimensions that are also geometric quantities. The generalized Biot–Savart equation is to minimize the 3-volume $\text{volume}(P) = \int_P dV_P$. As a result, the vortex-membranes always do the skew-mean-curvature flow that differs by the $\pi/2$ -rotation from the mean-curvature vector. The skew-mean-curvature flow does not stretch the subvortex-membrane while moving its points orthogonally to the mean curvatures. In Fig. 3, we compare Newton mechanics for mass-point with (geometric) Biot–Savart mechanics for vortices.

3. Biot–Savart Mechanics for a Vortex-Membrane: Pseudo-Quantum Mechanics for Single Particle

The issue of Biot–Savart mechanics for a vortex-membrane is about the dynamic evolution of windings for a vortex-membrane. We may ask a question — *how to characterize the evolution of a nonuniform helical vortex-membrane?* For example, a nonuniform helical vortex-line is shown in Fig. 4(b). We point out that the corresponding Biot–Savart mechanics is reduced to pseudo-quantum mechanics for single particle. Here, “pseudo” indicates that this mechanics just has similar structure to the traditional quantum mechanics but is different.

3.1. Kelvin wave and its dispersion

For 1D vortex-lines, there exists Kelvin wave, a transverse and circularly polarized wave, in which the vortex-line twists around its equilibrium position forming a helical structure. The plane-wave ansatz of Kelvin waves is described by a complex field,^{25,26}

$$z(x, t) = r_0 e^{\pm i k \cdot x - i \omega t + i \phi_0}, \quad (36)$$

where r_0 is the winding radius of vortex-line, $k = \frac{\pi}{a} > 0$ and a is a fixed length that denotes the half pitch of the windings. ϕ_0 is a constant angle. $\pm$ denotes

Fig. 4. (Color online) (a) A uniform helical vortex-line; (b) A nonuniform helical vortex-line.

two possible winding directions: left-hand with clockwise winding, or right-hand with counterclockwise winding. For Kelvin waves on vortex-lines, under LIA, the dispersion is obtained as

$$\omega \simeq \left(\frac{\kappa}{4\pi} \ln \epsilon \right) k^2, \quad (37)$$

where $\epsilon \sim \frac{1}{ka_0}$. In mathematics, we can generate a Kelvin wave by an operator $\hat{U}(x, t)$ on a flat vortex-line

$$z(x, t) = \hat{U}(x, t)z_0, \quad (38)$$

where

$$\hat{U}(x, t) = e^{i \int [\phi(x, t) \cdot \hat{K}] dx} \cdot \hat{F}(r_0). \quad (39)$$

Here, $z_0 = 0$ denotes a vortex-membrane with $r_0 = 0$. $\hat{U}(x, t) = e^{i \int [\phi(x, t) \cdot \hat{K}] dx} \cdot \hat{F}(r_0)$ is winding operator with $\hat{K} = -i \frac{d}{d\phi}$, $\phi(x, t) = \pm kx - \omega t$. $\hat{F}(r_0)$ is an expanding operator by shifting radius from 0 to r_0 on the membrane, i.e., $\hat{F}(r_0) \cdot 0 = r_0$. So $(\hat{F}(r_0))^{-1}$ is a shrinking operator by shifting radius from r_0 to 0 on the membrane, i.e., $(\hat{F}(r_0))^{-1} \cdot r_0 = 0$.

We then calculate the physical quantities of the Kelvin waves along winding direction (x -direction), i.e., the projected angular momentum J_x and the projected momentum p_x . The projected (Lamb impulse) momentum along the x -direction p_x of a vortex-line with a plane Kelvin wave is obtained as²⁶

$$p_x = \mathbf{P}_{\text{Lamb}} \cdot \mathbf{e}_x = \pm \frac{1}{2} \rho_0 \kappa l r_0^2 k \quad (40)$$

that leads to the projected (Lamb impulse) momentum density $\rho_{p_x} = \frac{1}{2} \rho_0 \kappa r_0^2 \cdot k$. l is the length of the system in x -direction. The projected (Lamb impulse) angular momentum along x -direction of a vortex-line with a plane Kelvin wave is given by²⁶

$$J_x = \mathbf{J}_{\text{Lamb}} \cdot \mathbf{e}_x = \pm \frac{1}{2} \rho_0 \kappa l r_0^2 \quad (41)$$

that leads to the projected (Lamb impulse) angular momentum density along x -direction $\rho_{J_x} = |\frac{J_x}{l}| = \rho_0 \kappa \frac{r_0^2}{2}$. Here, we use a mathematic result,

$$\int (\mathbf{s} \times d\mathbf{s}) \times \mathbf{s} \cdot \mathbf{e}_x \equiv V_{\text{cylinder}} \quad (42)$$

with a (uniform or nonuniform) helical curve on a cylinder with volume $V_{\text{cylinder}} = r_0^2 \cdot l$ (the symmetric axis is along x -direction, the radius of cross-section is r , the length along x -direction is l).

For Kelvin waves on a 3D helical vortex-membrane, the plane-wave ansatz is described by a complex field,

$$z(\mathbf{x}, t) = r_0 e^{\pm i \mathbf{k} \cdot \mathbf{x} - i \omega t + i \phi_0}, \quad (43)$$

where $\mathbf{k}$ is the winding wave vector on 3D vortex-membrane with $|\mathbf{k}| = \frac{\pi}{a}$ and a is a fixed length that denotes the half pitch of the windings.

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For the plane Kelvin waves on a 3D helical vortex-membrane, we have the Hamiltonian

$$H_{\text{volume}}(P) = (\kappa\alpha \ln \epsilon) \text{volume}(P) = (\kappa\alpha \ln \epsilon) \int_P dV_P = (\kappa\alpha \ln \epsilon) \iiint dx dy dz \cdot \sqrt{1 + \left| \frac{dz}{dx} \right|^2 + \left| \frac{dz}{dy} \right|^2 + \left| \frac{dz}{dz} \right|^2}. \quad (44)$$

Under local induction approximation and a simple geometrical constraint $|\frac{dz}{dx}|^2 + |\frac{dz}{dy}|^2 + |\frac{dz}{dz}|^2 \ll 1$, the Hamiltonian is reduced into

$$H_{\text{volume}}(P) \simeq \text{constant} + \frac{\kappa}{2} \iiint dx dy dz \cdot \left(\left| \frac{dz}{dx} \right|^2 + \left| \frac{dz}{dy} \right|^2 + \left| \frac{dz}{dz} \right|^2 \right). \quad (45)$$

Thus, we derive the dispersion of Kelvin waves as

$$\omega \simeq \frac{(\alpha\kappa \ln \epsilon)}{2} \mathbf{k}^2, \quad (46)$$

where $\mathbf{k}^2 = k_x^2 + k_y^2 + k_z^2$.

The (Lamb impulse) momentum along $\mathbf{e}$ -direction on vortex-membrane with a plane Kelvin wave $z(x, t) = r_0 \exp(-i\omega \cdot t + i\mathbf{k} \cdot \mathbf{x})$ is obtained as

$$\mathbf{p}_{\text{Lamb}} = \mathbf{P}_{\text{Lamb}} \cdot \mathbf{e} = \pm \frac{1}{2} \rho_0 \kappa V_P r_0^2 k, \quad (47)$$

where V_P is the total volume of the vortex-membrane. The (Lamb impulse) angular momentum of a plane Kelvin wave along $\mathbf{e}$ -direction is given by

$$|J_{\text{Lamb}}| = |\mathbf{J}_{\text{Lamb}} \cdot \mathbf{e}| = \frac{1}{2} \rho_0 \kappa V_P r_0^2. \quad (48)$$

The projected (Lamb impulse) angular momentum is a constant on the vortex-membrane.

3.2. Mapping to pseudo-quantum mechanics

We then map the Biot-Savart mechanics for a (constraint) helical vortex-membrane to a pseudo-quantum mechanics.

Firstly, we define the state-vector $\mathbf{z}_{\mathbf{k}}(\mathbf{x}, t)$ to denote the helical vortex-membrane as

$$\mathbf{z}_{\mathbf{k}}(\mathbf{x}, t) = r_0 \exp(-i\omega \cdot t + i\mathbf{k} \cdot \mathbf{x}). \quad (49)$$

With the help of the operator $\hat{U}(\vec{x}, t) = e^{i \int [\phi(x, t) \cdot \hat{K}] dx} \cdot \hat{F}(r_0)$, the Kelvin waves can also be generated from a straight one,

$$\mathbf{z}_{\mathbf{k}}(\mathbf{x}, t) = \hat{U}(\vec{x}, t) \cdot \mathbf{z}_0. \quad (50)$$

The Hilbert space of pseudo-quantum mechanics for a helical vortex-membrane consists of different states $\mathbf{z}_{\mathbf{k}}(\mathbf{x}, t)$. We use the state vector $|\mathbf{k}\rangle$ to denote the state

of Kelvin waves of vortex-membranes and the state vector $|\mathbf{k}\rangle$ to denote the state of knots in the following parts.

In general, under LIA, according to superposition principle, we construct the Kelvin wave as

$$z(\mathbf{x}, t) = \sum_{\mathbf{k}} r_{\mathbf{k}} \exp(-i\omega \cdot t + i\mathbf{k} \cdot \mathbf{x}), \quad (51)$$

where $r_{\mathbf{k}}$ is the radius of a partial wave $\exp(-i\omega \cdot t + i\mathbf{k} \cdot \mathbf{x})$. There exists a normalized volume condition,

$$\int z^*(\mathbf{x}, t) z(\mathbf{x}, t) dV_P = \int \sum_{\mathbf{k}} |r_{\mathbf{k}}|^2 dV_P = V_P \cdot r_0^2. \quad (52)$$

We may consider the value $V_P \cdot r_0^2$ to be the fixed volume of the system. For example, a 1D standing Kelvin wave denoted by $\frac{1}{\sqrt{2}}(|k\rangle + |-k\rangle)$ is a superposition Kelvin wave of two plane Kelvin waves with opposite wave vectors

$$z(x, t) = \frac{1}{\sqrt{2}}(r_0 e^{ik \cdot x - i\omega t + i\phi_0} + r_0 e^{-ik \cdot x - i\omega t + i\phi_0}) = \sqrt{2} r_0 \cos(kx) e^{-i\omega t + i\phi_0}. \quad (53)$$

For a plane wave, $z_{\mathbf{k}}(\mathbf{x}, t) = r_0 \exp(-i\omega \cdot t + i\mathbf{k} \cdot \mathbf{x})$, the (Lamb impulse) momentum is

$$\mathbf{p} = \hbar_{\text{eff}} \mathbf{k}, \quad (54)$$

where the (Lamb impulse) angular momentum J_{Lamb} is obtained as the effective Planck constant $\hbar_{\text{eff}}$,

$$\hbar_{\text{eff}} = J_{\text{Lamb}} = \frac{1}{2} \rho_0 \kappa V_P r_0^2.$$

The effective Planck constant is proportional to the total volume of the system. The effective energy of a Kelvin wave is

$$E_{\text{Lamb}} = \hbar_{\text{eff}} \omega. \quad (55)$$

As a result, we define the “wave-function” of a plane Kelvin wave as

$$\begin{aligned} \psi(\mathbf{x}, t) &= \sqrt{\frac{1}{V_P r_0^2}} z(\mathbf{x}, t) = \frac{1}{\sqrt{V_P}} \exp(-i\omega \cdot t + i\mathbf{k} \cdot \mathbf{x}) \\ &= \frac{1}{\sqrt{V_P}} \exp\left(\frac{-iE_{\text{Lamb}} t + i\mathbf{p}_{\text{Lamb}} \cdot \mathbf{x}}{\hbar_{\text{eff}}}\right) \end{aligned} \quad (56)$$

and a generalized constraint Kelvin waves as

$$\begin{aligned} \psi(\mathbf{x}, t) &= \sum_{\mathbf{k}} \sqrt{\frac{1}{V_P r_0^2}} r_{\mathbf{k}} \exp(-i\omega \cdot t + i\mathbf{k} \cdot \mathbf{x}) \\ &= \sum_{\mathbf{k}} a_{\mathbf{k}} \exp\left(\frac{-iE_{\text{Lamb}} t + i\mathbf{p}_{\text{Lamb}} \cdot \mathbf{x}}{\hbar_{\text{eff}}}\right), \end{aligned} \quad (57)$$

where $a_{\mathbf{k}} = \sqrt{\frac{1}{V_P r_0^2}} r_{\mathbf{k}}$.

For the average value of the effective energy $\langle E_{\text{Lamb}} \rangle$, we have

$$\begin{aligned}
 \langle E_{\text{Lamb}} \rangle &= \int \psi^*(\mathbf{x}, t) E_{\text{Lamb}} \psi(\mathbf{x}, t) dV_p \\
 &= \int \left[\sum_{\mathbf{p}} c_{\mathbf{p}}^* \exp \left(\frac{i(E_{\text{Lamb}} t - \mathbf{p}_{\text{Lamb}} \cdot \mathbf{x})}{\hbar_{\text{eff}}} \right) \right] \\
 &\quad \times E_{\text{Lamb}} \left[\sum_{\mathbf{p}'} c_{\mathbf{p}'} \exp \left(\frac{-i(E_{\text{Lamb}} t - \mathbf{p}'_{\text{Lamb}} \cdot \mathbf{x})}{\hbar_{\text{eff}}} \right) \right] dV_p \\
 &= \int \left[\sum_{\mathbf{p}} c_{\mathbf{p}}^* \exp \left(\frac{i(E_{\text{Lamb}} t - \mathbf{p}_{\text{Lamb}} \cdot \mathbf{x})}{\hbar_{\text{eff}}} \right) \right] \\
 &\quad \times \left(i\hbar_{\text{eff}} \frac{d}{dt} \right) \left[\sum_{\mathbf{p}'} c_{\mathbf{p}'} \exp \left(\frac{-i(E_{\text{Lamb}} t - \mathbf{p}'_{\text{Lamb}} \cdot \mathbf{x})}{\hbar_{\text{eff}}} \right) \right] dV_p \\
 &= \int \psi^*(\mathbf{x}, t) \left(i\hbar_{\text{eff}} \frac{d}{dt} \right) \psi(\mathbf{x}, t) dV_p. \tag{58}
 \end{aligned}$$

This result indicates that the effective energy becomes an operator

$$E_{\text{Lamb}} \rightarrow \hat{H}_{\text{Lamb}} = i\hbar_{\text{eff}} \frac{d}{dt}. \tag{59}$$

Using the similar approach, we derive

$$\begin{aligned}
 \langle \mathbf{p}_{\text{Lamb}} \rangle &= \int \psi^*(\vec{x}, t) \mathbf{p}_{\text{Lamb}} \psi(\vec{x}, t) dV_p \\
 &= \int \psi^*(\vec{x}, t) \left(-i\hbar_{\text{eff}} \frac{d}{d\vec{x}} \right) \psi(\vec{x}, t) dV_p. \tag{60}
 \end{aligned}$$

As a result, the (Lamb impulse) momentum also becomes an operator

$$\mathbf{p}_{\text{Lamb}} \rightarrow \hat{p}_{\text{Lamb}} = -i\hbar_{\text{eff}} \frac{d}{d\vec{x}}. \tag{61}$$

From the dispersion of Kelvin waves, we have

$$\omega \simeq \frac{(\alpha \kappa \ln \epsilon)}{2} \mathbf{k}^2.$$

The energy-momentum relationship $E_{\text{Lamb}} = H(\mathbf{p}_{\text{Lamb}}) = \frac{(\alpha \kappa \ln \epsilon)}{2\hbar_{\text{eff}}} \mathbf{p}_{\text{Lamb}}^2$ determines the equation of motion for vortex-membranes that becomes the ‘‘Schrödinger equation’’,

$$\hat{H}_{\text{Lamb}} \psi(\mathbf{x}, t) = \frac{\hat{p}_{\text{Lamb}}^2}{2m_{\text{pseudo}}} \psi(\mathbf{x}, t) \tag{62}$$

or

$$i\hbar_{\text{eff}} \frac{d\psi(\mathbf{x}, t)}{dt} = \frac{\hat{p}_{\text{Lamb}}^2}{2m_{\text{pseudo}}} \psi(\mathbf{x}, t) \quad (63)$$

with an effective mass

$$m_{\text{pseudo}} = \frac{\hbar_{\text{eff}}}{\alpha\kappa \ln \epsilon}. \quad (64)$$

For the eigenstate with eigenvalue E_{Lamb} , the wave-function is given by

$$\psi(\mathbf{x}, t) = f(\mathbf{x}) \exp\left(\frac{iE_{\text{Lamb}}t}{\hbar_{\text{eff}}}\right), \quad (65)$$

where $f(\mathbf{x})$ is spatial function. This corresponds to a twisting motion with fixed twisting angular velocity, $\omega = \frac{E_{\text{Lamb}}}{\hbar_{\text{eff}}}$. That means the excitations of the quantum state must have the quantized projected (Lamb impulse) energy,

$$\Delta E_{\text{Lamb}} = nE_{\text{Lamb}} = n\hbar_{\text{eff}}\omega, \quad (66)$$

where n is a positive integer number.

In pseudo-quantum mechanics, there are three conserved physical quantities for helical vortex-membranes: the energy $H(\mathbf{p}_{\text{Lamb}})$ that is proportional to the volume of the vortex-membrane $V_P = \text{volume}(P)$; the momentum $\mathbf{p}_{\text{Lamb}}$ that is proportional to the winding number along given direction $\mathbf{w}_{1D}$ (see below discussion); and the (Lamb impulse) angular momentum (the effective Planck constant $\hbar_{\text{eff}}$) that is proportional to the volume of the vortex-membrane in the 5D fluid $V_P \cdot \pi r_0^2$.

As a result, the pseudo-quantum mechanics describes the dynamics of a vortex-membrane with fixed volume in the 5D fluid. Under the constraint, the total degree of freedom of a vortex-membrane is reduced to 1. In Fig. 5, we show the comparison between pseudo-quantum mechanics for helical vortex-membrane and quantum mechanics for single quantum particle. The quantization of (constraint) Kelvin waves is similar to quantization of a matter wave in quantum mechanics: a constant J_{Lamb} plays the role of the Planck constant $\hbar_{\text{eff}}$, m_{pseudo} plays the role of mass in Schrödinger equation, and so on. However, they are different. The Kelvin waves are classical waves that obey deterministic classical mechanics, not probabilistic quantum mechanics. This is why we call it pseudo-quantum mechanics. In the following parts, we will show that the knots rather than vortex-membranes obey true quantum mechanics.

3.3. Winding number and winding-number density

In the above part, we derive the equation of motion of Kelvin waves. In principle, the evolution of the system can be solved. We then introduce winding number and winding-number density to describe the deformed helical vortex-membrane.

The winding number w_{1D} of two 1D vortex-lines is defined as

$$w_{1D} = \frac{-i}{2\pi r_0^2} \oint_C z^*(x, t) dz(x, t). \quad (67)$$

	Pseudo-quantum mechanics for helical vortex-membrane	Quantum mechanics for single particle
Physical objects	Vortex with constraint volume	Single particle
Motion	Skew-mean-curvature flow by rotating the system	Quantum motion
Equation of motion	Pseudo-Schrödinger equation: $i\hbar_{\text{eff}} \frac{d\psi(x,t)}{dt} = \frac{\hat{p}^2}{2m_{\text{pseudo}}} \psi(x,t)$	Schrödinger equation: $i\hbar \frac{d\psi(x,t)}{dt} = \frac{\hat{p}^2}{2m} \psi(x,t)$
Hamiltonian	Energy operator, $\hat{H} = \frac{\hat{p}^2}{2m_{\text{pseudo}}}$	Energy operator, $\hat{H} = \frac{\hat{p}^2}{2m}$
Wave function	Function for classical deterministic Kelvin waves: $z(x,t)$	Wave function for probability wave of particles: $\psi(x,t)$
Momentum	(Lamb impulse) momentum operator $-i\hbar_{\text{eff}} \frac{d}{dx}$	Momentum operator, $-i\hbar \frac{d}{dx}$
Mass	Pseudo-mass, m_{pseudo}	Particle's mass, m
Planck constant	(Lamb impulse) angular momentum J_{Lamb}	$\hbar$

Fig. 5. Comparison between pseudo-quantum mechanics for helical vortex-membrane and quantum mechanics for single quantum particle.

To locally characterize the winding behavior, we define the winding-number density ρ_{wind} , $\rho_{\text{wind}} = \frac{\Delta w_{1D}}{\Delta x}$. For example, for a vortex-line described by a plane Kelvin wave $z(x,t) = r_0 e^{-i\omega \cdot t + ik \cdot x}$, the winding number $w_{1D} = \frac{k}{2\pi} l$ is proportional to the length of the system l and the density of winding number ρ_{wind} is uniform, i.e., $\rho_{\text{wind}} = \frac{w_{1D}}{l} = \frac{k}{2\pi}$. From this equation, the momentum p_{Lamb} is proportional of the winding number w_{1D} .

According to pseudo-quantum mechanics, the local winding can be described by the density operator of winding number,

$$\rho_{\text{wind}} \rightarrow \hat{\rho}_{\text{wind}} = \frac{\hat{k}}{2\pi} = \frac{1}{2\pi \hbar_{\text{eff}}} \hat{p} = -i \frac{1}{2\pi} \frac{d}{dx}. \quad (68)$$

For a vortex-line described by an arbitrary plane Kelvin wave $z(x,t) = \sum_k c_k e^{-i\omega \cdot t + ik \cdot x}$, the total 1D winding number is obtained by the following equation:

$$\begin{aligned} w_{1D} &= \langle x, t | \frac{l \cdot \hat{k}}{2\pi} [x, t] = \frac{1}{r_0^2} \int z^*(x, t) \frac{\hat{k}}{2\pi} z(x, t) dx \\ &= \frac{1}{r_0^2} \int z^*(x, t) \left(-i \frac{1}{2\pi} \frac{d}{dx} \right) z(x, t) dx. \end{aligned} \quad (69)$$

The winding-number density becomes

$$\rho_{\text{wind}}(x, t) = z^*(x, t) \left(-i \frac{1}{2\pi} \frac{1}{r_0^2} \frac{d}{dx} \right) z(x, t). \quad (70)$$

For example, a local winding at $x = x_0$ is defined by $\rho_{\text{wind}}(x, t) = \delta(x - x_0)$ that corresponds to a sharp changing of vortex-line; for the case of $z(x, t) = \text{constant}$, there is no winding at all $\rho_{\text{wind}}(x, t) = 0$.

In 5D space, we also define the winding number and the winding-number density for a helical vortex-membrane along different directions.

Along the given direction $\mathbf{e}^I$, winding number of the vortex-membrane (r_0 is a constant) described by plane Kelvin wave $z(\mathbf{x}, t) = r_0 e^{-i\omega \cdot t + ik^I \cdot x^I}$ is well defined. Because an arbitrary Kelvin wave can be described by $z(\mathbf{x}, t) = \sum_k c_k e^{-i\omega \cdot t + i\mathbf{k} \cdot \mathbf{x}}$, we can use a vector of winding-number density to describe local winding of a vortex-membrane $z(x^I, t)$. So in 3D space, there are three winding numbers w_{1D}^I ($I = X, Y, Z$) along different directions as

$$w_{1D}^I = \frac{-i}{2\pi r_0^2} \oint_{C^I} z^*(\mathbf{x}, t) dz(\mathbf{x}, t), \quad (71)$$

where C^I denotes the closed path around a flat membrane along different directions. For a vortex-membrane described by a plane Kelvin wave $z(\mathbf{x}, t) = r_0 e^{-i\omega \cdot t + i\mathbf{k} \cdot \mathbf{x}}$, the winding number along direction $\mathbf{e}^I$ is given by

$$w_{1D}^I = \frac{k^I}{2\pi} L^I \quad (72)$$

that is proportional to the length of the system L^I along $\mathbf{e}^I$ -direction and the winding-number density is $\rho_{\text{wind}}^I = \frac{k^I}{2\pi}$.

Thus, the local winding along direction $\mathbf{e}$ can be described by the operator of winding-number density,

$$\hat{\rho}_{\text{wind}} = \frac{\hat{k}}{2\pi} = -i \frac{1}{2\pi} \frac{d}{dx^I}, \quad (73)$$

where $\hat{\rho}_{\text{wind}} = \hat{\rho}_{\text{wind}} \cdot \mathbf{e}^I$. With the help of $\hat{\rho}_{\text{wind}}$, the winding number of a vortex-membrane is obtained as

$$\mathbf{w}_{1D} = \langle x, t | l \cdot \hat{\rho}_{\text{wind}} | x, t \rangle = \int z^*(\mathbf{x}, t) \left(-i \frac{1}{2\pi r_0^2} \frac{d}{d\mathbf{x}} \right) z(\mathbf{x}, t) d\mathbf{x}. \quad (74)$$

The vector of winding-number density becomes

$$\boldsymbol{\rho}_{\text{wind}} = z^*(\mathbf{x}, t) \left(-i \frac{1}{2\pi r_0^2} \frac{d}{d\mathbf{x}} \right) z(\mathbf{x}, t). \quad (75)$$

Therefore, we answer the question — *how to characterize the evolution of a nonuniform helical vortex-membrane?* We consider a deformed vortex-membrane $z(\mathbf{x})$ as the initial condition. We expand $z(\mathbf{x})$ by the eigenstates of Kelvin waves as $z(\mathbf{x}) = \sum_k c_k e^{i\mathbf{k} \cdot \mathbf{x}}$. Under time evolution, the function of vortex-membranes

becomes $z(\mathbf{x}, t) = \sum_k c_k e^{i\mathbf{k} \cdot \mathbf{x}} e^{-i\omega \cdot t}$. Finally, the winding-number density for the deformed vortex-membrane is obtained as

$$\begin{aligned} \rho_{\text{wind}}(\mathbf{x}, t) &= z^*(\mathbf{x}, t) \left(-i \frac{1}{2\pi r_0^2} \frac{d}{d\mathbf{x}} \right) z(\mathbf{x}, t) \\ &= \frac{1}{2\pi r_0^2} \sum_{k, k'} \mathbf{k} \cdot c_{k'}^* c_k \cdot e^{-i(\mathbf{k}' - \mathbf{k}) \cdot \mathbf{x}} \cdot e^{i(\omega' - \omega) \cdot t}. \end{aligned} \quad (76)$$

We emphasize that under time evolution, the energy $H(\mathbf{p}_{\text{Lamb}})$, the momentum $\mathbf{p}_{\text{Lamb}}$ (the total winding number along given direction w_{1D}) and the (Lamb impulse) angular momentum (the effective Planck constant $\hbar_{\text{eff}}$) are all conserved.

In particular, we have the topological representation on action S , i.e.,

$$\begin{aligned} S &= \int p_{\text{Lamb}} \cdot dx = \hbar_{\text{eff}} \int k \cdot dx = \hbar_{\text{eff}} \int 2\pi \rho_{\text{wind}} \cdot dx = 2\pi \hbar_{\text{eff}} \cdot w_{1D} \\ &= \hbar_{\text{eff}} \cdot w_{1D}. \end{aligned} \quad (77)$$

From the point of view of topology, we explain the Sommerfeld quantization condition

$$\oint p_{\text{Lamb}} \cdot dx = n \hbar_{\text{eff}}, \quad (78)$$

where $\hbar_{\text{eff}} = 2\pi \hbar_{\text{eff}}$ and n is an integer number. The Sommerfeld quantization condition is just the topological condition of the winding number, i.e.,

$$\oint p_{\text{Lamb}} \cdot dx = \hbar_{\text{eff}} \cdot w_{1D},$$

where w_{1D} is the winding number of the helical vortex-membrane.

3.4. Path integral formulation for winding evolution

In 1948, Feynman derived path integral formulation of quantum mechanics, based on the fact that the propagator can be written as a sum over all possible paths between the initial point and the final point. With the help of the path integral, people can calculate the probability amplitude $K(\mathbf{x}', t_f; \mathbf{x}, t_i)$ from an initial position $\mathbf{x}$ at time $t = t_i$ (that is described by a state $|t_i, \mathbf{x}\rangle$) to position $\mathbf{x}'$ at a later time $t = t_f$ ($|t_f, \mathbf{x}'\rangle$).

Using a similar approach, we derive the path integral formulation for pseudo-quantum mechanics. A state $|t, \mathbf{x}\rangle$ in pseudo-quantum mechanics denotes a local winding at point $\mathbf{x}$ and time t . Letting the state evolve with time and project on the state $|\mathbf{x}'\rangle$, we have the transition amplitude of the process as

$$K(\mathbf{x}', t_f; \mathbf{x}, t_i) = \langle t_f, \mathbf{x}' | [t_i, \mathbf{x}] = \sum_n e^{iS_n/\hbar_{\text{eff}}} = \int \mathcal{D}\mathbf{p}_{\text{Lamb}}(t) \mathcal{D}\mathbf{x}(t) e^{iS/\hbar_{\text{eff}}}, \quad (79)$$

where $S = \int [\mathbf{p}_{\text{Lamb}} \cdot \frac{d\mathbf{x}}{dt} - \hat{H}_{\text{Lamb}}(\mathbf{p}_{\text{Lamb}}, \mathbf{x})] dt$ and $\hat{H}_{\text{Lamb}}(\mathbf{p}_{\text{Lamb}}, \mathbf{x}) = \frac{\mathbf{p}_{\text{Lamb}}^2}{2m_{\text{pseudo}}}$. Each path contributes $e^{iS_n/\hbar_{\text{eff}}}$ where S_n is the n th classical action i.e., $\sum_n e^{iS_n/\hbar_{\text{eff}}}$.

Let us discuss the implication of path-integral formulation in pseudo-quantum mechanics.

We firstly consider a state with a local winding at the position $(\mathbf{x}_i, \phi_0)$ and time $t = t_i$, where $\mathbf{x}_i$ denotes the original position and ϕ_0 the phase angle. To calculate the transition amplitude $K(\mathbf{x}', t_f; \mathbf{x}, t_i) = \langle t_f, \mathbf{x}' | [t_i, \mathbf{x}] \rangle$, the winding splits into pieces. For a winding-piece, the action is denoted by S_n for an arbitrary possible classical path from $(\mathbf{x}_i, \phi_0)$ to $(\mathbf{x}_f, \phi_n)$ within time $T = t_f - t_i$. In particular, the phase angle ϕ may be different for a winding-piece moving along different classical paths. The phase-changing of a possible classical path is given by

$$\Delta\phi = \phi_n - \phi_0 = \frac{S_n}{\hbar_{\text{eff}}}. \quad (80)$$

Because the path may be not closed, $\frac{S_n}{\hbar_{\text{eff}}}$ may be not an integer number. As a result, we have the transition amplitude as $e^{iS_n/\hbar_{\text{eff}}}$. We summarize the contribution from all windings and get the final transition amplitude as

$$\sum_n e^{i\Delta\phi_n} = \sum_n e^{iS_n/\hbar_{\text{eff}}} \quad (81)$$

that is just the Feynman's path integral formulation.

In summary, pseudo-quantum mechanics becomes a toy mechanics to learn the mechanism of true quantum mechanics.

4. Biot–Savart Mechanics for Two Entangled Vortex-Membranes: Pseudo-Quantum Mechanics for a Two-Component Particle

In this section, we discuss the Biot–Savart mechanics for two entangled vortex-membranes. The issue of Biot–Savart mechanics for two entangled vortex-membranes is about the dynamic evolution of entanglement between them. Here, we ask a question — *how to characterize the evolution of nonuniform entangled vortex-membranes?* For example, a nonuniform entangled vortex-line is shown in Fig. 6(b). We point out that to characterize the two entangled vortex-membranes, the corresponding Biot–Savart mechanics is mapped to pseudo-quantum mechanics for a two-component particle.

4.1. Biot–Savart mechanics for two entangled vortex-membranes with leapfrogging motion

In this part, we discuss the Biot–Savart mechanics for two entangled vortex-membranes.

In five-dimensional space $\{\mathbf{x}, \xi, \eta\}$ ($\mathbf{x} = (x, y, z)$), we introduce a complex description on ξ - η complex plane, $z_{A/B}(\mathbf{x}, t) = \xi_{A/B}(\mathbf{x}, t) + i\eta_{A/B}(\mathbf{x}, t) = r_{A/B}(\mathbf{x}, t)e^{i\phi_{A/B}(\mathbf{x}, t)}$, where $r_{A/B}(\mathbf{x}, t)$ is amplitude and $\phi_{A/B}(\mathbf{x}, t)$ is angle in ξ - η complex plane. The index A/B denotes vortex-line-A or vortex-line-B. Under a

Fig. 6. (Color online) (a) Uniform entangled vortex-lines; (b) nonuniform entangled vortex-lines. The index A/B denotes vortex-line-A or vortex-line-B. The red/green spots denote the zeros between two projected vortex-lines.

simple geometrical constraint, $z'_{A/B}(\mathbf{x}) \sim \frac{|z_{A/B}(\mathbf{x}_1) - z_{A/B}(\mathbf{x}_2)|}{|\mathbf{x}_1 - \mathbf{x}_2|} \ll 1$, the generalized Biot-Savart equation can be written into

$$i \frac{dz_{A/B}}{dt} = \frac{\delta H(z_{A/B})}{\delta z_{A/B}^*}, \quad (82)$$

where the Hamiltonian is given by Ref. 27

$$\begin{aligned} H &= \text{Kinetic term} + \text{interaction term} = H_{\text{volume}} + H_{\text{Kirchhoff}} \\ &\simeq \int \left[\frac{(\alpha \kappa \ln \epsilon)}{2} (|z'_A|^2 + |z'_B|^2) - (\alpha \kappa \ln \epsilon) \ln |z_A(\mathbf{x}) - z_B(\mathbf{x})| \right] dV_P \\ &\quad + \text{constant}. \end{aligned} \quad (83)$$

Here, r_0 is the inter-vortex distance and a_0 denotes the vortex filament radius, i.e., $r_0 \gg a_0$.

Because the vortex-membranes are almost straight, the nonlocal interactions, varying on vortex-membranes, are approximated to be similar to those of two straight vortex-membranes that is described by Kirchhoff Hamiltonian, $H_{\text{Kirchhoff}}$.

Now, the Biot-Savart equation of motion for each vortex-membrane is given by Ref. 27

$$\begin{aligned} i \frac{\partial z_A}{\partial t} &= \frac{\delta H[z_A, z_B]}{\delta z_A^*}, \\ i \frac{\partial z_B}{\partial t} &= \frac{\delta H[z_A, z_B]}{\delta z_B^*}, \end{aligned} \quad (84)$$

where

$$\begin{aligned} \frac{\partial z_A}{\partial t} &= i \frac{(\alpha \kappa \ln \epsilon)}{2} \frac{\partial^2 z_A}{\partial \mathbf{x}^2} + i(\alpha \kappa \ln \epsilon) \frac{z_A - z_B}{|z_A - z_B|^2}, \\ \frac{\partial z_B}{\partial t} &= i \frac{(\alpha \kappa \ln \epsilon)}{2} \frac{\partial^2 z_B}{\partial \mathbf{x}^2} - i(\alpha \kappa \ln \epsilon) \frac{z_A - z_B}{|z_A - z_B|^2}. \end{aligned} \quad (85)$$

By introducing $z_A = \frac{z_u + z_v}{2}$ and $z_B = \frac{z_v - z_u}{2}$, the above equations transform into Ref. 27

$$\begin{aligned}\frac{\partial z_u}{\partial t} &= i \frac{(\alpha \kappa \ln \epsilon)}{2} \frac{\partial^2 z_u}{\partial \mathbf{x}^2} + i(\alpha \kappa \ln \epsilon) \frac{2z_u}{|z_u|^2}, \\ \frac{\partial z_v}{\partial t} &= i \frac{(\alpha \kappa \ln \epsilon)}{2} \frac{\partial^2 z_v}{\partial \mathbf{x}^2}.\end{aligned}\quad (86)$$

The plane Kelvin wave solutions are obtained as

$$z_u = r_0 e^{i(\mathbf{k} \cdot \mathbf{x} - \omega_u t + \phi_u)}, \quad z_v = r_0 e^{i(\mathbf{k} \cdot \mathbf{x} - \omega_v t + \phi_v)}, \quad (87)$$

where ϕ_u and ϕ_v are constant phase angles, and the angular frequencies are

$$\omega_u = \frac{(\alpha \kappa \ln \epsilon)}{2} \mathbf{k}^2 - (\alpha \kappa \ln \epsilon) \frac{2}{r_0^2} \quad (88)$$

and

$$\omega_v = \frac{(\alpha \kappa \ln \epsilon)}{2} \mathbf{k}^2. \quad (89)$$

In this paper, we set $\phi_u = 0$ and $\phi_v = 0$. For two entangled vortex-membranes, the nonlocal interaction leads to leapfrogging motion. Above solutions of the Biot-Savart equation can be written into

$$\begin{pmatrix} z_A(\mathbf{x}, t) \\ z_B(\mathbf{x}, t) \end{pmatrix} = \frac{r_0}{2} \begin{pmatrix} e^{i\omega^* t} + 1 \\ 1 - e^{i\omega^* t} \end{pmatrix} e^{i\mathbf{k} \cdot \mathbf{x} - i\omega t} = \begin{pmatrix} r_A \\ r_B \end{pmatrix} e^{i\mathbf{k} \cdot \mathbf{x} - i\omega t + i\omega^* t/2}, \quad (90)$$

where the winding radii of two vortex-membranes are $r_A = r_0 \cos(\frac{\omega^* t}{2})$ and $r_B = -r_0 \sin(\frac{\omega^* t}{2})$, respectively. $\omega = \omega_v = \frac{(\alpha \kappa \ln \epsilon)}{2} \mathbf{k}^2$ and $\omega^* = \omega_v - \omega_u = (\alpha \kappa \ln \epsilon) \frac{2}{r_0^2}$ denote (global) rotating velocity frequency and (internal) leapfrogging angular frequency, respectively. From above solutions, we have a constraint condition of total volume, i.e.,

$$|r_A|^2 + |r_B|^2 \equiv r_0^2. \quad (91)$$

For rotating motion with finite ω , the entangled vortex-membranes are described by plane waves $e^{i\mathbf{k} \cdot \mathbf{x} - i\omega t}$. For leapfrogging motion with finite ω^* , the entangled vortex-membranes exchange energy in a periodic fashion. The winding radii of two vortex-membranes oscillate with a period $T = \frac{2\pi}{\omega^*}$: At $t = 0$, helical vortex-membrane-A winds around straight vortex-membrane-B clockwise (the state $[A]$); At $t = \frac{T}{4}$, the system becomes a symmetric double helix vortex-membranes (the state $((i+1)[A] - (i-1)[B])/2$); At $t = \frac{T}{2}$, helical vortex-membrane-B winds around straight vortex-membrane-A (the state $[B]$), ...

During leapfrogging process, the Lamb impulse (momentum) and the angular Lamb impulse (angular momentum) are conserved. The effective Planck constant is obtained as projected (Lamb impulse) angular momentum as $\hbar_{\text{eff}} = J_{\text{Lamb}} = \frac{1}{2} \rho_0 \kappa r_0^2 \cdot V_P$, where V_P is the total volume of the system.

We then map the Biot-Savart mechanics to the pseudo-quantum mechanics. The system with two entangled vortex-membranes is mapped to a two-component particle (not a two-particle case), i.e.,

$$\begin{pmatrix} i\hbar_{\text{eff}} \frac{dz_A(\mathbf{x}, t)}{dt} \\ i\hbar_{\text{eff}} \frac{dz_B(\mathbf{x}, t)}{dt} \end{pmatrix} = \hat{H}_{\text{Lamb}} \begin{pmatrix} z_A(\mathbf{x}, t) \\ z_B(\mathbf{x}, t) \end{pmatrix} \\ = \left[\frac{\hat{p}_{\text{Lamb}}^2}{2m_{\text{pseudo}}} + \frac{\hbar_{\text{eff}}\omega^*}{2} \cdot (\tau_x - 1) \right] \begin{pmatrix} z_A(\mathbf{x}, t) \\ z_B(\mathbf{x}, t) \end{pmatrix} \quad (92)$$

with a constraint

$$|z_A(\mathbf{x}, t)|^2 + |z_B(\mathbf{x}, t)|^2 = 1. \quad (93)$$

Here, $\tau_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is Pauli matrix and $\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The leapfrogging process is characterized by $\frac{\hbar_{\text{eff}}\omega^*}{2}(\tau_x - 1)$. As a result, the pseudo-quantum mechanics describes the dynamics of two entangled vortex-membranes with fixed volume in the 5D fluid. Under the constraint, the total degree of freedom of a vortex-membrane is reduced to 2.

In pseudo-quantum mechanics, there are also three conserved physical quantities for entangled vortex-membranes: the total energy $H(\mathbf{p}_{\text{Lamb}})$ that is proportional to the total volume of the two vortex-membranes $V_P = \text{volume}(P)$; the total momentum $\mathbf{p}_{\text{Lamb}}$ that is proportional to the linking number between two vortex-membranes (see below discussion); and the total (Lamb impulse) angular momentum (the effective Planck constant $\hbar_{\text{eff}}$) that is proportional to the total volume of the two vortex-membranes in the 5D fluid $V_P \cdot r_0^2$.

4.2. Linking number and linking-number density

In the above part, we derive the equation of motion of two entangled vortex-membranes. However, due to the leapfrogging process, the winding number and winding-number density are not well defined. Instead, it is the linking number and linking-number density that characterize the entanglement between two vortex-membranes.

Firstly, we introduce the linking number to characterize entanglement between two vortex-membranes.

The (Gauss) linking-number ζ_{1D} for two 1D vortex-lines is a topological invariable to characterize the entanglement that is defined as Ref. 28

$$\zeta_{1D} = \frac{1}{4\pi} \oint_{C_A} \oint_{C_B} \frac{(\mathbf{s}_A - \mathbf{s}_B) \cdot d\mathbf{s}_A \times d\mathbf{s}_B}{|\mathbf{s}_A - \mathbf{s}_B|^3}. \quad (94)$$

The 1D linking number is also integer number. The system with different linking numbers has different entanglement patterns between two vortex-lines.

To locally characterize the entanglement, we define the density of linking-number ρ_{link} ,

$$\rho_{\text{link}} = \frac{\Delta \zeta_{1D}}{\Delta x}. \quad (95)$$

For the entangled vortex-membranes, owing to the intrinsic reversal symmetry between vortex-membrane-A and vortex-membrane-B, we can obtain the linking-number density from the winding-number density at special time of the leapfrogging motion, $t = 0$, i.e.,

$$\begin{pmatrix} z_A(x, t) \\ z_B(x, t) \end{pmatrix} = \begin{pmatrix} r_0 e^{ik \cdot x - i\omega t} \\ 0 \end{pmatrix}. \quad (96)$$

At this time, the winding-number density of vortex-membrane-A is $\rho_{\text{wind},A} = \frac{k}{2\pi}$. The linking-number density is equal to $\rho_{\text{wind},A}$,

$$\rho_{\text{link}} = \rho_{\text{wind},A} = \frac{\zeta_{1D}}{l} = \frac{k}{2\pi}. \quad (97)$$

Because the linking number is a conserved quantity and does not change during leapfrogging motion, the corresponding operator of linking-number density is given by

$$\rho_{\text{linking}} \rightarrow \hat{\rho}_{\text{linking}} = \frac{\hat{k}}{2\pi} = -i \frac{1}{2\pi} \frac{d}{dx}. \quad (98)$$

For two entangled 1D vortex-line, the linking-number is defined by

$$\begin{aligned} \zeta_{1D} &= \langle \mathbf{Z}(x, t) | l \cdot \hat{\rho}_{\text{linking}} | \mathbf{Z}(x, t) \rangle = \frac{1}{r_0^2} \int \mathbf{Z}^*(x, t) \hat{\rho}_{\text{linking}} \mathbf{Z}(x, t) dx \\ &= \int z_A^*(x, t) \left(-i \frac{1}{2\pi} \frac{1}{r_0^2} \frac{d}{dx} \right) z_A(x, t) dx \\ &\quad + \int z_B^*(x, t) \left(-i \frac{1}{2\pi} \frac{1}{r_0^2} \frac{d}{dx} \right) z_B(x, t) dx, \end{aligned} \quad (99)$$

where $\mathbf{Z}(x, t) = \begin{pmatrix} z_A(\mathbf{x}, t) \\ z_B(\mathbf{x}, t) \end{pmatrix}$. Thus, the linking-number density that is proportional to the total momentum p_{Lamb} is

$$\begin{aligned} \rho_{\text{linking}} &= \mathbf{Z}^*(x, t) \hat{\rho}_{\text{linking}} \mathbf{Z}(x, t) = z_A^*(x, t) \left(-i \frac{1}{2\pi} \frac{1}{r_0^2} \frac{d}{dx} \right) z_A(x, t) \\ &\quad + z_B^*(x, t) \left(-i \frac{1}{2\pi} \frac{1}{r_0^2} \frac{d}{dx} \right) z_B(x, t). \end{aligned} \quad (100)$$

In 5D space, the 2D Gauss linking number can only be defined between two 2D closed, oriented, submanifolds given by Ref. 29

$$\zeta_{2D} = \frac{1}{\text{vol } S^n} \int_{K \times L} \frac{\Omega_{k,\ell}(\alpha)}{|x|^{k+1} |y|^{\ell+1} \sin^n \alpha} [x, dx, y, dy], \quad (101)$$

where $\Omega_{k,\ell}(\alpha) = \int_{\theta=\alpha}^{\pi} \sin^k(\theta-\alpha) \sin^\ell \theta d\theta$. Here, $\alpha(x, y)$ is the angle between $x \in X$ and $y \in Y$, thought of as vectors in 5D space. So there is no 3D linking number to characterize the entanglement between two 3D vortex-membranes in 5D fluid.

Instead, we discuss entanglement between two 3D vortex-membranes via (projected) 1D linking number. There are three 1D linking-numbers ζ_{1D}^I ($I = X, Y, Z$) along different directions that are

$$\zeta_{1D}^I = \frac{1}{4\pi} \oint_{C_{x^I,A}} \oint_{C_{x^I,B}} \frac{(\mathbf{s}_A^I - \mathbf{s}_B^I) \cdot d\mathbf{s}_A^I \times d\mathbf{s}_B^I}{|\mathbf{s}_A^I - \mathbf{s}_B^I|^3}, \quad (102)$$

where $\mathbf{s}^I = \mathbf{r} \cdot \mathbf{e}^I$. The vector of linking number is defined as $\zeta_{1D} = (\zeta_{1D}^X, \zeta_{1D}^Y, \zeta_{1D}^Z)$. We define the vector of linking-number densities and linking-number density operators for 3D vortex-membranes,

$$\rho_{\text{linking}}^I(x, t) = \frac{1}{r_0^2} \mathbf{Z}^*(x, t) \hat{\rho}_{\text{linking}}^I \mathbf{Z}(x, t), \quad (103)$$

and

$$\hat{\rho}_{\text{linking}} = \left(-\frac{i}{2\pi r_0^2} \frac{d}{dx^X}, -\frac{i}{2\pi r_0^2} \frac{d}{dx^Y}, -\frac{i}{2\pi r_0^2} \frac{d}{dx^Z} \right), \quad (104)$$

respectively. For a vortex-membrane described by a plane Kelvin wave $z_A(x^I, t) = r_0 e^{-i\omega \cdot t + ik^I \cdot x^I}$ and a constant vortex-line described by $z_B(x^I, t) \equiv 0$, the 1D linking-number along direction $\mathbf{e}^I$ is given by $\zeta_{1D}^I = \frac{|k^I|}{2\pi} L_I$ and the density of linking-number ρ_{link}^I is $\rho_{\text{link}}^I = \frac{\zeta_{1D}^I}{L_I}$.

As a result, we answer the question — *how to characterize the evolution of nonuniform entangled vortex-membranes?* For a given initial state $\mathbf{Z}(\mathbf{x})$, the linking-number density for the local deformation of entangled vortex-membranes is obtained as $\rho_{\text{linking}}^I(\mathbf{x}, t) = \frac{1}{r_0^2} \mathbf{Z}^*(\mathbf{x}, t) \hat{\rho}_{\text{linking}}^I \mathbf{Z}(\mathbf{x}, t)$, where

$$\mathbf{Z}(\mathbf{x}) = \sum_k c_k e^{i\mathbf{k} \cdot \mathbf{x}} \rightarrow \mathbf{Z}(\mathbf{x}, t) = \sum_k c_k e^{i\mathbf{k} \cdot \mathbf{x}} e^{-i\omega \cdot t}. \quad (105)$$

From the point of view of topology, the Sommerfeld quantization condition for two-component particle is the topological condition of the linking number, i.e.,

$$\oint p_{\text{Lamb}} \cdot dx = h_{\text{eff}} \cdot \zeta_{1D},$$

where ζ_{1D} is the linking number of the two entangled vortex-membranes.

4.3. Tensor representation for entangled vortex-membranes

4.3.1. Tensor states for plane Kelvin waves

In this part, we introduce the tensor representation for Kelvin waves by separating the wave vector along a given direction into positive part and negative part, $\pm \mathbf{k}$.

For a plane Kelvin wave with fixed wavelength $2a = \frac{2\pi}{|\mathbf{k}|}$ along $\mathbf{e}$ -direction, there are two helical degrees of freedom: $z_{\mathbf{k}}(\mathbf{x}, t)$ at $\mathbf{k}$ or $z_{-\mathbf{k}}(\mathbf{x}, t)$ at $-\mathbf{k}$. After considering the vortex degrees of freedom A or B that characterize the Kelvin waves on different vortex-membranes, there are four degenerate states to characterize a plane Kelvin wave with fixed wavelength $2a$, i.e., the basis is

$$\begin{aligned} |\mathbf{k}, A\rangle &= \begin{pmatrix} z_{\mathbf{k}, A}(\mathbf{x}) \\ 0 \\ 0 \\ 0 \end{pmatrix}, & |\mathbf{k}, B\rangle &= \begin{pmatrix} 0 \\ z_{\mathbf{k}, B}(\mathbf{x}) \\ 0 \\ 0 \end{pmatrix}, \\ |-\mathbf{k}, A\rangle &= \begin{pmatrix} 0 \\ 0 \\ z_{-\mathbf{k}, A}(\mathbf{x}) \\ 0 \end{pmatrix}, & |-\mathbf{k}, B\rangle &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ z_{-\mathbf{k}, B}(\mathbf{x}) \end{pmatrix}. \end{aligned} \quad (106)$$

Here, $|\mathbf{k}, A\rangle$ denotes a plane Kelvin wave with wave vector $\mathbf{k}$ on vortex-membrane-A,

$$z_{\mathbf{k}, A}(\mathbf{x}) = r_0 e^{i\mathbf{k} \cdot \mathbf{x}}; \quad (107)$$

$|-\mathbf{k}, A\rangle$ denotes a plane Kelvin wave with wave vector $-\mathbf{k}$ on vortex-membrane-A,

$$z_{-\mathbf{k}, A}(\mathbf{x}) = r_0 e^{-i\mathbf{k} \cdot \mathbf{x}}; \quad (108)$$

$|\mathbf{k}, B\rangle$ denotes a plane Kelvin wave with wave vector $\mathbf{k}$ on vortex-membrane-B,

$$z_{\mathbf{k}, B}(\mathbf{x}) = r_0 e^{i\mathbf{k} \cdot \mathbf{x}}; \quad (109)$$

$|-\mathbf{k}, B\rangle$ denotes a plane Kelvin wave with wave vector $-\mathbf{k}$ on vortex-membrane-B,

$$z_{-\mathbf{k}, B}(\mathbf{x}) = r_0 e^{-i\mathbf{k} \cdot \mathbf{x}}. \quad (110)$$

See the illustration in Fig. 7.

Thus, an arbitrary plane Kelvin wave can be characterized by a superposed state of the basis along different spatial directions. We introduce a tensor representation to define the plane Kelvin waves with fixed wavelength,

$$|\mathbf{Z}\rangle = \prod_I |\mathbf{Z}^I\rangle \quad (111)$$

with

$$|\mathbf{Z}^I\rangle = \begin{pmatrix} \alpha_{1,1}^I \\ \alpha_{1,-1}^I \\ \alpha_{-1,1}^I \\ \alpha_{-1,-1}^I \end{pmatrix}, \quad (112)$$

Fig. 7. (Color online) An illustration of basis of four internal degrees of freedom for plane Kelvin waves in pseudo-quantum mechanics, $[\mathbf{k}, \mathbf{B}]$, $[-\mathbf{k}, \mathbf{B}]$, $[\mathbf{k}, \mathbf{A}]$, $[-\mathbf{k}, \mathbf{A}]$. The lower figures illustrate linear dispersion for the four elementary states. The red dots denote the four possible elementary states with opposite wave vectors of the Kelvin waves.

where I denotes the spatial indices, $X, Y, Z, \dots$; $i = 1, -1$ denotes helical degrees of freedom $\mathbf{k}$ or $-\mathbf{k}$; $j = 1, -1$ denotes the vortex-degrees of freedom A or B. In general, there are $d \times 2 \times 2$ elements $\alpha_{i,j}^I$. Each element $\alpha_{i,j}^I$ is proportional to the winding radius of the corresponding plane Kelvin waves. According to the geometric constraint condition, we have

$$\begin{aligned}
 |\alpha_{1,1}^X|^2 + |\alpha_{1,-1}^X|^2 + |\alpha_{-1,1}^X|^2 + |\alpha_{-1,-1}^X|^2 &= 1, \\
 |\alpha_{1,1}^Y|^2 + |\alpha_{1,-1}^Y|^2 + |\alpha_{-1,1}^Y|^2 + |\alpha_{-1,-1}^Y|^2 &= 1, \\
 |\alpha_{1,1}^Z|^2 + |\alpha_{1,-1}^Z|^2 + |\alpha_{-1,1}^Z|^2 + |\alpha_{-1,-1}^Z|^2 &= 1, \\
 &\dots
 \end{aligned} \tag{113}$$

So the resulting function for a Kelvin wave with fixed wavelength is given by

$$Z^I(\mathbf{x}) = \alpha_{1,1}^I z_{\mathbf{k},\mathbf{A}}(\mathbf{x}) + \alpha_{1,-1}^I z_{\mathbf{k},\mathbf{B}}(\mathbf{x}) + \alpha_{-1,1}^I z_{-\mathbf{k},\mathbf{A}}(\mathbf{x}) + \alpha_{-1,-1}^I z_{-\mathbf{k},\mathbf{B}}(\mathbf{x}). \tag{114}$$

Owing to the fact of the energy degeneracy, all superposed states that consist of four degenerate states along different spatial directions denoted by different weights $\alpha_{i,j}^I$ have the same energy.

4.3.2. Tensor operator

The tensor states of a plane Kelvin wave can be classified by operator representation.

We define the $d \times 3 \times 4$ tensor operators $\hat{\Gamma}_{i,j}^I = \sigma_i^I \otimes \tau_j^I$ ($I = X, Y, Z, \dots; i = x, y, z$ for helical degrees of freedom; $j = 0, x, y, z$ for the vortex-degrees of freedom) to be

$$\begin{aligned}
 \hat{\Gamma}_{x,x}^X &= \sigma_x^X \otimes \tau_x^X, & \hat{\Gamma}_{x,x}^Y &= \sigma_x^Y \otimes \tau_x^Y, & \hat{\Gamma}_{x,x}^Z &= \sigma_x^Z \otimes \tau_x^Z, & \hat{\Gamma}_{x,0}^X &= \sigma_x^X \otimes \tau_0^X, \\
 \hat{\Gamma}_{x,y}^X &= \sigma_x^X \otimes \tau_y^X, & \hat{\Gamma}_{x,y}^Y &= \sigma_x^Y \otimes \tau_y^Y, & \hat{\Gamma}_{x,y}^Z &= \sigma_x^Z \otimes \tau_y^Z, & \hat{\Gamma}_{x,0}^Y &= \sigma_x^Y \otimes \tau_0^Y, \\
 \hat{\Gamma}_{x,z}^X &= \sigma_x^X \otimes \tau_z^X, & \hat{\Gamma}_{x,z}^Y &= \sigma_x^Y \otimes \tau_z^Y, & \hat{\Gamma}_{x,z}^Z &= \sigma_x^Z \otimes \tau_z^Z, & \hat{\Gamma}_{x,0}^Z &= \sigma_x^Z \otimes \tau_0^Z, \\
 \hat{\Gamma}_{y,x}^X &= \sigma_y^X \otimes \tau_x^X, & \hat{\Gamma}_{y,x}^Y &= \sigma_y^Y \otimes \tau_x^Y, & \hat{\Gamma}_{y,x}^Z &= \sigma_y^Z \otimes \tau_x^Z, & \hat{\Gamma}_{y,0}^X &= \sigma_y^X \otimes \tau_0^X, \\
 \hat{\Gamma}_{y,y}^X &= \sigma_y^X \otimes \tau_y^X, & \hat{\Gamma}_{y,y}^Y &= \sigma_y^Y \otimes \tau_y^Y, & \hat{\Gamma}_{y,y}^Z &= \sigma_y^Z \otimes \tau_y^Z, & \hat{\Gamma}_{y,0}^Y &= \sigma_y^Y \otimes \tau_0^Y, \\
 \hat{\Gamma}_{y,z}^X &= \sigma_y^X \otimes \tau_z^X, & \hat{\Gamma}_{y,z}^Y &= \sigma_y^Y \otimes \tau_z^Y, & \hat{\Gamma}_{y,z}^Z &= \sigma_y^Z \otimes \tau_z^Z, & \hat{\Gamma}_{y,0}^Z &= \sigma_y^Z \otimes \tau_0^Z, \\
 \hat{\Gamma}_{z,x}^X &= \sigma_z^X \otimes \tau_x^X, & \hat{\Gamma}_{z,x}^Y &= \sigma_z^Y \otimes \tau_x^Y, & \hat{\Gamma}_{z,x}^Z &= \sigma_z^Z \otimes \tau_x^Z, & \hat{\Gamma}_{z,0}^X &= \sigma_z^X \otimes \tau_0^X, \\
 \hat{\Gamma}_{z,y}^X &= \sigma_z^X \otimes \tau_y^X, & \hat{\Gamma}_{z,y}^Y &= \sigma_z^Y \otimes \tau_y^Y, & \hat{\Gamma}_{z,y}^Z &= \sigma_z^Z \otimes \tau_y^Z, & \hat{\Gamma}_{z,0}^Y &= \sigma_z^Y \otimes \tau_0^Y, \\
 \hat{\Gamma}_{z,z}^X &= \sigma_z^X \otimes \tau_z^X, & \hat{\Gamma}_{z,z}^Y &= \sigma_z^Y \otimes \tau_z^Y, & \hat{\Gamma}_{z,z}^Z &= \sigma_z^Z \otimes \tau_z^Z, & \hat{\Gamma}_{z,0}^Z &= \sigma_z^Z \otimes \tau_0^Z, \\
 & \dots & & & & & &
 \end{aligned} \tag{115}$$

Here, $\tau_0^{X/Y/Z} = \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Different tensor states have different tensor operators

$$\langle \hat{\Gamma}^I \rangle = \langle \mathbf{Z}_{\text{knot-crystal}} | \hat{\Gamma}^I | \mathbf{Z}_{\text{knot-crystal}} \rangle = \mathbf{\Gamma}^I = \mathbf{n}_\sigma^I \otimes (\mathbf{n}_\tau^I, \tau), \tag{116}$$

where $\hat{\Gamma}^I = \sigma^I \otimes (\tau^I, \mathbf{1})$ and σ^I, τ^I are 2×2 Pauli matrices for helical and vortex degrees of freedom, respectively. On the basis of pseudo-spin base $(\tau_x^I, \tau_y^I, \tau_z^I, \mathbf{1})$, the tensor states are determined by the “direction” of pseudo-spin order for helical degrees of freedom

$$\langle \tau^I \rangle = \langle z | (\mathbf{1} \otimes \tau_i^I) | z \rangle = (\langle \mathbf{1} \otimes \tau_x^I \rangle, \langle \mathbf{1} \otimes \tau_y^I \rangle, \langle \mathbf{1} \otimes \tau_z^I \rangle) = \mathbf{n}_\tau^I, \langle \mathbf{1} \rangle = \langle \mathbf{Z} | (\mathbf{1} \otimes \mathbf{1}) | \mathbf{Z} \rangle = \tau \tag{117}$$

In general, we have $|\mathbf{n}_\tau^I|^2 + |\tau|^2 = 1$. On the basis of pseudo-spin base $(\sigma_x^I, \sigma_y^I, \sigma_z^I)$, the tensor states are determined by the “direction” of pseudo-spin order for helical degrees of freedom,

$$\langle \sigma^I \rangle = \langle \mathbf{Z} | (\sigma_i^I \otimes \mathbf{1}) | \mathbf{Z} \rangle = (\langle \sigma_x^I \otimes \mathbf{1} \rangle, \langle \sigma_y^I \otimes \mathbf{1} \rangle, \langle \sigma_z^I \otimes \mathbf{1} \rangle) = \mathbf{n}_\sigma^I. \tag{118}$$

In general, we have $|\mathbf{n}_\sigma^I|^2 = 1$. For example, $\mathbf{n}_\sigma^X = (1, 0, 0)$ is a tensor state along spatial x -direction and helical x -direction, $\langle \sigma_x^X \otimes \mathbf{1} \rangle = 1$, $\langle \sigma_y^X \otimes \mathbf{1} \rangle = 0$, $\langle \sigma_z^X \otimes \mathbf{1} \rangle = 0$. Because the vortex-degrees of freedom are always trivial, the vortex-vector is a constant vector along different directions, i.e., $\mathbf{n}_\tau^X = \mathbf{n}_\tau^Y = \mathbf{n}_\tau^Z = \mathbf{n}_\tau$ and $\tau = \tau_0$. Therefore in the following parts, we focus on the tensor states $\mathbf{n}_\sigma^I$; helical degrees of freedom.

4.3.3. Example

For a 1D travelling Kelvin wave $Z(\mathbf{X}) = \begin{pmatrix} Z_A(\mathbf{X}) \\ Z_B(\mathbf{X}) \end{pmatrix} = r_0 e^{i\mathbf{k} \cdot \mathbf{x}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ or $Z(\mathbf{X}) = \begin{pmatrix} Z_A(\mathbf{X}) \\ Z_B(\mathbf{X}) \end{pmatrix} = r_0 e^{-i\mathbf{k} \cdot \mathbf{x}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, the tensor state is represented by

$$[\mathbf{Z}^X] = \begin{pmatrix} \alpha_{1,1}^I \\ \alpha_{1,-1}^I \\ \alpha_{-1,1}^I \\ \alpha_{-1,-1}^I \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad (119)$$

or

$$[\mathbf{Z}^X] = \begin{pmatrix} \alpha_{1,1}^I \\ \alpha_{1,-1}^I \\ \alpha_{-1,1}^I \\ \alpha_{-1,-1}^I \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad (120)$$

of which the tensor state is denoted by

$$\langle \sigma^X(x) \otimes 1 \rangle = \mathbf{n}_\sigma^X = (0, 0, \pm 1). \quad (121)$$

We call it plane σ_z -Kelvin wave.

For a standing Kelvin wave described by $Z(\mathbf{X}) = \begin{pmatrix} Z_A(\mathbf{X}) \\ Z_B(\mathbf{X}) \end{pmatrix} = \sqrt{2}r_0 \cos(k \cdot x) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ or $Z(\mathbf{X}) = \begin{pmatrix} Z_A(\mathbf{X}) \\ Z_B(\mathbf{X}) \end{pmatrix} = i\sqrt{2}r_0 \sin(k \cdot x) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, the tensor state is represented by

$$[\mathbf{Z}^X] = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad (122)$$

or

$$[\mathbf{Z}^X] = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \quad (123)$$

of which the tensor state is denoted by

$$\langle \sigma^X(x) \otimes \mathbf{1} \rangle = \mathbf{n}_\sigma^X = (\pm 1, 0, 0). \quad (124)$$

We call it σ_x -Kelvin wave.

Next, we discuss the 2D plane Kelvin waves.

For a 2D σ_z -Kelvin wave described by $\mathbf{Z}(\mathbf{x}) = \begin{pmatrix} z_A(\mathbf{x}) \\ z_B(\mathbf{x}) \end{pmatrix} = r_0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{\pm i\mathbf{k} \cdot \mathbf{x}}$, the tensor state can be represented by

$$[\mathbf{Z}^X] = [\mathbf{Z}^Y] = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad (125)$$

or

$$[\mathbf{Z}^X] = [\mathbf{Z}^Y] = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}. \quad (126)$$

The tensor state is denoted by

$$\langle \sigma^X \otimes \mathbf{1} \rangle = \mathbf{n}_\sigma^X = (0, 0, \pm 1), \quad \langle \sigma^Y \otimes \mathbf{1} \rangle = \mathbf{n}_\sigma^Y = (0, 0, \pm 1). \quad (127)$$

Another 2D plane Kelvin wave with fixed wavelength is the standing Kelvin wave. Along x -direction, the function of plane Kelvin wave becomes

$$z(x) = \begin{pmatrix} z_A(x) \\ z_B(x) \end{pmatrix} = \frac{1}{\sqrt{2}} r_0 (e^{ik \cdot x} + e^{-ik \cdot x}) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \sqrt{2} r_0 \cos(\mathbf{k} \cdot \mathbf{x}) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (128)$$

along y -direction, the function of the plane Kelvin wave becomes

$$z(y) = \begin{pmatrix} z_A(y) \\ z_B(y) \end{pmatrix} = \frac{1}{\sqrt{2}} r_0 (e^{ik \cdot y} + ie^{-ik \cdot y}) \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (129)$$

The tensor state is represented by

$$[\mathbf{Z}^X] = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad (130)$$

or

$$[\mathbf{Z}^Y] = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ i \\ i \end{pmatrix}. \quad (131)$$

The tensor state is denoted by

$$\langle \sigma^X \otimes \mathbf{1} \rangle = \mathbf{n}_\sigma^X = (1, 0, 0), \quad \langle \sigma^Y \otimes \mathbf{1} \rangle = \mathbf{n}_\sigma^Y = (0, 1, 0). \quad (132)$$

For the plane Kelvin wave along x -direction, we have,

$$\langle \sigma^X \otimes \mathbf{1} \rangle = (1, 0, 0); \quad (133)$$

For the plane Kelvin wave along y -direction, we have,

$$\langle \sigma^Y \otimes \mathbf{1} \rangle = (0, 1, 0). \quad (134)$$

Thirdly, we discuss 3D plane Kelvin waves with fixed wavelength.

For a 3D plane σ_z -Kelvin waves with fixed wavelength described by $\mathbf{Z}(\mathbf{x}) = \begin{pmatrix} z_A(\mathbf{x}) \\ z_B(\mathbf{x}) \end{pmatrix} = r_0 e^{\pm i \mathbf{k} \cdot \mathbf{x}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, the tensor state is represented by

$$[\mathbf{Z}^X] = [\mathbf{Z}^Y] = [\mathbf{Z}^Z] = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad (135)$$

or

$$[\mathbf{Z}^X] = [\mathbf{Z}^Y] = [\mathbf{Z}^Z] = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad (136)$$

of which the tensor state is also denoted by

$$\langle \sigma^X \otimes \mathbf{1} \rangle = \mathbf{n}_\sigma^X = \mathbf{n}_\sigma^Y = \mathbf{n}_\sigma^Z = (0, 0, \pm 1). \quad (137)$$

Another 3D plane Kelvin waves with fixed wavelength is described by the following tensor state:

$$\begin{aligned} [\mathbf{Z}^X] &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \\ [\mathbf{Z}^Y] &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ i \\ i \end{pmatrix}, \\ [\mathbf{Z}^Z] &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}. \end{aligned} \quad (138)$$

Along x -direction, the function of the plane Kelvin wave becomes

$$z(x) = \begin{pmatrix} z_A(x) \\ z_B(x) \end{pmatrix} = \frac{1}{\sqrt{2}} r_0 (e^{i\mathbf{k} \cdot \mathbf{x}} + e^{-i\mathbf{k} \cdot \mathbf{x}}) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \sqrt{2} r_0 \cos(\mathbf{k} \cdot \mathbf{x}) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (139)$$

along y -direction, the function of the plane Kelvin wave becomes

$$z(y) = \begin{pmatrix} z_A(y) \\ z_B(y) \end{pmatrix} = \frac{1}{\sqrt{2}} r_0 (e^{i\mathbf{k} \cdot \mathbf{y}} + ie^{-i\mathbf{k} \cdot \mathbf{y}}) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (140)$$

along z -direction, the function of the plane Kelvin wave becomes

$$z(z) = \begin{pmatrix} z_A(z) \\ z_B(z) \end{pmatrix} = r_0 e^{i\mathbf{k} \cdot \mathbf{z}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (141)$$

For the tensor state along x -direction, we have,

$$\langle \sigma^X \otimes \mathbf{1} \rangle = \mathbf{n}_\sigma^X = (1, 0, 0); \quad (142)$$

For the tensor state along y -direction, we have,

$$\langle \sigma^Y \otimes \mathbf{1} \rangle = \mathbf{n}_\sigma^Y = (0, 1, 0); \quad (143)$$

For the tensor state along z -direction, we have,

$$\langle \sigma^Z \otimes \mathbf{1} \rangle = \mathbf{n}_\sigma^Z = (0, 0, 1). \quad (144)$$

4.3.4. Generation operator

In this part, we define generator operators for Kelvin waves of different tensor states.

Firstly, we generate the four degenerate states along a given direction $[\mathbf{k}, A]$, $[-\mathbf{k}, A]$, $[\mathbf{k}, B]$, $[-\mathbf{k}, B]$ by four operators

$$\begin{aligned} [\mathbf{k}, A] &= \hat{U}(\mathbf{k}, A)[0], & [-\mathbf{k}, A] &= \hat{U}(-\mathbf{k}, A)[0], \\ [\mathbf{k}, B] &= \hat{U}(\mathbf{k}, B)[0], & [-\mathbf{k}, B] &= \hat{U}(-\mathbf{k}, B)[0] \end{aligned} \quad (145)$$

with

$$\begin{aligned} \hat{U}(\mathbf{k}, A) &= e^{i \int [\phi_{\mathbf{k}, A}(x) \cdot \hat{K}] dx} \cdot \hat{F}(r_0), & \hat{U}(-\mathbf{k}, A) &= e^{i \int [\phi_{-\mathbf{k}, A}(x) \cdot \hat{K}] dx} \cdot \hat{F}(r_0), \\ \hat{U}(\mathbf{k}, B) &= e^{i \int [\phi_{\mathbf{k}, B}(x) \cdot \hat{K}] dx} \cdot \hat{F}(r_0), & \hat{U}(-\mathbf{k}, B) &= e^{i \int [\phi_{-\mathbf{k}, B}(x) \cdot \hat{K}] dx} \cdot \hat{F}(r_0), \end{aligned} \quad (146)$$

where $\hat{F}(r_0)$ is an expanding operator by shifting radius from 0 to r_0 on the membrane, $\hat{K} = -i \frac{d}{d\phi}$, $\phi(x) = \pm kx$, and x is coordinate along $\mathbf{x}$ direction. Here, $[0]$ denotes a vortex-membrane with $r_0 = 0$.

Based on the basis $\begin{pmatrix} [\mathbf{k}, A] \\ [\mathbf{k}, B] \\ [-\mathbf{k}, A] \\ [-\mathbf{k}, B] \end{pmatrix}$, the generator operator is simplified into

$$\hat{U}(\mathbf{k}, \Gamma^I) = e^{i \int \Gamma^I [\phi_{\mathbf{k}}(x) \cdot \hat{K}] dx} \cdot \hat{F}(r_0) \quad (147)$$

with $\mathbf{\Gamma}^I = (\mathbf{n}_\sigma^I \sigma^I) \otimes (\mathbf{n}_\tau \tau + \mathbf{1}\tau_0)$. Then we get

$$\begin{pmatrix} [\mathbf{k}, A] \\ [\mathbf{k}, B] \\ [-\mathbf{k}, A] \\ [-\mathbf{k}, B] \end{pmatrix} = \hat{U}(\mathbf{k}, \mathbf{\Gamma}^I) \begin{pmatrix} [0] \\ [0] \\ [0] \\ [0] \end{pmatrix}. \quad (148)$$

For different Kelvin waves with fixed wavelength, the generation operators can be changed from a basis $\begin{pmatrix} [\mathbf{k}, A] \\ [\mathbf{k}, B] \\ [-\mathbf{k}, A] \\ [-\mathbf{k}, B] \end{pmatrix}$ to another $\begin{pmatrix} [\mathbf{k}, A]' \\ [\mathbf{k}, B]' \\ [-\mathbf{k}, A]' \\ [-\mathbf{k}, B]' \end{pmatrix}$ by an $SU(2) \otimes SU(2)$ rotation operator

$$\hat{S} = \hat{S} \otimes \hat{V}, \quad (149)$$

$\hat{S} = \prod_I \hat{S}^I$ with $\hat{S}^I$ the $SU(2)$ operation on helical degrees of freedom

$$\hat{S}^I \begin{pmatrix} [\mathbf{k}] \\ [-\mathbf{k}] \end{pmatrix} = \begin{pmatrix} [\mathbf{k}]' \\ [-\mathbf{k}]' \end{pmatrix} \quad (150)$$

and $\hat{V} = \prod_I \hat{V}^I$ with $\hat{V}^I$ the $SU(2)$ operation on vortex degrees of freedom

$$\hat{V}^I \begin{pmatrix} [A] \\ [B] \end{pmatrix} = \begin{pmatrix} [A]' \\ [B]' \end{pmatrix}. \quad (151)$$

In this paper, we consider a trivial $\hat{V}$ operation, $\hat{V} \equiv 1$ and focus on the effect from $\hat{S}^I$, i.e.,

$$\hat{S} = \hat{S} \otimes \mathbf{1} = \prod_I \hat{S}^I. \quad (152)$$

By doing $\hat{S}^I$ operation, we have

$$\sigma^I \otimes (\mathbf{n}_\tau \tau + \mathbf{1}\tau_0) \rightarrow (\sigma^I)' \otimes (\mathbf{n}_\tau \tau + \mathbf{1}\tau_0) = \hat{S}[\sigma^I \otimes (\mathbf{n}_\tau \tau + \mathbf{1}\tau_0)](\hat{S}^I)^{-1} \quad (153)$$

or

$$\Gamma^I = \sigma^I \otimes (\mathbf{n}_\tau \tau + \mathbf{1}\tau_0) \rightarrow (\Gamma^I)' = \hat{S}\Gamma^I\hat{S}^{-1} = (\sigma^I)' \otimes (\mathbf{n}_\tau \tau + \mathbf{1}\tau_0), \quad (154)$$

By doing $SU(2) \otimes SU(2)$ rotation operation $\hat{S}$, the generation operator rotates as

$$\hat{U}(\mathbf{k}, \mathbf{\Gamma}^I) \rightarrow [\hat{U}(\mathbf{k}, \mathbf{\Gamma}^I)]' = \hat{S}\hat{U}(\mathbf{k}, \mathbf{\Gamma}^I)\hat{S}^{-1} = \hat{U}(\mathbf{k}, (\mathbf{\Gamma}^I)'). \quad (155)$$

As a result, after $SU(2) \otimes SU(2)$ rotation operation, a tensor state described by $\mathbf{n}_\sigma^I$ changes to another described by $(\mathbf{n}_\sigma^I)'$. Thus, a new basis becomes

$$\begin{pmatrix} [\mathbf{k}, A] \\ [\mathbf{k}, B] \\ [-\mathbf{k}, A] \\ [-\mathbf{k}, B] \end{pmatrix} \rightarrow \begin{pmatrix} [\mathbf{k}, A]' \\ [\mathbf{k}, B]' \\ [-\mathbf{k}, A]' \\ [-\mathbf{k}, B]' \end{pmatrix} = [\hat{U}^I(\mathbf{k}, \mathbf{\Gamma}^I)]' \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \hat{S}\hat{U}^I(\mathbf{k}, \mathbf{\Gamma}^I)\hat{S}^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \hat{\mathcal{S}} \begin{pmatrix} |\mathbf{k}, A\rangle \\ |\mathbf{k}, B\rangle \\ |-\mathbf{k}, A\rangle \\ |-\mathbf{k}, B\rangle \end{pmatrix}. \quad (156)$$

For example, by using the $SU(2) \otimes SU(2)$ rotation operation $\hat{\mathcal{S}}$, we can change a 3D σ_z -Kelvin waves to another standing Kelvin waves by the rotating generation operators

$$\hat{S}^X = e^{i\frac{\pi}{2}\sigma_y \otimes \mathbf{1}}, \quad \hat{S}^Y = e^{-i\frac{\pi}{2}\sigma_x \otimes \mathbf{1}}, \quad \hat{S}^Z = 1. \quad (157)$$

4.3.5. Generalized spatial translation symmetry

We discuss the spatial translation symmetry of plane Kelvin waves with fixed wavelength.

Different plane Kelvin waves of different tensor states $\langle \hat{\mathbf{T}}^I \rangle = \mathbf{\Gamma}^I$ have different translation symmetries. We define translation operator $\hat{\mathcal{T}}(\Delta \mathbf{x})$ as

$$\hat{\mathcal{T}}(\Delta \mathbf{x}) \mathbf{Z}(\mathbf{x}) = \mathbf{Z}'(\mathbf{x} + \Delta \mathbf{x}). \quad (158)$$

For the case of

$$\mathcal{T}(2a) \mathbf{Z}(\mathbf{x}) = \mathbf{Z}(\mathbf{x}), \quad (159)$$

the system has translation symmetry.

The generalized translation symmetry of the plane Kelvin waves is characterized by three translation operators,

$$\hat{\mathcal{T}}_x(2a) = e^{i2a(\hat{k}^X \cdot \mathbf{\Gamma}^X)}, \quad \hat{\mathcal{T}}_y(2a) = e^{i2a(\hat{k}^Y \cdot \mathbf{\Gamma}^Y)}, \quad \hat{\mathcal{T}}_z(2a) = e^{i2a(\hat{k}^Z \cdot \mathbf{\Gamma}^Z)}, \quad (160)$$

where $\mathbf{\Gamma}^I$ is a four-by-four matrix,

$$\mathbf{\Gamma}^I = (\mathbf{n}_\sigma^I \sigma^I) \otimes (\mathbf{n}_\tau \tau + \mathbf{1} \tau_0) \quad (161)$$

$\hat{k}^X = -i \frac{d}{dx}$ (or $\hat{k}^Y = -i \frac{d}{dy}$, $\hat{k}^Z = -i \frac{d}{dz}$) is wave vector operator. For the plane Kelvin waves described by tensor state $\mathbf{n}_\sigma^I$, we have

$$\mathbf{\Gamma}^I \mathbf{Z}(\mathbf{x}) = \pm \mathbf{I} \cdot \mathbf{Z}(\mathbf{x}), \quad (162)$$

where

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

It looks like all plane Kelvin waves with different tensor states $\mathbf{\Gamma}^I$ have the same translation symmetries as

$$\hat{\mathcal{T}}(2a) \mathbf{Z}(\mathbf{x}) = \mathbf{Z}(\mathbf{x}). \quad (163)$$

However, in addition to the translation symmetry, there exists generalized translation symmetry by doing a translation operation $\mathcal{T}(\Delta x) = e^{i\Delta x(\hat{k}^I \cdot \mathbf{\Gamma}^I)}$

$$\mathbf{Z}(\mathbf{x}) \rightarrow \mathbf{Z}'(\mathbf{x}) = \mathcal{T}(\Delta x)\mathbf{Z}(\mathbf{x}) = e^{\pm i \cdot \mathbf{I} \cdot k \cdot \Delta x} \mathbf{Z}(\mathbf{x}). \quad (164)$$

Thus, an arbitrary continuous spatial translation operation is a combination of a discrete spatial translation operation

$$|\mathbf{Z}(\mathbf{x}, t)\rangle \rightarrow \mathcal{T}(\Delta \mathbf{x})\mathbf{Z}(\mathbf{x}) = \mathcal{T}(|\Delta \mathbf{x}| = 2a)\mathbf{Z}(\mathbf{x}) = \mathbf{Z}(\mathbf{x}) \quad (165)$$

and a global gauge transformation operation

$$\mathbf{Z}(\mathbf{x}) \rightarrow \mathbf{Z}(\mathbf{x})e^{i \cdot \mathbf{I} \cdot \Delta \phi}, \quad (166)$$

where $\Delta \phi = ik \cdot [2a[(\Delta x) \bmod 2a]]$.

4.3.6. Biot–Savart equation in tensor representation

Then we derive the Biot–Savart equation in tensor representation.

For perturbative Kelvin waves, the Biot–Savart equation $i \frac{dz_{A/B}(\mathbf{x}, t)}{dt} = \frac{\delta \hat{H}(z_{A/B}(\mathbf{x}, t))}{\delta z_{A/B}^*}$ turns into the pseudo-Schrödinger equation

$$i\hbar_{\text{eff}} \frac{d\mathbf{Z}(\mathbf{x}, t)}{dt} = \hat{\mathbf{H}} \cdot \mathbf{Z}(\mathbf{x}, t), \quad (167)$$

where

$$\mathbf{Z}(\mathbf{x}, t) = \begin{pmatrix} \alpha_{1,1}^I(\mathbf{x}, t) \\ \alpha_{1,-1}^I(\mathbf{x}, t) \\ \alpha_{-1,1}^I(\mathbf{x}, t) \\ \alpha_{-1,-1}^I(\mathbf{x}, t) \end{pmatrix}$$

denotes perturbative Kelvin waves around the four degenerate Kelvin states and $\hbar_{\text{eff}}$ is the effective Planck constant in pseudo-quantum mechanics. Based on the basis $\begin{pmatrix} [\mathbf{k}] \\ [-\mathbf{k}] \end{pmatrix} \otimes \begin{pmatrix} [\mathbf{A}] \\ [\mathbf{B}] \end{pmatrix}$ ($k > 0$), we obtain the total Hamiltonian as

$$\hat{\mathbf{H}} = \hat{T} + \hat{V} = \sum_I \left[\frac{(\hat{p}_{\text{Lamb}}^I(\sigma^I \otimes \mathbf{1}))^2}{2m_{\text{pseudo}}} \right] + \frac{\hbar_{\text{eff}}\omega^*}{2}(\mathbf{1} \otimes (\tau_x - \mathbf{1})). \quad (168)$$

Thus, in tensor representation, the Biot–Savart equation is changed into

$$\begin{aligned} i\hbar_{\text{eff}} \frac{d\mathbf{Z}(\mathbf{x}, t)}{dt} &= \left\{ \sum_I \left[\frac{(\hat{p}_{\text{Lamb}}^I(\sigma^I \otimes \mathbf{1}))^2}{2m_{\text{pseudo}}} \right] + \frac{\hbar_{\text{eff}}\omega^*}{2}(\mathbf{1} \otimes (\tau_x - \mathbf{1})) \right\} \cdot \mathbf{Z}(\mathbf{x}, t) \\ &= \left\{ \sum_I \left[\frac{(\hat{p}_{\text{Lamb}}^I)^2}{2m_{\text{pseudo}}} \right] + \frac{\hbar_{\text{eff}}\omega^*}{2}(\mathbf{1} \otimes (\tau_x - \mathbf{1})) \right\} \cdot \mathbf{Z}(\mathbf{x}, t) \end{aligned} \quad (169)$$

where $(\hat{p}_{\text{Lamb}}^I(\sigma^I \otimes \mathbf{1}))^2 = (\hat{p}_{\text{Lamb}}^I)^2$ ($p_{\text{Lamb}}^I > 0$).

4.3.7. Zeros and zero-density

In the previous section, we have defined linking number and linking-number density to characterize the deformed two vortex-membranes. However, the winding number and winding-number density are not well defined for standing waves. To locally characterize different Kelvin waves, we introduce the zeros between two projected entangled vortex-membranes. Thus, it is the zero number and zero-number density that characterize the deformation of Kelvin waves in tensor representation.

(a) Projection

A d-dimensional vortex-membrane in d+2D space $\{\mathbf{x}, \xi(\mathbf{x}), \eta(\mathbf{x})\}$ can be described by the two-component function $\begin{pmatrix} \xi(\mathbf{x}) \\ \eta(\mathbf{x}) \end{pmatrix}$. We then introduce the concept of *projection*: a projection of a vortex-membrane along a given direction θ on $\{\xi, \eta\}$ space. In mathematics, the projection is defined by

$$\hat{P}_\theta \begin{pmatrix} \xi(\mathbf{x}) \\ \eta(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} \xi_\theta(\mathbf{x}) \\ [\eta_\theta(\mathbf{x})]_0 \end{pmatrix}, \quad (170)$$

where $\xi_\theta(\mathbf{x})$ is variable and $[\eta_\theta(\mathbf{x})]_0$ is constant. In the following parts, we use $\hat{P}$ to denote the projection operators. Because the projection direction out of vortex-membrane is characterized by an angle θ in $\{\xi, \eta\}$ space, we have

$$\begin{pmatrix} \xi_\theta \\ \eta_\theta \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \quad (171)$$

where θ is angle, i.e., $\theta \bmod 2\pi = 0$. So the projected vortex-membrane is described by the function

$$\xi_\theta(\mathbf{x}) = \xi(\mathbf{x}) \cos \theta + \eta(\mathbf{x}) \sin \theta. \quad (172)$$

(b) Zeros between two projected entangled vortex-lines

We then define the projection between two entangled vortex-lines $\{\xi_{A/B}(x, t), \eta_{A/B}(x, t)\}$ along a given direction θ in 3D space by

$$\hat{P}_\theta \begin{pmatrix} \xi_{A/B}(x, t) \\ \eta_{A/B}(x, t) \end{pmatrix} = \begin{pmatrix} \xi_{A/B, \theta}(x, t) \\ [\eta_{A/B, \theta}(x, t)]_0 \end{pmatrix}, \quad (173)$$

where $\xi_{A/B, \theta}(x, t) = \xi_{A/B}(x, t) \cos \theta - \eta_{A/B}(x, t) \sin \theta$ is variable and $[\eta_{A/B, \theta}(x, t)]_0 = \xi_{A/B}(x, t) \sin \theta - \eta_{A/B}(x, t) \cos \theta$ is constant. So the projected vortex-line is described by the function $\xi_{A/B, \theta}(x, t)$. For two projected vortex-lines described by $\xi_{A, \theta}(x, t)$ and $\xi_{B, \theta}(x, t)$, a zero is solution of the equation

$$\hat{P}_\theta[z_A(x, t)] \equiv \xi_{A, \theta}(x, t) = \hat{P}_\theta[z_B(x, t)] \equiv \xi_{B, \theta}(x, t). \quad (174)$$

We call the equation to be zero-equation and its solutions to be zero-solution (see the below discussion).

For vortex-lines described by plane wave $\mathbf{Z}(x, t) = \begin{pmatrix} r_0 e^{-i\omega \cdot t + ik \cdot x} \\ 0 \end{pmatrix}$, there exist periodic zeros. From the zero-equation $\xi_{0,A,\theta}(x, t) = \xi_{0,B,\theta}(x, t)$ or $\cos(kx - \omega t - \theta) = 0$, we get the zero-solutions to be

$$x(t) = a \cdot n + \frac{a}{\pi} \omega t + \frac{a}{\pi} \left(\theta + \frac{\pi}{2} \right), \quad (175)$$

where n is an integer number and $\theta = -\frac{\pi}{2}$. From the projection, we have a 1D crystal of zeros, of which the zero density ρ_{zero} is

$$\rho_{\text{zero}} = \frac{k}{\pi} = \frac{1}{a}. \quad (176)$$

Figure 8 shows zeros between two projected vortex-lines. In addition, for this case with $z_B = 0$, the zero-number density is twice as the linking-number density between two entangled vortex-membranes that is also equal to the winding-number density for vortex-membrane-A, i.e.,

$$\rho_{\text{zero}} \equiv 2\rho_{\text{linking}} \equiv 2\rho_{\text{wind}}. \quad (177)$$

The zeros from the zero-solution do not change for a plane Kelvin wave with different tensor states. For a plane Kelvin wave with clockwise winding characterized by $[\mathbf{k}]$ and another plane Kelvin wave with counterclockwise winding characterized by $[-\mathbf{k}]$, the functions for plane Kelvin waves are $r_0 e^{-i\omega \cdot t + ik \cdot x}$ and $r_0 e^{-i\omega \cdot t - ik \cdot x}$, respectively. For both cases with fixed time $t = 0$, up to a constant phase factor, we have the same zero-equation $\cos(kx) = 0$ and the same zero-solution $x(t) = a \cdot n$, respectively. For different plane Kelvin waves with different helical degrees of freedom, we have $[\mathbf{Z}] = \alpha[\mathbf{k}] + \beta[-\mathbf{k}]$ with $|\alpha|^2 + |\beta|^2 = 1$. As a result, up to a constant phase θ_0 , a plane Kelvin wave with the same wavelength but different tensor orders has the same zero-equation

$$\cos(kx - \theta_0) = 0 \quad (178)$$

and the same zero-solution

$$x(t) = a \cdot n + \frac{a}{\pi} \left(\theta_0 + \frac{\pi}{2} \right), \quad (179)$$

respectively.

Next, we define the zeros from the projection of 3D vortex-membranes in 5D space. A 3D vortex-membrane in 5D space $\{\mathbf{x}, \xi(\mathbf{x}), \eta(\mathbf{x})\}$ ($\mathbf{x} = (x, y, z)$) can be described by the two-component function $\begin{pmatrix} \xi(\mathbf{x}) \\ \eta(\mathbf{x}) \end{pmatrix}$. Along $\mathbf{e}$ -direction, for the helical vortex-membrane described by $\mathbf{z}(x, t) = r_0 e^{-i\omega \cdot t + ik \cdot \mathbf{x}}$, there exist periodic zeros between them. From the zero-equation $\xi_{0,A,\theta}(\mathbf{x}, t) = \xi_{0,B,\theta}(\mathbf{x}, t)$ or $\cos(\mathbf{k} \cdot \mathbf{x} - \omega t + \phi_0) = 0$, we get the zero-solutions to be

$$\mathbf{x}(t) = \left(a \cdot n + \frac{a}{\pi} \omega t \right) \mathbf{e}, \quad (180)$$

where n is an integer number and $\theta = -\frac{\pi}{2}$. So there exist the vector of zero density $\boldsymbol{\rho}_{\text{zero}} = (\rho_{x,\text{zero}}, \rho_{y,\text{zero}}, \rho_{z,\text{zero}})$ along three different directions ($\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$).

(c) *Zero-density operator*

We define zero-density operator to characterize two vortex-membranes.

According to pseudo-quantum mechanics, for each plane Kelvin wave $e^{-i\omega \cdot t + ik \cdot x}$, the linking-number density is given by $\rho_{\text{zero}} = 2\rho_{\text{linking}} = \frac{k}{\pi}$. Thus, the corresponding zero-density operator is given by

$$\rho_{\text{zero}} \rightarrow \hat{\rho}_{\text{zero}} = \frac{\hat{k}}{\pi} = -i \frac{1}{\pi} \frac{d}{dx}. \quad (181)$$

In tensor representation, we can classify different types of zeros. For two entangled 1D vortex-line, the zero-number along σ_z -internal direction is given by

$$\begin{aligned} N_{\text{zero}}^{\sigma_z} &= \int \tilde{\mathbf{Z}}^*(x, t) \left[(\sigma^z \otimes \mathbf{1}) \cdot \left(-i \frac{1}{\pi r_0^2} \frac{d}{dx} \right) \right] \tilde{\mathbf{Z}}(x, t) dx \\ &= \int z_{\uparrow, A}^*(x, t) \left(-i \frac{1}{\pi r_0^2} \frac{d}{dx} \right) z_{\uparrow, A}(x, t) dx \\ &\quad + \int z_{\uparrow, B}^*(x, t) \left(-i \frac{1}{\pi r_0^2} \frac{d}{dx} \right) z_{\uparrow, B}(x, t) dx \\ &\quad + \int z_{\downarrow, A}^*(x, t) \left(-i \frac{1}{\pi r_0^2} \frac{d}{dx} \right) z_{\downarrow, A}(x, t) dx \\ &\quad + \int z_{\downarrow, B}^*(x, t) \left(-i \frac{1}{\pi r_0^2} \frac{d}{dx} \right) z_{\downarrow, B}(x, t) dx, \end{aligned} \quad (182)$$

where

$$\tilde{\mathbf{Z}}(x, t) = \begin{pmatrix} z_{\uparrow A}(x, t) \\ z_{\uparrow B}(x, t) \\ z_{\downarrow A}(x, t) \\ z_{\downarrow B}(x, t) \end{pmatrix}.$$

The zero-number density is given by

$$\frac{1}{r_0^2} \tilde{\mathbf{Z}}^*(x, t) [(\sigma^z \otimes \mathbf{1}) \cdot \hat{\rho}_{\text{zero}}] \tilde{\mathbf{Z}}(x, t). \quad (183)$$

Using similar approach, we define the zero-density for standing wave along σ_x -internal direction

$$N_{\text{zero}}^{\sigma_x} = \frac{1}{r_0^2} \int \tilde{\mathbf{Z}}^*(x, t) [(\sigma^x \otimes \mathbf{1}) \cdot \hat{\rho}_{\text{zero}}] \tilde{\mathbf{Z}}(x, t) dx. \quad (184)$$

The corresponding zero-number density along σ_x -internal direction is given by

$$\rho_{\text{zero}}^{\sigma_x} = \frac{1}{r_0^2} \tilde{\mathbf{Z}}^*(x, t) [(\sigma^x \otimes \mathbf{1}) \cdot \hat{\rho}_{\text{zero}}] \tilde{\mathbf{Z}}(x, t). \quad (185)$$

In general, the zero density and zero-density operator are given by

$$\rho_{\text{zero}}^{\sigma} = \frac{1}{r_0^2} \tilde{\mathbf{Z}}^*(x, t) [(\sigma \otimes \mathbf{1}) \cdot \hat{\rho}_{\text{zero}}] \tilde{\mathbf{Z}}(x, t), \quad (186)$$

We also define the tensor of zero densities $\rho_{\text{zero}}^{I,\sigma}(x^I, t)$ and its operators $\hat{\rho}_{\text{zero}}^{I,\sigma}$ for 3D vortex-membranes, i.e.,

$$\rho_{\text{zero}}^{I,\sigma}(x^I, t) = \tilde{\mathbf{Z}}^*(x^I, t) \left[(\boldsymbol{\sigma} \otimes \mathbf{1}) \cdot \left(-i \frac{1}{\pi r_0^2} \frac{d}{dx^I} \right) \right] \tilde{\mathbf{Z}}(x^I, t), \quad (187)$$

and

$$\hat{\rho}_{\text{zero}}^{I,\sigma} = (\boldsymbol{\sigma} \otimes \mathbf{1}) \cdot \left(-\frac{i}{\pi r_0^2} \frac{d}{dx^I} \right), \quad (188)$$

respectively.

Finally, we could use the zero-density tensor $\rho_{\text{zero}}^{I,\sigma}(x^I, t)$ to describe the time-evolution of two nonuniform entangled vortex-membranes: for two entangled vortex-membranes with given initial state $\tilde{\mathbf{Z}}(\mathbf{x})$, the time-dependent zero-density tensor $\rho_{\text{zero}}^{I,\sigma}(x^I, t)$ is obtained as $\rho_{\text{zero}}^{I,\sigma}(x^I, t) = \tilde{\mathbf{Z}}^*(x^I, t) [(\boldsymbol{\sigma} \otimes \mathbf{1}) \cdot (-i \frac{1}{\pi r_0^2} \frac{d}{dx^I})] \tilde{\mathbf{Z}}(x^I, t)$, where

$$\tilde{\mathbf{Z}}(\mathbf{x}) = \sum_k c_k e^{i\mathbf{k} \cdot \mathbf{x}} \rightarrow \tilde{\mathbf{Z}}(\mathbf{x}, t) = \sum_k c_k e^{i\mathbf{k} \cdot \mathbf{x}} e^{-i\omega \cdot t}. \quad (189)$$

From the point of view of information (see the discussion below), the Sommerfeld quantization condition for four-component particle is the condition of the zero number, i.e.,

$$\oint p_{\text{Lamb}} \cdot dx = h_{\text{eff}} \cdot 2N_{\text{zero}}^{\sigma_z}.$$

5. Emergent Quantum Mechanics in Knot Physics

The issue of emergent quantum mechanics is about the dynamic evolution for an object of a zero (a knot) with volume-changing, another types of local deformation of two entangled vortex-membranes. Here, we ask another question — *how to characterize the evolution of the zeros with volume-changing?*

In last section, we found that the deformed knot-crystal is described by the pseudo-quantum mechanics, of which two entangled vortex-membranes with fixed volume are mapped onto a particle with two internal degrees of freedom in tensor representation. It is obvious that the physical degrees of freedom for a vortex-membrane (the size $V_P \rightarrow \infty$) are seriously reduced. Because a vortex-membrane is an extended object, in general, each vortex-piece (vortex filament) with infinitesimal size ($\Delta V_P \rightarrow 0$) has a degree of freedom. So the pseudo-quantum mechanics cannot characterize the zero fluctuations from volume-changing on entangled vortex-membranes. In the following parts, from point of view of information, we consider the smallest volume-changing on entangled vortex-membranes as an object with a single zero ($\Delta V_{\text{knot}} \equiv a^3 r_0^2$). We call such object a knot. To characterize knot dynamics, the Biot-Savart mechanics becomes emergent quantum mechanics, the true quantum mechanics.

5.1. Basic theory for knots

Before developing emergent quantum mechanics for knots, we review the concept of “information”.

A generalized definition of the concept “information” is “*An answer to a specific question*”. In general, the answer is related to data that represents values attributed to parameters. In other words, information can be interpreted as a data pattern or an ordered sequence of symbols from an alphabet. Take DNA as an example. The sequence of nucleotides is a pattern that influences the formation and development of an organism. In addition, information processing consists of an input–output function that maps any input sequence into an output sequence. Therefore, information reduces uncertainty. The uncertainty of an event is measured by its probability of occurrence. The more uncertain an event, the more information is required to resolve uncertainty of that event.

To represent information, information unit must be defined. The bit is a typical unit of information. Example: information in one “fair” coin flip: $\log_2(\frac{2}{1}) = 1$ bit, and in two “fair” coin flips is $\log_2(\frac{4}{1}) = 2$ bits. In summary, the information is defined by the following equation

$$\text{Information} = \text{a data pattern of information unit.}$$

The use of knotted strings for record-keeping purposes in China dates back to ancient times. In 2003, J. D. Bekenstein claimed that a growing trend in physics was to define the physical world as being made up of information itself (and thus information is defined in this way).

By using projected representation, we show that different entanglement patterns between two vortex-membranes correspond to different patterns of zeros that are obtained from different zero-solutions, i.e.,

$$\text{Information} = \text{a pattern of zeros between projected vortex-membranes.} \quad (190)$$

Fig. 8. (Color online) (a) Perfect entangled vortex-membranes; (b) The information of perfect entangled vortex-membranes by projection $\dots 1111111\dots$; (c) The information from projected entangled vortex-membranes with a hole-like knot that is described by the information, $\dots 1110111\dots$; (d) The information from projected entangled vortex-membranes with a particle-like knot that is described by the information, $\dots 1112111\dots$.

In physics, the information is characterized by the distribution of zeros. Here, the information unit is a zero between projected vortex-membranes,

$$\text{Information unit} = \text{A zero between two projected vortex-membranes.} \quad (191)$$

For the case of an extra zero, we have a knot; for the case of missing zero, we have an anti-knot. See the illustration in Fig. 8. The colors of the dots in Fig. 8 denote L and R types of zeros.

5.1.1. Definition of a unified knot

Each zero corresponds to a piece of vortex-membranes. We call the object to be a *knot*, a piece of plane Kelvin wave with fixed wavelength. As a result, the system (plane Kelvin wave with fixed wavelength) can be regarded as a crystal of knots. I call the system *knot-crystal*. In this section, we will study knot and show its properties on a knot-crystal.

We give the mathematic definition of knot.

From the point of view of information, a knot is an information unit with fixed geometric properties. For example, for 1D knot, people divide the knot-crystal into N identical pieces, each of which is just a knot. As a result, some properties of knot can be obtained by scaling the properties of a knot-crystal; From the point of view of a classical field theory, a knot is elementary topological defect of a knot-crystal that is always anti-phase changing along arbitrary direction $\mathbf{e}$, $\mathbf{Z}(\mathbf{x}, t) \rightarrow \mathbf{Z}_{\text{knot}}(\mathbf{x}, t)$, i.e.,

$$\frac{\mathbf{Z}_{\text{knot}}(\mathbf{x} \rightarrow -\infty, t)}{\mathbf{Z}_{\text{knot}}(\mathbf{x} \rightarrow \infty, t)} = -\frac{\mathbf{Z}(\mathbf{x} \rightarrow -\infty, t)}{\mathbf{Z}(\mathbf{x} \rightarrow \infty, t)}. \quad (192)$$

Let us show the fact that an information unit must be an anti-phase changing on Kelvin waves.

Based on the projected vortex-membranes, we define a knot by a monotonic function $F_\theta(x) = \xi_{A,\theta}(x) - \xi_{B,\theta}(x)$ with

$$\text{sgn}[F_\theta(x \rightarrow -\infty) \cdot F_\theta(x \rightarrow \infty)] = -1, \quad (193)$$

where x denotes the position along the given direction $\mathbf{e}$. A knot can be regarded as domain wall of π -phase shifting. When there exists a knot, the periodic boundary condition of Kelvin waves along arbitrary direction is changed into anti-periodic boundary condition. So the sign-switching character can be labeled by winding number w_{1D} . The winding number w_{1D} for a knot along given direction is $\pm \frac{1}{2}$, i.e.,

$$\begin{aligned} w_{1D}^I &= \frac{-i}{2\pi(r_0)^2} \int_{-\infty}^{\infty} \mathbf{Z}_{\text{knot}}^*(x^I) d\mathbf{Z}_{\text{knot}}(x^I) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\phi(x^I)}{dx^I} dx^I \\ &= \frac{1}{2\pi} \int d\phi(x^I) = \pm \frac{1}{2}, \end{aligned} \quad (194)$$

where x is the coordinate along the given direction. Because the zero number is twice of winding number, each zero $\Delta N_{\text{zero}} = \pm 1$ corresponds to a knot with half winding number, $\Delta w_{1D} = \pm \frac{1}{2}$. A knot is an object with a fractional winding number. As a result, the quantum number for a 3D knot can be denoted by

$$N_{\text{knot}} = 8\Delta w_{1D}^X \cdot \Delta w_{1D}^Y \cdot \Delta w_{1D}^Z = \pm 1. \quad (195)$$

On the other hand, from the topological character of a knot, there must exist a point, at which $\xi_A(x)$ is equal to $\xi_B(x)$. The position of the point x is determined by a local solution of the *zero-equation*

$$F_\theta(x) = 0 \quad (196)$$

or

$$\xi_{A,\theta}(x) = \xi_{B,\theta}(x). \quad (197)$$

So each knot corresponds to a zero between two vortex-membranes along the given direction.

In summary, for a knot, there are three properties:

- (1) Conservation condition: The conservation condition indicates that as an information unit, the existence of a zero for a knot is independent on the directions of projection angle θ . When one gets a zero-solution along a given direction θ , it will never split or disappear whatever changing the projection direction, $\theta \rightarrow \theta'$. The type of zeros is switched $L \leftrightarrow R$ by rotating the projection angle $\theta \rightarrow \theta + \pi$;
- (2) Isotropic property: According to isotropic condition, there exists a single knot solution along arbitrary direction $\mathbf{e}$. During rotation operation $\mathbf{e} \rightarrow \mathbf{e}' = R\mathbf{e}$, the zero of the knot still exists but its spin degrees of freedom are rotating synchronously, $\sigma \rightarrow \sigma' = R\sigma$;
- (3) Z2 topological property: Each knot is π -phase changing along the arbitrary direction $\mathbf{e}$. When there exists a knot, for all Kelvin waves, the periodic boundary condition along the given direction is changed into anti-periodic boundary condition.

For a knot, there are three conserved physical quantities: the energy of a (static) knot that is proportional to the volume of the knot on vortex-membrane $V_P = a^3$, where a is a characteristic winding length along different directions; the (Lamb impulse) angular momentum (the effective Planck constant $\hbar_{\text{knot}}$) that is proportional to the volume of the knot in the 5D fluid $V_{\text{knot}} = a^3 \cdot r_0^2$, where r_0 is a characteristic winding radius; the winding number $|\Delta w_{1D}| = \frac{1}{2}$ or the zero number $|\Delta N_{\text{zero}}| = 1$. According to the geometric character of the three conserved physical quantities, the shape of the knot will never be changed. However, the knot can split and the three physical quantities are conserved for all knot-pieces. Quantum mechanics is a mechanics to determine the distribution of knot-pieces.

5.1.2. Internal degrees of freedom for a knot

For a knot, there exists a zero satisfying conservation condition, isotropic condition and Z2 topological property. A knot (a zero) has four degrees of freedom: two spin degrees of freedom $\uparrow$ or $\downarrow$ from the helicity degrees of freedom, the other two vortex degrees of freedom from the vortex degrees of freedom that characterize the vortex-membranes, A or B. The basis to define the microscopic structure of a knot is given by

$$\begin{aligned} |\uparrow, A\rangle &= \begin{pmatrix} z_{\text{knot}, \uparrow, A}(\mathbf{x}, t) \\ 0 \\ 0 \\ 0 \end{pmatrix}, & |\uparrow, B\rangle &= \begin{pmatrix} 0 \\ z_{\text{knot}, \uparrow, B}(\mathbf{x}, t) \\ 0 \\ 0 \end{pmatrix}, \\ |\downarrow, A\rangle &= \begin{pmatrix} 0 \\ 0 \\ z_{\text{knot}, \downarrow, A}(\mathbf{x}, t) \\ 0 \end{pmatrix}, & |\downarrow, B\rangle &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ z_{\text{knot}, \downarrow, B}(\mathbf{x}, t) \end{pmatrix}, \end{aligned} \quad (198)$$

where A/B denotes the degrees of freedom of vortex-membranes; $\uparrow/\downarrow$ denotes the degrees of freedom of (pseudo-) spin. $z_{\text{knot}, \uparrow, A/B}(\mathbf{x}, t)$ is the knot-function for a knot on vortex-membrane A/B with clockwise winding; $z_{\text{knot}, \downarrow, A/B}(\mathbf{x}, t)$ is the knot-function for a knot on vortex-membrane A/B with counterclockwise winding.

Firstly, we discuss spin degrees of freedom. We have shown that there exist two types of winding directions for a knot

$$\mathbf{Z}_{\text{knot}}(\mathbf{x}, t) = \begin{pmatrix} z_{\text{knot}, \uparrow}(\mathbf{x}, t) \\ z_{\text{knot}, \downarrow}(\mathbf{x}, t) \end{pmatrix} \rightarrow \begin{pmatrix} |\uparrow\rangle \\ |\downarrow\rangle \end{pmatrix}. \quad (199)$$

So from the Sommerfeld quantization condition, $\oint p_{\text{Lamb}} \cdot dx = h_{\text{eff}} \cdot w_{1D}$, a knot with half winding number is an object with spin-1/2. A knot with the eigenstates of σ_z described by $|\uparrow\rangle$ or $|\downarrow\rangle$ is really a piece of traveling Kelvin waves; a knot with the eigenstates of σ_x described by $|\rightarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$ or $|\leftarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)$ is a piece of standing Kelvin waves; a knot with the eigenstates of σ_y described by $\frac{1}{\sqrt{2}}(|\uparrow\rangle + i|\downarrow\rangle)$ or $\frac{1}{\sqrt{2}}(|\uparrow\rangle - i|\downarrow\rangle)$ is a piece of another type of standing Kelvin waves. An important property of spin degrees of freedom is time-reversal symmetry. In physics, under time-reversal operation, a clockwise winding knot (spin- $\uparrow$ particle) turns into a counterclockwise winding knot (spin- $\downarrow$ particle). In addition, in the next sections, we will show that owing to the existence of spin-orbital coupling (SOC) in the Dirac model, the degrees of freedom $\uparrow/\downarrow$ indeed play the role of spin degrees of freedom as those in quantum mechanics.

Next, we discuss the vortex degrees of freedom of knots. The vortex-degrees of freedom of knots are denoted by

$$\mathbf{Z}_{\text{knot}}(\mathbf{x}, t) = \begin{pmatrix} z_{\text{knot},A}(\mathbf{x}, t) \\ z_{\text{knot},B}(\mathbf{x}, t) \end{pmatrix} \rightarrow \begin{pmatrix} |A\rangle \\ |B\rangle \end{pmatrix}. \quad (200)$$

For knots, we identify $z_{\text{knot},A}(\mathbf{x}, t)$ the function of A-vortex-membrane to be the wave-function of $|A\rangle$, $z_{\text{knot},B}(\mathbf{x}, t)$ the function of an B-vortex-membrane to be the wave-function of $|B\rangle$.

Let us give the functions of four knot states:

(1) The up-spin knot on vortex-membrane-A, $|\uparrow, A\rangle$ is defined by

$$z_{\text{knot},\uparrow,A}(x, t) = r_{\uparrow,A}(x) \exp[i\phi_{\text{knot},\uparrow,A}(x, t)], \quad (201)$$

where $\phi_{\text{knot},\uparrow,A}(x)$ is a monotonic function changing π -phase. For unified knot, we have

$$r_{\uparrow,A}(x) \rightarrow \begin{cases} 0, & x \in (-\infty, x_0] \\ r_0, & x \in (x_0, x_0 + a] \\ 0, & x \in (x_0 + a, \infty) \end{cases} \quad (202)$$

and

$$\phi_{\text{knot},\uparrow,A}(x) \rightarrow \phi_{\text{unified-knot},\uparrow,A}(x) = \begin{cases} -\phi_0 - \frac{\pi}{2}, & x \in (-\infty, x_0] \\ -\phi_0 - \frac{\pi}{2} + k_0(x - x_0), & x \in (x_0, x_0 + a] \\ -\phi_0 + \frac{\pi}{2}, & x \in (x_0 + a, \infty) \end{cases}, \quad (203)$$

where $x = \mathbf{x} \cdot \mathbf{e}$ is the coordination on the axis along a given direction $\mathbf{e}$ and ϕ_0 is an arbitrary constant angle, $k_0 = \frac{\pi}{a}$. The center of the unified knot is at $x = x_0 + a$. For fragmentized knot, we have

$$\phi_{\text{knot},\uparrow,A}(x) \rightarrow \phi_{\text{fragmentized},\uparrow,A}(x). \quad (204)$$

We will give the definition of $\phi_{\text{fragmentized},\uparrow,A}(x)$ in the following section. For different types of knots, there exists a linear relationship between $\phi(x)$ and x , i.e., $\frac{d\phi(x)}{dx} \propto k_0$ in the phase-changing region of $x_0 < x \leq x_0 + a$. The linear character comes from the minimization of the volume $\text{volume}(P)$ of vortex-membrane. See the illustration of unified knots in Fig. 9. In the limit of $x \rightarrow -\infty$, we have $\phi(x) = \phi_0 - \frac{\pi}{2}$; in the limit of $x \rightarrow \infty$, we have $\phi(x) = \phi_0 + \frac{\pi}{2}$. So we obtain

$$\phi_{\text{knot},\uparrow,A}(x \rightarrow \infty) - \phi_{\text{knot},\uparrow,A}(x \rightarrow -\infty) = \pi. \quad (205)$$

Along the arbitrary direction $\mathbf{e}$, we have a sign-switching as

$$\frac{z_{\text{knot},\uparrow,A}(x \rightarrow -\infty, t)}{z_{\text{knot},\uparrow,A}(x \rightarrow \infty, t)} = -1; \quad (206)$$

Fig. 9. (Color online) (a) The relation between phase and coordination of a unified knot; (b) The relation between radius and coordination of a unified knot; (c) The illustration of a unified knot; (d) In complex field representation, a unified knot is an anti-phase domain wall (the red arrows denote the directions of phase angles).

(2) The down-spin knot on vortex-membrane-A, $|\downarrow, A\rangle$ is defined by

$$z_{\text{knot}, \downarrow, A}(x, t) = r_{\downarrow, A} \exp[i\phi_{\text{knot}, \downarrow, A}(x, t)], \quad (207)$$

where

$$r_{\downarrow, A}(x) \rightarrow \begin{cases} 0, & x \in (-\infty, x_0] \\ r_0, & x \in (x_0, x_0 + a] \\ 0, & x \in (x_0 + a, \infty) \end{cases} \quad (208)$$

and

$$\phi_{\text{knot}, \downarrow, A}(x) \rightarrow \phi_{\text{unified-knot}, \downarrow, A}(x) = \begin{cases} \phi_0 + \frac{\pi}{2}, & x \in (-\infty, x_0] \\ \phi_0 + \frac{\pi}{2} - k_0(x - x_0), & x \in (x_0, x_0 + a] \\ \phi_0 - \frac{\pi}{2}, & x \in (x_0 + a, \infty) \end{cases}. \quad (209)$$

Thus, we have

$$\phi_{\text{knot}, \downarrow, A}(x \rightarrow \infty) - \phi_{\text{knot}, \downarrow, A}(x \rightarrow -\infty) = -\pi. \quad (210)$$

Along arbitrary direction $\mathbf{e}$, we have a sign-switching as

$$\frac{z_{\text{knot}, \downarrow, A}(x \rightarrow -\infty, t)}{z_{\text{knot}, \downarrow, A}(x \rightarrow \infty, t)} = -1; \quad (211)$$

(3) The up-spin knot on vortex-membrane-B, $|\uparrow, B\rangle$ is defined by

$$z_{\text{knot}, \uparrow, B}(x, t) = r_{\uparrow, B} \exp[i\phi_{\text{knot}, \uparrow, B}(x, t)], \quad (212)$$

where

$$r_{\uparrow, B}(x) \rightarrow \begin{cases} 0, & x \in (-\infty, x_0] \\ r_0, & x \in (x_0, x_0 + a] \\ 0, & x \in (x_0 + a, \infty) \end{cases} \quad (213)$$

and

$$\phi_{\text{knot}, \uparrow, B}(x) \rightarrow \phi_{\text{unified-knot}, \uparrow, B}(x) = \begin{cases} -\phi_0 - \frac{\pi}{2}, & x \in (-\infty, x_0] \\ -\phi_0 - \frac{\pi}{2} + k_0(x - x_0), & x \in (x_0, x_0 + a] \\ -\phi_0 + \frac{\pi}{2}, & x \in (x_0 + a, \infty) \end{cases}. \quad (214)$$

Thus, we have

$$\phi_{\text{knot}, \uparrow, B}(x \rightarrow \infty) - \phi_{\text{knot}, \uparrow, B}(x \rightarrow -\infty) = \pi. \quad (215)$$

Along arbitrary direction $\mathbf{e}$, we have a sign-switching as

$$\frac{z_{\text{knot}, \uparrow, B}(x \rightarrow -\infty, t)}{z_{\text{knot}, \uparrow, B}(x \rightarrow \infty, t)} = -1; \quad (216)$$

(4) The down-spin knot on vortex-membrane-B, $|\downarrow, B\rangle$ is defined by

$$z_{\text{knot}, \downarrow, B}(x, t) = r_{\downarrow, B} \exp[i\phi_{\text{knot}, \downarrow, B}(x, t)], \quad (217)$$

where

$$r_{\downarrow, B}(x) \rightarrow \begin{cases} 0, & x \in (-\infty, x_0] \\ r_0, & x \in (x_0, x_0 + a] \\ 0, & x \in (x_0 + a, \infty) \end{cases} \quad (218)$$

and

$$\phi_{\text{knot}, \downarrow, B}(x) \rightarrow \phi_{\text{unified-knot}, \downarrow, B}(x) = \begin{cases} \phi_0 + \frac{\pi}{2}, & x \in (-\infty, x_0] \\ \phi_0 + \frac{\pi}{2} - k_0(x - x_0), & x \in (x_0, x_0 + a] \\ \phi_0 - \frac{\pi}{2}, & x \in (x_0 + a, \infty) \end{cases}. \quad (219)$$

Thus, we have

$$\phi_{\text{knot}, \downarrow, B}(x \rightarrow \infty) - \phi_{\text{knot}, \downarrow, B}(x \rightarrow -\infty) = -\pi. \quad (220)$$

Along arbitrary direction $\mathbf{e}$, we have a sign-switching as

$$\frac{z_{\text{knot}, \downarrow, B}(x \rightarrow -\infty, t)}{z_{\text{knot}, \downarrow, B}(x \rightarrow \infty, t)} = -1. \quad (221)$$

The knots with different internal degrees of freedom are described by different superposed states of the basis

$$\begin{pmatrix} |\uparrow, A\rangle \\ |\uparrow, B\rangle \\ |\downarrow, A\rangle \\ |\downarrow, B\rangle \end{pmatrix} = \begin{pmatrix} |\uparrow\rangle \\ |\downarrow\rangle \end{pmatrix} \otimes \begin{pmatrix} |A\rangle \\ |B\rangle \end{pmatrix}. \quad (222)$$

Along arbitrary direction $\mathbf{e}$, due to sign-switching character for each of element states $|\uparrow, A\rangle, |\downarrow, A\rangle, |\uparrow, B\rangle, |\downarrow, B\rangle$, all superposed states $|\psi\rangle = \alpha|\uparrow, A\rangle + \beta|\downarrow, A\rangle + \gamma|\uparrow, B\rangle + \kappa|\downarrow, B\rangle$ change sign as

$$\frac{\psi(\mathbf{x} \rightarrow -\infty, t)}{\psi(\mathbf{x} \rightarrow \infty, t)} = -1.$$

On the other hand, for each of element states $|\uparrow, A\rangle, |\downarrow, A\rangle, |\uparrow, B\rangle, |\downarrow, B\rangle$ along a given projection direction $\mathbf{e}$ and a fixed θ , we get a zero-equation

$$\phi(x) = \phi_0 \mp \frac{\pi}{2} \pm k_0(x - x_0) = \pm \frac{\pi}{2} + \theta. \quad (223)$$

Due to the existence of single zero between two projected vortex-membranes for each state $|\uparrow, A\rangle, |\downarrow, A\rangle, |\uparrow, B\rangle, |\downarrow, B\rangle$, all superposed states $|\psi\rangle = \alpha|\uparrow, A\rangle + \beta|\downarrow, A\rangle + \gamma|\uparrow, B\rangle + \kappa|\downarrow, B\rangle$ have a single zero. According to the conservation of the information unit, there exists a constraint

$$\int \psi^\dagger(\mathbf{x}, t) \cdot \psi(\mathbf{x}, t) dx \equiv 1 \quad (224)$$

that gives a normalization condition as

$$|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\kappa|^2 = 1. \quad (225)$$

The normalization implies a normalized volume condition,

$$\int z^*(\mathbf{x}, t) z(\mathbf{x}, t) dx = l \cdot [|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\kappa|^2] = V_{\text{knot}} \quad (226)$$

or

$$|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\kappa|^2 = r_0^2. \quad (227)$$

The total volume of the knot (volume-changing of two entangled vortex-membranes) $V_{\text{knot}} = V_P r_0^2$ is always fixed.

In addition to the four degrees of freedom, we have another degree of freedom — chiral degrees of freedom by projecting a knot. Under projection with given projected angle θ , each projected knot state corresponds to two possible zeros — one zero with $\text{sgn}[\Upsilon] = 1$ is denoted by $|L\rangle$; the other with $\text{sgn}[\Upsilon] = -1$ is denoted by $|R\rangle$ ($\Upsilon = \eta_{A,\theta}(\mathbf{x}, t) - \eta_{B,\theta}(\mathbf{x}, t)$).

5.1.3. Tensor states for a knot

For a 3D knot with four internal degrees of freedom,

$$\begin{pmatrix} |\uparrow, A, x_0\rangle \\ |\uparrow, B, x_0\rangle \\ |\downarrow, A, x_0\rangle \\ |\downarrow, B, x_0\rangle \end{pmatrix} \quad (228)$$

We introduce a tensor representation to define a knot,

$$|\mathbf{Z}_{\text{knot}}\rangle = \prod_I |\mathbf{Z}_{\text{knot}}^I\rangle \quad (229)$$

with

$$|\mathbf{Z}_{\text{knot}}^I\rangle = \begin{pmatrix} \beta_{1,1}^I \\ \beta_{1,-1}^I \\ \beta_{-1,1}^I \\ \beta_{-1,-1}^I \end{pmatrix}. \quad (230)$$

For $\beta_{i,j}^I$, I denotes the spatial indices X, Y, Z ; $i = 1, -1$ denotes helical degrees of freedom; $j = 1, -1$ denotes the vortex-degrees of freedom. According to the constraint condition, we have

$$\langle \mathbf{Z}_{\text{knot}}^I | \mathbf{Z}_{\text{knot}}^I \rangle = 1, \quad (231)$$

So the function for a knot is given by

$$\mathbf{Z}_{\text{knot}}^I(\mathbf{x} - \mathbf{x}_0, t) = \prod_{I, \mathbf{x}_0} \mathbf{Z}_{\text{knot}}^I(\mathbf{x} - \mathbf{x}_0, t) \quad (232)$$

The tensor states for a knot can also be classified by operator representation. We define the $3 \times 3 \times 3$ tensor operators $\hat{\Gamma}_{\text{knot},i,j}^I = \sigma_{\text{knot},i}^I \otimes (\tau_{\text{knot},j}^I, \mathbf{1})$ ($I = X, Y, Z$; for different spatial directions; $i = x, y, z$ for spin degrees of freedom; $j = x, y, z$ for the vortex-degrees of freedom). Different tensor states are described by

$$\langle \hat{\Gamma}_{\text{knot}}^I \rangle = \langle \mathbf{Z}_{\text{knot}} | \hat{\Gamma}^I | \mathbf{Z}_{\text{knot}} \rangle = \mathbf{\Gamma}_{\text{knot}}^I = \mathbf{n}_{\text{knot},\sigma}^I \otimes (\mathbf{n}_\tau \tau + \mathbf{1}\tau_0). \quad (233)$$

One type of 3D knot is a σ_z -type knot that is characterized by

$$\langle \sigma_{\text{knot}}^X \otimes \mathbf{1} \rangle = \langle \sigma_{\text{knot}}^Y \otimes \mathbf{1} \rangle = \langle \sigma_{\text{knot}}^Z \otimes \mathbf{1} \rangle = (0, 0, \pm 1). \quad (234)$$

Another 3D knot is called SOC knot that is described as

$$\langle \sigma_{\text{knot}}^X \otimes \mathbf{1} \rangle = (1, 0, 0), \quad \langle \sigma_{\text{knot}}^Y \otimes \mathbf{1} \rangle = (0, 1, 0), \quad \langle \sigma_{\text{knot}}^Z \otimes \mathbf{1} \rangle = (0, 0, 1). \quad (235)$$

5.1.4. *Tensor-network state for a knot-crystal*

Periodic entangled vortex-membranes form knot-crystal. In pseudo-quantum mechanics, a knot-crystal corresponds to two entangled vortex-membranes described by a special pure state of Kelvin waves with fixed wavelength. In pseudo-quantum mechanics, we consider flat vortex-membrane as the ground state, i.e.,

$$\mathbf{Z}(\mathbf{x}, t) = \begin{pmatrix} z_A(\mathbf{x}, t) \\ z_B(\mathbf{x}, t) \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow |\text{vacuum}\rangle \quad (236)$$

A Kelvin wave becomes excited state around $|\text{vacuum}\rangle$; in emergent quantum mechanics, we consider knot-crystal as a ground state,

$$\mathbf{Z}_{\text{knot-crystal}}(\mathbf{x}, t) = \begin{pmatrix} z_A(\mathbf{x}, t) \\ z_B(\mathbf{x}, t) \end{pmatrix} = \begin{pmatrix} r_A \\ r_B \end{pmatrix} e^{i\mathbf{k}_0 \cdot \mathbf{x} - i\omega_0 t + i\omega^* t/2} \rightarrow |\text{vacuum}\rangle, \quad (237)$$

the knots become topological excitations on it.

Because a knot-crystal is a plane Kelvin wave with fixed wave vector k_0 , we can use the tensor representation to characterize knot-crystals, $\mathbf{\Gamma}_{\text{knot-crystal}}^I = (\mathbf{n}_\sigma^I \sigma^I) \otimes \mathbf{1}$ and define corresponding generation operator $\hat{U}_{\text{knot-crystal}}(\mathbf{k}_0, \mathbf{\Gamma}_{\text{knot-crystal}}^I)$ for it.

Firstly, we define generate operator by knot-crystal

$$\hat{U}_{\text{knot-crystal}}(\mathbf{k}, \mathbf{\Gamma}_{\text{knot-crystal}}^I) = e^{i \int \mathbf{\Gamma}^I [\phi_{\mathbf{k}_0}(x) \cdot \hat{K}] dx} \cdot \hat{F}(r_0), \quad (238)$$

where $\hat{F}(r_0)$ is an expanding operator by shifting radius from 0 to r_0 on the membrane, $\hat{K} = -i \frac{d}{d\phi}$, $\phi(x) = \pm kx$, and x is coordinate along $\mathbf{x}$ direction. Here, $|0\rangle$ denotes a vortex-membrane with $r_0 = 0$. The tensor state of the knot-crystal is determined by $\mathbf{\Gamma}_{\text{knot-crystal}}^I = (\mathbf{n}_\sigma^I \sigma^I) \otimes \mathbf{1}$.

For example, a particular knot-crystal is called SOC knot-crystal that is described by a 3D plane Kelvin waves with fixed wavelength $\mathbf{Z}_{\text{knot-crystal}}(\mathbf{x})$. Its tensor state is

$$\langle \sigma^X \otimes \mathbf{1} \rangle = \mathbf{n}_\sigma^X = (1, 0, 0), \quad \langle \sigma^Y \otimes \mathbf{1} \rangle = \mathbf{n}_\sigma^Y = (0, 1, 0), \quad \langle \sigma^Z \otimes \mathbf{1} \rangle = \mathbf{n}_\sigma^Z = (0, 0, 1). \quad (239)$$

On the one hand, a knot is a piece of knot-crystal; On the other hand, a knot-crystal can be regarded as a composite system with multi-knot, each of which is described by same tensor state. So the knot-crystal is a tensor-network state $\mathbf{\Gamma}_{\text{knot-crystal}}^{\text{Tensor-network}}$ for multi-knot. The tensor-network state comes from the product of the basis of all knots

$$\prod_{\mathbf{x}_0} (|L, \mathbf{x}_0\rangle) \otimes (|R, \mathbf{x}_0 - a \cdot \mathbf{e}_x\rangle) \otimes (|R, \mathbf{x}_0 - a \cdot \mathbf{e}_y\rangle) \otimes (|R, \mathbf{x}_0 - a \cdot \mathbf{e}_z\rangle) \dots \quad (240)$$

The summation of $\mathbf{x}_0$ is obtained by summing the zeros between projected vortex-membranes. The generalized translation symmetry for a knot-crystal becomes

$$(|R, \mathbf{x}_0 - a \cdot \mathbf{e}\rangle) = \mathcal{T}(\Delta \mathbf{x} = -a \cdot \mathbf{e}) \iota^- \cdot (|L, \mathbf{x}_0\rangle), \quad (241)$$

where $\iota^\pm$ is an operator denoting switching of the chiral (sublattice) degrees of freedom as

$$\iota^+ = (R \rightarrow L), \quad \iota^- = (L \rightarrow R). \quad (242)$$

There exist topological defects of the tensor-network states that are just extra zeros (the knots). The chiral degrees of freedom become the sublattice degrees of freedom for knots on knot-crystal! In particular, the emergent Lorentz invariance for knots comes from the two-sublattice structure.

By considering the local degrees of freedom, the state space (the Hilbert space) becomes suddenly enlarged from 2 in pseudo-quantum mechanics to $4^{N_{\text{knot}}}$ in emergent quantum mechanics. The chiral degrees of freedom become the freedom of “lattice” and label the position of the sublattices.

5.1.5. Knot operator and knot number operator

To characterize knots (the elementary volume-changing of two entangled vortex-membranes), we introduce its operator-representation.

For 1D unified knot in Dirac representation, we define it by

$$\mathbf{Z}_{\text{knot}}(x, t) = \hat{U}_{\text{knot}} \mathbf{Z}_{\text{knot-crystal}}(x, t) = \hat{U}_{\text{knot}} \hat{U}_{\text{knot-crystal}} \begin{pmatrix} [0] \\ [0] \\ [0] \\ [0] \end{pmatrix}, \quad (243)$$

where $\hat{U}_{\text{knot}}$ is a generation operator for a knot that is an eigenstate of spin operator σ_z ,

$$\hat{U}_{\text{knot}} = e^{i \int (\sigma_z \otimes \mathbf{1}) [\phi_{\text{knot}}(x) \cdot \hat{K}] dx} \cdot \hat{F}_{\text{knot}}(r_0), \quad (244)$$

where $\hat{F}_{\text{knot}}(r_0)$ is an expanding operator by shifting 0 to r_0 in the winding region of a knot (for example, $x \in (x_0, x_0 + a]$). $\hat{U}_{\text{knot-crystal}}$ denotes the generation operator for a 1D knot-crystal that is the plane Kelvin wave with fixed wavelength and becomes the same tensor state for a knot.

We define the generation operator for a 3D knot by doing a knot-operation

$$\hat{U}_{\text{knot}} = \hat{U}_{\text{knot}}^X \hat{U}_{\text{knot}}^Y \hat{U}_{\text{knot}}^Z, \quad (245)$$

where

$$\begin{aligned} \hat{U}_{\text{knot}}^X &= e^{i \int (\sigma^X \otimes \mathbf{1}) [\phi_{\text{knot}}(x) \cdot \hat{K}] dx} \cdot \hat{F}_{\text{knot}}(r_0), \\ \hat{U}_{\text{knot}}^Y &= e^{i \int (\sigma^Y \otimes \mathbf{1}) [\phi_{\text{knot}}(y) \cdot \hat{K}] dy} \cdot \hat{F}_{\text{knot}}(r_0), \\ \hat{U}_{\text{knot}}^Z &= e^{i \int (\sigma^Z \otimes \mathbf{1}) [\phi_{\text{knot}}(z) \cdot \hat{K}] dz} \cdot \hat{F}_{\text{knot}}(r_0). \end{aligned} \quad (246)$$

For example, 3D SOC knot is defined by

$$\begin{aligned}\hat{U}_{\text{knot}}^X &= e^{\int i(\sigma_x \otimes \mathbf{1})[\phi_{\text{knot}}(x) \cdot \hat{K}]dx} \cdot \hat{F}_{\text{knot}}(r_0), \\ \hat{U}_{\text{knot}}^Y &= e^{\int i(\sigma_y \otimes \mathbf{1})[\phi_{\text{knot}}(y) \cdot \hat{K}]dy} \cdot \hat{F}_{\text{knot}}(r_0), \\ \hat{U}_{\text{knot}}^Z &= e^{\int i(\sigma_z \otimes \mathbf{1})[\phi_{\text{knot}}(z) \cdot \hat{K}]dz} \cdot \hat{F}_{\text{knot}}(r_0);\end{aligned}\quad (247)$$

3D σ_z -knot is defined by

$$\begin{aligned}\hat{U}_{\text{knot}}^X &= e^{\int i(\sigma_z \otimes \mathbf{1})[\phi_{\text{knot}}(x) \cdot \hat{K}]dx} \cdot \hat{F}_{\text{knot}}(r_0), \\ \hat{U}_{\text{knot}}^Y &= e^{\int i(\sigma_z \otimes \mathbf{1})[\phi_{\text{knot}}(y) \cdot \hat{K}]dy} \cdot \hat{F}_{\text{knot}}(r_0), \\ \hat{U}_{\text{knot}}^Z &= e^{\int i(\sigma_z \otimes \mathbf{1})[\phi_{\text{knot}}(z) \cdot \hat{K}]dz} \cdot \hat{F}_{\text{knot}}(r_0).\end{aligned}\quad (248)$$

In addition, the knot-number of a 3D knot is defined by

$$\begin{aligned}\langle \hat{K} \rangle &= \langle \hat{K}_x \hat{K}_y \hat{K}_z \rangle \\ &= \frac{1}{\pi^3 r_0^2} \left[\int \mathbf{Z}_{\text{knot}}^*(\mathbf{x}, t) \cdot \hat{K}_x \hat{K}_y \hat{K}_z \cdot \mathbf{Z}_{\text{knot}}(\mathbf{x}, t) \right] d\phi(x) d\phi(y) d\phi(z).\end{aligned}\quad (249)$$

5.2. Emergent quantum mechanics

In the above section, we show that a knot (the elementary volume-changing of two entangled vortex-membranes) is an elementary topological defect of knot-crystal and also fundamental information carrier. In this section, we will show that knots obey the emergent quantum mechanics.

5.2.1. Information scaling between knot and knot-crystal

From the point of view of pseudo-quantum mechanics, knot-crystal is a special entangled vortex-membranes that is described by a tensor state. The total degrees of freedom for a knot-crystal is 2. From the point of view of emergent quantum mechanics, a knot-crystal is periodic zeros between projected vortex-membranes that is described by tensor-network state for knots. The total degrees of freedom of a knot-crystal become $4^{N_{\text{knot}}}$. In particular, there exists a holographic principle: the pseudo-spin-tensor state for a knot on a knot-crystal is same to the tensor state for the knot-crystal in emergent quantum mechanics.

On the other hand, a knot corresponds to the changing of one zero on a knot-crystal. When a knot is generated on a knot-crystal, the boundary of all Kelvin waves changes — periodic boundary condition changes into anti-periodic boundary condition, vice versa. Because the Hilbert space is never changed during the

changing of the boundary condition, the Hilbert space of knots

$$\begin{pmatrix} |\psi_{\uparrow,A}\rangle \\ |\psi_{\uparrow,B}\rangle \\ |\psi_{\downarrow,A}\rangle \\ |\psi_{\downarrow,B}\rangle \end{pmatrix}$$

corresponds to the Hilbert space of Kelvin waves

$$\begin{pmatrix} [\psi_{\uparrow,A}] \\ [\psi_{\uparrow,B}] \\ [\psi_{\downarrow,A}] \\ [\psi_{\downarrow,B}] \end{pmatrix}$$

by perturbing the knot-crystal, i.e.,

$$\begin{pmatrix} |\psi_{\uparrow,A}\rangle \\ |\psi_{\uparrow,B}\rangle \\ |\psi_{\downarrow,A}\rangle \\ |\psi_{\downarrow,B}\rangle \end{pmatrix} \Longleftrightarrow \begin{pmatrix} [\psi_{\uparrow,A}] \\ [\psi_{\uparrow,B}] \\ [\psi_{\downarrow,A}] \\ [\psi_{\downarrow,B}] \end{pmatrix}. \quad (250)$$

According to the correspondence, several quantum quantities of knots (momentum and angular momentum) can be easily obtained by using the information scaling,

$$\hbar_{\text{knot-crystal}} \longrightarrow \hbar_{\text{knot}} = \frac{\hbar_{\text{eff}}}{N_{\text{knot}}}, \quad (251)$$

where $\hbar_{\text{knot}}$ is the Planck constant for knots in emergent quantum mechanics and $\hbar_{\text{eff}}$ is the Planck constant for plane Kelvin wave with fixed wavelength in pseudo-quantum mechanics. N_{knot} is the total knot number in the knot-crystal.

5.2.2. Fragmentized knot

According to the geometric character of the three conserved physical quantities, the shape of knot will never be changed. However, the knot can split and the three physical quantities are conserved for all knot-pieces. To characterize this property, we introduce the concept of “fragmentized knot”. Quantum mechanics is a mechanics to determine the distribution of knot-pieces.

We take 1D knot with fixed spin and vortex degrees of freedom as an example to show the concept of “fragmentized knot” ($\sigma \rightarrow \uparrow, \tau \rightarrow B$). Before introducing the fragmentized knot, we give the definition of unified knot that is described by

$$z_{\text{knot}}(\phi(x)) = r_{\text{knot}} \exp[i\phi_{\text{knot}}(x)], \quad (252)$$

Fig. 10. (Color online) (a) A knot. The inset illustrates the unified knot; (b) A fragmentized knot that is split into two pieces; (c) A fragmentized knot that is split three pieces; (d) A fragmentized knot that is split into infinite pieces. The blue spots denote the fragmentized knot with $N \rightarrow \infty$. For fragmentized knots with $N = 1, 2, 3, \infty$, there always exists a single knot solution and the knot number is 1.

where

$$r_{\text{knot}}(x) \rightarrow \begin{pmatrix} 0, & x \in (-\infty, x_0] \\ r_0, & x \in (x_0, x_0 + a] \\ 0, & x \in (x_0 + a, \infty) \end{pmatrix}. \quad (253)$$

x is coordinate along an arbitrary direction $\mathbf{e}$ and $\phi_{\text{knot}}(x)$ is the corresponding phase angle.

We split a unified knot into two pieces, each of them is a half-knot with $\frac{\pi}{2}$ phase-changing. As shown in Fig. 10, we split the knot into two pieces. The knot-function of two-piece fragmentized knot is $z(\phi_{\text{knot}}(x))$, where

$$r_{\text{knot}}(x) \rightarrow \begin{pmatrix} 0, & x \in (-\infty, x_0] \\ r_0, & x \in \left(x_0, x_0 + \frac{a}{2}\right] \\ 0, & x \in \left(x_0 + \frac{a}{2}, x_0 + x'_0\right] \\ r_0, & x \in \left(x_0 + x'_0, x_0 + x'_0 + \frac{a}{2}\right] \\ 0, & x \in \left(x_0 + x'_0 + \frac{a}{2}, \infty\right] \end{pmatrix} \quad (254)$$

and

$$\phi_{\text{knot}}(x) = \begin{pmatrix} -\frac{\pi}{2}, & x \in (-\infty, x_0] \\ -\frac{\pi}{2} + \frac{\pi}{a}(x - x_0), & x \in \left(x_0, x_0 + \frac{a}{2}\right] \\ 0, & x \in \left(x_0 + \frac{a}{2}, x_0 + x'_0\right] \\ \frac{\pi}{a}(x - x_0 - x'_0), & x \in \left(x_0 + x'_0, x_0 + x'_0 + \frac{a}{2}\right] \\ \frac{\pi}{2}, & x \in \left(x_0 + x'_0 + \frac{a}{2}, \infty\right] \end{pmatrix} \quad (255)$$

with a condition of $x'_0 > \frac{a}{2}$. It is obvious that for the fragmented knot, there exists only single zero-solution due to topological condition. And we have 50% probability to find a zero-solution in the region of $x_0 < x \leq x_0 + \frac{a}{2}$ and 50% probability to find a zero-solution in the region of $x_0 + x'_0 < x \leq x_0 + x'_0 + \frac{a}{2}$. We had defined a knot with half-winding to be

$$z_{N=1}(\phi(x), x_0) = \hat{U}(\phi_{\text{knot}}, x_0)z, \quad (256)$$

of which the knot is at x_0 . As a result, The knot-function of two-piece fragmented knot is defined by

$$[z(\phi)]_{\text{fragment}, N=2} = \hat{U}(\Delta\phi_{\frac{1}{2}\text{-knot}}, x_0) \cdot \hat{U}(\Delta\phi_{\frac{1}{2}\text{-knot}}, x_0 + x'_0)z_0, \quad (257)$$

of which the two half-knots are at x_0 and $x_0 + x'_0$, respectively.

Similarly, we may split a knot into N pieces, each of which is an identical knot-piece with $\frac{\pi}{N}$ phase-changing. The knot-function of N knot-pieces is

$$\begin{aligned} [z(\phi)]_{\text{fragment}, N} &= \hat{U}(\Delta\phi_{\frac{1}{N}\text{-knot}}, (x'_0)_1) \cdot \hat{U}(\Delta\phi_{\frac{1}{N}\text{-knot}}, (x'_0)_2) \dots \\ &\quad \cdot \hat{U}(\Delta\phi_{\frac{1}{N}\text{-knot}}, (x'_0)_N)z_0 \\ &= \prod_{i=1}^N \hat{U}(\Delta\phi_{\frac{1}{N}\text{-knot}}, (x'_0)_i)z_0, \end{aligned} \quad (258)$$

where N identical knot-pieces are at $(x'_0)_1, (x'_0)_2, \dots, (x'_0)_N$, respectively. For N knot-pieces, there also exists a zero and the knot-number is conserved. We have $\frac{1}{N}$ probability to find a zero in a knot-piece. By using similar argument, an arbitrary superposed states of knots can be fragmented. Figure 10(d) is an illustration of a fragmented knot.

The generation operator for a fragmented knot with uniform distribution of knot-pieces is defined as

$$\tilde{\mathbf{Z}}_{\text{fragment}} = \hat{U}_{\text{fragment}}|0\rangle, \quad (259)$$

where $\hat{U}_{\text{fragment}} = \hat{U}_{\text{fragment}}^X \hat{U}_{\text{fragment}}^Y \hat{U}_{\text{fragment}}^Z$ denotes a uniform distributed knot-pieces as

$$\begin{aligned}\hat{U}_{\text{fragment}}^X &= \prod_{i=1}^{N \rightarrow \infty} \hat{U}_{\text{fragment}}^X(\Delta\phi_{\frac{1}{N}\text{-knot}}(x_0)_i), \\ \hat{U}_{\text{fragment}}^Y &= \prod_{i=1}^{N \rightarrow \infty} \hat{U}_{\text{fragment}}^Y(\Delta\phi_{\frac{1}{N}\text{-knot}}(y_0)_i), \\ \hat{U}_{\text{fragment}}^Z &= \prod_{i=1}^{N \rightarrow \infty} \hat{U}_{\text{fragment}}^Z(\Delta\phi_{\frac{1}{N}\text{-knot}}(z_0)_i).\end{aligned}\quad (260)$$

with

$$\begin{aligned}\hat{U}_{\text{fragment}}^X(\Delta\phi_{\frac{1}{N}\text{-knot}}(x_0)_i) &= e^{\int i(\sigma_X \otimes \mathbf{1})[\Delta\phi_{\frac{1}{N}\text{-knot}}((x_0)_i) \cdot \hat{K}]dx} \cdot \hat{F}_{\frac{1}{N}\text{-knot}}(r_0, (x_0)_i), \\ \hat{U}_{\text{fragment}}^Y(\Delta\phi_{\frac{1}{N}\text{-knot}}(y_0)_i) &= e^{\int i(\sigma_Y \otimes \mathbf{1})[\Delta\phi_{\frac{1}{N}\text{-knot}}((y_0)_i) \cdot \hat{K}]dy} \cdot \hat{F}_{\frac{1}{N}\text{-knot}}(r_0, (y_0)_i), \\ \hat{U}_{\text{fragment}}^Z(\Delta\phi_{\frac{1}{N}\text{-knot}}(z_0)_i) &= e^{\int i(\sigma_Z \otimes \mathbf{1})[\Delta\phi_{\frac{1}{N}\text{-knot}}((z_0)_i) \cdot \hat{K}]dz} \cdot \hat{F}_{\frac{1}{N}\text{-knot}}(r_0, (z_0)_i),\end{aligned}\quad (261)$$

where $\hat{F}_{\frac{1}{N}\text{-knot}}(r_0, (x_0)_i)$ is an operator by shifting 0 to r_0 in the winding region of $x \in ((x_0)_i, (x_0)_i + \frac{a}{N}]$, $\hat{F}_{\frac{1}{N}\text{-knot}}(r_0, (y_0)_i)$ is an operator by shifting 0 to r_0 in the winding region of $y \in ((y_0)_i, (y_0)_i + \frac{a}{N}]$, $\hat{F}_{\frac{1}{N}\text{-knot}}(r_0, (z_0)_i)$ is an operator by shifting 0 to r_0 in the winding region of $z \in ((z_0)_i, (z_0)_i + \frac{a}{N}]$, and

$$\begin{aligned}\Delta\phi_{\frac{1}{N}\text{-knot}}((x_0)_i) &= \left\{ \begin{array}{ll} -\phi_0 - \frac{\pi}{2N}, & x \in (-\infty, x_0] \\ -\phi_0 - \frac{\pi}{2N} + k_0(x - (x_0)_i), & x \in \left((x_0)_i, (x_0)_i + \frac{a}{N}\right] \\ -\phi_0 + \frac{\pi}{2N}, & x \in \left((x_0)_i + \frac{a}{N}, \infty\right) \end{array} \right\}, \\ \Delta\phi_{\frac{1}{N}\text{-knot}}((y_0)_i) &= \left\{ \begin{array}{ll} -\phi_0 - \frac{\pi}{2N}, & y \in (-\infty, y_0] \\ -\phi_0 - \frac{\pi}{2N} + k_0(y - (y_0)_i), & y \in \left((y_0)_i, (y_0)_i + \frac{a}{N}\right] \\ -\phi_0 + \frac{\pi}{2N}, & y \in \left((y_0)_i + \frac{a}{N}, \infty\right) \end{array} \right\},\end{aligned}\quad (262)$$

$$\Delta\phi_{\frac{1}{N}\text{-knot}}((z_0)_i) = \left\{ \begin{array}{ll} -\phi_0 - \frac{\pi}{2N}, & z \in (-\infty, x_0] \\ -\phi_0 - \frac{\pi}{2N} + k_0(z - (z_0)_i), & z \in \left((z_0)_i, (z_0)_i + \frac{a}{N}\right] \\ -\phi_0 + \frac{\pi}{2N}, & z \in \left((z_0)_i + \frac{a}{N}, \infty\right) \end{array} \right\}, \quad (263)$$

where $k_0 = \frac{\pi}{a}$. The center of the knot-piece with $\frac{\pi}{N}$ phase-changing is at $(x_0, y_0, z_0)_i$. For this case, we assume the distribution of the knot-pieces is same as the zeros of the knot-crystal,

$$\begin{aligned}(x_0)_i &= a \cdot i, \quad i = 0, 1, 2, \dots N_x = \frac{L_x}{a} \\ (y_0)_i &= a \cdot i, \quad i = 0, 1, 2, \dots N_y = \frac{L_y}{a} \\ (z_0)_i &= a \cdot i, \quad i = 0, 1, 2, \dots N_z = \frac{L_z}{a}.\end{aligned}\tag{264}$$

An important issue is to obtain the spatial and temporal distribution of the knot-pieces that has the information of a knot. In the next part, we will show that the quantum states of a knot — the wave-functions describe the spatial and temporal distribution of the knot-pieces determined by the perturbative Kelvin waves on a knot-crystal.

5.2.3. Quantum states of fragmentized knots

Firstly, we consider a knot with uniform knot-pieces, of which the state is denoted by

$$\hat{U}_{\text{fragment}} |0\rangle \rightarrow |\psi\rangle = c_{\mathbf{k}=0}^\dagger |\text{vacuum}\rangle,\tag{265}$$

where $c_{\mathbf{k}=0}^\dagger$ means that the knot has the same wave vectors to those of its background. The density of knot-pieces is given by

$$\rho_{\text{knot}} = \frac{1}{V_P},\tag{266}$$

where V_P is volume of the system,

$$V_P = \begin{pmatrix} L_x, & d = 1 \\ L_x L_y, & d = 2 \\ L_x L_y L_z, & d = 3 \end{pmatrix}.\tag{267}$$

$L_x/L_y/L_z$ denotes the size of vortex-membrane along $X/Y/X$ direction. It is obvious that the function for a knot with $\mathbf{k} = 0$ is same as that of a knot-crystal by scaling the linking number along given direction.

We take 1D knot with fixed spin and vortex degrees of freedom (for example, $\sigma = \uparrow$, $\tau = B$) as an example to prove the knot density $\rho_{\text{knot}} = \frac{1}{L_x}$ for a uniform distribution of knot-pieces. For 1D knot, the knot-number is defined as

$$N_{\text{knot}} = \langle \hat{K} \rangle = \frac{1}{\pi |r_0|^2} \int z_{\text{fragment}}^* \hat{K} z_{\text{fragment}} d\phi(x),\tag{268}$$

where $\hat{K} = -i \frac{d}{d\phi}$. To calculate the knot-number, we introduce two types of integrals: one is *on-vortex-membrane integral* with a sorting of knot-positions in on vortex-membrane, the other is *out-vortex-membrane integral* with a sorting of knot-phases in extra space (ξ, η) .

We can label a knot-piece by its position $(x_0)_{i_x}$ on vortex-membrane space or label a knot-piece by its position $(x_0)_{i_\phi}$ in (ξ, η) space. Here, i_ϕ denotes a sorting of ordering of phase ϕ from small to bigger and i_x denotes a sorting of coordination x with a given order. Each i_ϕ corresponds to an i_x . As a result, there exists an *integral-transformation* by reorganizing all knot-pieces with different rules as

$$\begin{aligned}
 N_{\text{knot}} &= \frac{1}{|r_0|^2 \pi} \int z_{\text{fragment}}^* \hat{K} z_{\text{fragment}} d\phi(x) \\
 &= \frac{1}{|r_0|^2} \sum_{i_\phi=1}^{i_\phi=N} z_{\text{fragment},N}^* \hat{K} z_{\text{fragment},N} \\
 &= \frac{1}{|r_0|^2} \sum_{i_x=1}^{i_x=N} z_{\text{fragment},N}^* \hat{K} z_{\text{fragment},N} \\
 &\quad \cdot \sum_{i_y=1}^{i_y=N} z_{\text{fragment},N}^* \hat{K} z_{\text{fragment},N} \\
 &\quad \cdot \sum_{i_z=1}^{i_z=N} z_{\text{fragment},N}^* \hat{K} z_{\text{fragment},N} \\
 &= \frac{1}{|r_0|^2} \frac{1}{L_x} \int z_{\text{fragment}}^* \hat{K} z_{\text{fragment}} dx.
 \end{aligned} \tag{269}$$

By transforming an out-vortex-membrane integral to on-vortex-membrane integral, the knot-number is obtained as

$$N_{\text{knot}} = \langle \hat{K} \rangle = \frac{1}{\pi |r_0|^2} \int z_{\text{fragment}}^* \hat{K} z_{\text{fragment}} dx,$$

where

$$z_{\text{fragment}} = \hat{U}_{\text{fragment}} z_{\text{knot-crystal}}. \tag{270}$$

As a result, we get

$$\begin{aligned}
 N_{\text{knot}} &= \langle \hat{K} \rangle = \frac{1}{\pi |r_0|^2} \int z_{\text{knot-crystal}}^* \{ \hat{U}_{\text{fragment}}^{-1} \hat{K} \hat{U}_{\text{fragment}} \} z_{\text{knot-crystal}} dx \\
 &= \frac{1}{L_x} \int dx = 1.
 \end{aligned} \tag{271}$$

In the limit of $N \rightarrow \infty$, we indeed have a uniform distribution of the N identical knot-pieces on vortex-membrane,

$$\rho_{\text{knot}} = \frac{1}{V_P}. \tag{272}$$

Next, we consider a plane Kelvin wave with an extra fragmentized knot. Now, the system is described by

$$\mathbf{Z}_{\text{knot-crystal}}(\mathbf{x}, t) \rightarrow \mathbf{Z}'_{\text{knot-crystal}}(\mathbf{x}, t) = e^{i \cdot \mathbf{1} \cdot (\mathbf{k} \cdot \mathbf{x}) - i \omega \cdot t} \mathbf{Z}_{\text{knot-crystal}}(\mathbf{x}, t). \tag{273}$$

Here, to simplify, we take a fragmentized knot with fixed spin (for example, $\uparrow$) and vortex degrees of freedom (for example, B) as an example. Then we consider the effect of an extra knot,

$$z_{\text{fragment}} = \frac{1}{\sqrt{V_P}} \exp[i\Delta\phi(\mathbf{x}, t)], \quad (274)$$

where $\Delta\phi(\mathbf{x}, t) = \mathbf{k} \cdot \mathbf{x} - \omega \cdot t$. We may also denote the state by

$$|\psi_{\mathbf{k}}(\mathbf{x}, t)\rangle = \hat{U}_{\text{fragment}} |0\rangle = c_{\mathbf{k}}^\dagger |0\rangle. \quad (275)$$

The wave-function $|\psi_{\mathbf{k}}(\mathbf{x}, t)\rangle$ and the function $[\psi_{\mathbf{k}}(\mathbf{x}, t)]$ correspond to anti-periodic boundary condition and periodic boundary condition for the plane Kelvin waves. $\hat{U}_{\mathbf{k}}$ is the operator changing the boundary condition. So the number of zeros is changed by ± 1 . We say that $|\psi_{\mathbf{k}}(\mathbf{x}, t)\rangle$ becomes plane waves for knots.

From the superposition principle of Kelvin waves, an arbitrary quantum state for a knot is given by

$$|\mathbf{Z}_{\text{knot}}(\mathbf{x}, t)\rangle = \begin{pmatrix} z_{\uparrow, A}(\mathbf{x}, t) \\ z_{\uparrow, B}(\mathbf{x}, t) \\ z_{\downarrow, A}(\mathbf{x}, t) \\ z_{\downarrow, B}(\mathbf{x}, t) \end{pmatrix}, \quad (276)$$

where

$$z_{\sigma, \tau}(\mathbf{x}, t) = r_{\sigma, \tau}(\mathbf{x}, t) \cdot e^{i\phi_{\sigma, \tau}(\mathbf{x}, t)} = \sum_{\mathbf{k}} (r_{\sigma, \tau, \mathbf{k}}) \cdot e^{\pm i(\mathbf{k} \cdot \mathbf{x}) \pm i(\omega \cdot t)}, \quad (277)$$

where $r_{\sigma, \tau, \mathbf{k}}$ is the amplitude of given plane wave $\mathbf{k}$ and $\sigma = \uparrow/\downarrow$ denotes spin degrees of freedom, $\tau = A/B$ denotes vortex degrees of freedom. The results illustrate superposition principle for fragmentized knot. We may denote the equation by

$$|\psi(\mathbf{x}, t)\rangle = c^\dagger(\mathbf{x}, t) |\text{vacuum}\rangle.$$

To simplify, in this part, we discuss the properties of quantum states for knots by fixing the spin and vortex degrees of freedom, (for example, $\sigma \rightarrow \uparrow$, $\tau \rightarrow B$).

We point out that the function of a Kelvin wave with an extra fragmentized knot describes the distribution of the N identical knot-pieces and plays the role of the wave-function in quantum mechanics as

$$\frac{1}{\sqrt{V_P}} \frac{z(\mathbf{x}, t)}{r_0} = \sqrt{\rho_{\text{knot}}(\vec{x}, t)} e^{i\Delta\phi(\mathbf{x}, t)} \Longleftrightarrow \psi(\mathbf{x}, t), \quad (278)$$

and

$$\rho_{\text{knot}}(\mathbf{x}, t) \Longleftrightarrow n_{\text{knot}}(\mathbf{x}, t), \quad (279)$$

where the function of Kelvin wave with a fragmentized knot $\frac{z(\mathbf{x}, t)}{\sqrt{V_P} r_0}$ becomes the wave-function $\psi(\mathbf{x}, t)$ in emergent quantum mechanics, the angle $\Delta\phi(\mathbf{x}, t)$ becomes the quantum phase angle of wave-function, the knot density $\rho_{\text{knot}} = \langle \frac{\Delta\hat{K}}{\Delta V_P} \rangle$ becomes

the density of knot-pieces $n_{\text{knot}}(\mathbf{x})$. Thus, the density of knot-pieces in a given region is obtained as $\psi^*(\mathbf{x}, t)\psi(\mathbf{x}, t)$.

We then prove that the wave-function $\psi(\mathbf{x}, t) = \sqrt{\rho_{\text{knot}}(x, t)}e^{i\phi(\mathbf{x}, t)}$ is the function of Kelvin wave for knots (elementary particles). The knot density $\rho_{\text{knot}} = \langle \frac{\Delta \hat{K}}{\Delta V_P} \rangle$ is equal to

$$\begin{aligned}
 \rho_{\text{knot}} &= \frac{1}{r_0^2 \pi^3} z_{\text{fragment}, N}^* \hat{K} z_{\text{fragment}, N} \cdot d\phi(x) d\phi(y) d\phi(z) \\
 &= \frac{1}{r_0^2} \sum_{i_{\phi(x)}, i_{\phi(y)}, i_{\phi(z)}} z_{\text{fragment}, N}^*(t, (x_0)_{i_{\phi}}) \hat{K} z_{\text{fragment}, N}(t, (x_0)_{i_{\phi}}) \\
 &= \frac{1}{r_0^2} \sum_{i_x} z_{\text{fragment}, N}^*(t, (x_0)_{i_x}) \hat{K} z_{\text{fragment}, N}(t, (x_0)_{i_x}) \\
 &\quad \cdot \sum_{i_y} z_{\text{fragment}, N}^*(t, (x_0)_{i_y}) \hat{K} z_{\text{fragment}, N}(t, (x_0)_{i_y}) \\
 &\quad \cdot \sum_{i_z}^* z_{\text{fragment}, N}^*(t, (x_0)_{i_z}) \hat{K} z_{\text{fragment}, N}(t, (x_0)_{i_z}) \\
 &= \frac{1}{\Delta V_P \cdot r_0^2} z_{\text{fragment}, N}^*(\mathbf{x}, t) \hat{K} z_{\text{fragment}, N}(\mathbf{x}, t) dV_P \\
 &= \frac{1}{\Delta V_P} \psi^*(\mathbf{x}, t) \left(-i \frac{d}{d\phi} \right) \psi(\mathbf{x}, t) dV_P \\
 &= \psi^*(\mathbf{x}, t) \psi(\mathbf{x}, t). \tag{280}
 \end{aligned}$$

For this case, we have set the bare density for finding a knot to be zero $\rho_{\text{knot}}(\mathbf{x}, t) \equiv 0$. As a result, $\psi(\mathbf{x}, t)$ indeed plays the role of wave-function in quantum mechanics, $\rho_{\text{knot}}(x, t)$ is the knot density and $\phi(\mathbf{x}, t)$ is the phase (the angle in extra space). For a single knot, we have (volume) normalization condition $\int \rho_{\text{knot}}(\mathbf{x}, t) dV_P = 1$. In the following part, we will show the probability interpretation for wave-functions according to the dynamic projection with fast-clock effect.

Thus, the quantum state is invariant under a global gauge transformation, i.e.,

$$\psi(\mathbf{x}, t) \rightarrow \psi'(\mathbf{x}, t) = \psi(\mathbf{x}, t) e^{i\phi_0} = \sqrt{\rho_{\text{knot}}(\mathbf{x}, t)} e^{i\phi(\mathbf{x}, t)} e^{i\phi_0}, \tag{281}$$

where ϕ_0 is constant. Such invariant comes from a global rotation symmetry in (ξ, η) space.

5.2.4. Momentum operator and energy operator for knots

For a plane wave, $\psi(\mathbf{x}, t) = \frac{1}{\sqrt{V_P}} e^{-i\omega \cdot t + i\mathbf{k} \cdot \mathbf{x}}$ (to simplify, we have fixed the spin and vortex degrees of freedom, $\sigma \rightarrow \uparrow, \tau \rightarrow B$), the projected (Lamb impulse) energy of a knot is

$$E_{\text{knot}} = \hbar_{\text{knot}} \omega \tag{282}$$

and the projected (Lamb impulse) momentum of a knot is

$$\mathbf{p}_{\text{knot}} = \hbar_{\text{knot}} \mathbf{k}, \quad (283)$$

where the effective Planck constant $\hbar_{\text{knot}}$ is obtained as projected (Lamb impulse) angular momentum of a knot (the elementary volume-changing of two entangled vortex-membranes)

$$\hbar_{\text{knot}} = J_{\text{knot}} = \frac{1}{2} \rho_0 \kappa V_{\text{knot}}.$$

As a result, we have

$$\psi(\mathbf{x}, t) = \frac{1}{\sqrt{V_P}} \exp \left(\frac{-iE_{\text{knot}}t + i\mathbf{p}_{\text{knot}} \cdot \mathbf{x}}{\hbar_{\text{knot}}} \right). \quad (284)$$

From the fact of superposition principle of Kelvin waves, a generalized wavefunction can be

$$\psi(\mathbf{x}, t) = \sqrt{\rho_{\text{knot}}(\vec{x}, t)} e^{i\phi(\mathbf{x}, t)} = \sum_{\mathbf{p}} c_{\mathbf{p}} \exp \left(\frac{-iE_{\text{knot}}t + i\mathbf{p}_{\text{knot}} \cdot \mathbf{x}}{\hbar_{\text{knot}}} \right). \quad (285)$$

Next, we calculate the expect values of the projected (Lamb impulse) energy $\langle E_{\text{knot}} \rangle$ and the projected (Lamb impulse) momentum $\langle \mathbf{p}_{\text{knot}} \rangle$, respectively.

From the wavefunction $\psi(\mathbf{x}, t) = \sum_{\mathbf{p}} c_{\mathbf{p}} \exp \left(\frac{-i(E_{\text{knot}}t - \mathbf{p}_{\text{knot}} \cdot \mathbf{x})}{\hbar_{\text{knot}}} \right)$, we have

$$\begin{aligned} \langle E \rangle &= \int E_{\text{knot}} \rho_{\text{knot}}(x) dV_P = \int \psi^*(x, t) E_{\text{knot}} \psi(x, t) dV_P \\ &= \int \left[\sum_{\mathbf{p}} c_{\mathbf{p}}^* \exp \left(\frac{i(E_{\text{knot}}t - \mathbf{p}_{\text{knot}} \cdot \mathbf{x})}{\hbar_{\text{knot}}} \right) \right] \\ &\quad \times \left(i\hbar_{\text{knot}} \frac{d}{dt} \right) \left[\sum_{\mathbf{p}'} c_{\mathbf{p}'} \exp \left(\frac{-i(E_{\text{knot}}t - \mathbf{p}'_{\text{knot}} \cdot \mathbf{x})}{\hbar_{\text{knot}}} \right) \right] dV_P \\ &= \int \psi^*(\mathbf{x}, t) \left(i\hbar_{\text{knot}} \frac{d}{dt} \right) \psi(\mathbf{x}, t) dV_P. \end{aligned} \quad (286)$$

This result indicates that the projected (Lamb impulse) energy becomes operator

$$E_{\text{knot}} \rightarrow \hat{E}_{\text{knot}} = i\hbar_{\text{knot}} \frac{d}{dt}. \quad (287)$$

Using the similar approach, we derive

$$\begin{aligned} \langle \mathbf{p}_{\text{knot}} \rangle &= \int \mathbf{p}_{\text{knot}} \rho_{\text{knot}}(\mathbf{x}, t) dV_P = \int \psi^*(x, t) \mathbf{p}_{\text{knot}} \psi(\mathbf{x}, t) dV_P \\ &= \int \psi^*(\mathbf{x}, t) \left(-i\hbar_{\text{knot}} \frac{d}{d\mathbf{x}} \right) \psi(\mathbf{x}, t) dV_P. \end{aligned} \quad (288)$$

As a result, the projected (Lamb impulse) momentum also becomes operators

$$\mathbf{p}_{\text{knot}} \rightarrow \hat{\mathbf{p}}_{\text{knot}} = -i\hbar_{\text{knot}} \frac{d}{d\mathbf{x}}. \quad (289)$$

5.2.5. The Schrödinger equation

For deriving the Hamiltonian of knots, we treat the Hamiltonian of perturbative Kelvin waves around the knot-crystal. For knot-crystal of two entangled vortex-membranes, the Biot-Savart equation $i\frac{dz_{A/B}}{dt} = \frac{\delta\hat{H}(z_{A/B})}{\delta z_{A/B}^*}$ turns into the Schrödinger equation for constraint Kelvin waves

$$i\hbar_{\text{knot-crystal}} \frac{d\mathbf{Z}_{\text{knot-crystal}}(\mathbf{x}, t)}{dt} = \hat{H} \cdot \mathbf{Z}_{\text{knot-crystal}}(\mathbf{x}, t), \quad (290)$$

where $\mathbf{Z}_{\text{knot-crystal}}(\mathbf{x}, t)$ denote perturbative Kelvin waves around the knot-crystal (a special plane Kelvin wave) and the Hamiltonian density is given by

$$\hat{H}_{\text{knot-crystal}} = \sum_I \left[\frac{(\hat{p}_{\text{Lamb}}^I)^2}{2m_{\text{pseudo}}} \right] + \frac{\hbar_{\text{eff}}\omega^*}{2}(\mathbf{1} \otimes (\tau_x - \mathbf{1})), \quad (291)$$

where $m_{\text{pseudo}} = \frac{\hbar_{\text{eff}}}{\alpha\kappa \ln \epsilon}$ and $\hat{p}_{\text{lamb}}^I = -i\hbar_{\text{eff}} \frac{d}{dx^I}$.

For a knot, by scaling the Planck constant, we derive the Schrödinger equation for knots as

$$i\hbar_{\text{knot}} \frac{d\psi(\mathbf{x}, t)}{dt} = \hat{H}_{\text{knot}}\psi(\mathbf{x}, t), \quad (292)$$

where

$$\psi(\mathbf{x}, t) = \begin{pmatrix} \psi_{\uparrow, A}(\mathbf{x}, t) \\ \psi_{\uparrow, B}(\mathbf{x}, t) \\ \psi_{\downarrow, A}(\mathbf{x}, t) \\ \psi_{\uparrow, B}(\mathbf{x}, t) \end{pmatrix}$$

denotes quantum states of a knot with four degrees of freedom and $\hat{H}_{\text{knot}}$ is the Hamiltonian of knots. On flat vortex-membranes, the Hamiltonian of SOC knots is obtained by scaling that of plane Kelvin wave with same tensor state as

$$\hat{H}_{\text{knot}} = \sum_I \left[\frac{(\hat{p}_{\text{knot}}^I)^2}{2m_{\text{pseudo}}} \right] + \frac{\hbar_{\text{knot}}\omega^*}{2}(\mathbf{1} \otimes (\tau_x - \mathbf{1})), \quad (293)$$

where $m_{\text{pseudo}} = \frac{\hbar_{\text{knot}}}{\alpha\kappa \ln \epsilon}$ and $\hat{p}_{\text{knot}}^I = -i\hbar_{\text{knot}} \frac{d}{dx^I}$. In next section, we will derive the knot Hamiltonian $\hat{H}_{\text{knot}}$ on the SOC knot-crystal, that is different from above Hamiltonian.

For the eigenstate with eigenvalue E_{knot} ,

$$\hat{H}_{\text{knot}}\psi(\mathbf{x}, t) = E_{\text{knot}}\psi(\mathbf{x}, t) \quad (294)$$

the wavefunction becomes

$$\psi(\mathbf{x}, t) = f(\mathbf{x}) \exp\left(\frac{iE_{\text{knot}}t}{\hbar}\right), \quad (295)$$

where $f(\mathbf{x})$ is spatial function. This state corresponds to a vortex-membrane with fixed angular velocity, $\omega = \frac{E_{\text{knot}}}{\hbar_{\text{knot}}}$. As a result, the quantized condition for knot is

due to the conservation of angular momentum in extra space (the volume of the knot in 5D space) for information-unit, $\hbar_{\text{knot}}$.

In particular, we have the topological representation on action S for knots, i.e.,

$$\begin{aligned} S &= \int p_{\text{knot}} \cdot dx = \hbar_{\text{knot}} \int k \cdot dx = \hbar_{\text{knot}} \int 2\pi\rho_{\text{wind}} \cdot dx = 2\pi\hbar_{\text{knot}} \cdot w_{1D} \\ &= h_{\text{knot}} \cdot w_{1D}. \end{aligned} \quad (296)$$

From the point of view of topology, we explain the Sommerfeld quantization condition for knot-pieces

$$\oint p_{\text{knot}} \cdot dx = nh_{\text{knot}}, \quad (297)$$

where $h_{\text{knot}} = 2\pi\hbar_{\text{knot}}$ and n is an integer number. The Sommerfeld quantization condition is just the topological condition of the winding number for a process of “static” knot-pieces, i.e.,

$$\oint p_{\text{knot}} \cdot dx = h_{\text{knot}} \cdot w_{1D},$$

where w_{1D} is the winding number of the process that can be regarded as the winding number for corresponding Kelvin waves.

5.3. Emergent quantum field theory

In this part, we regard knot-crystal as a multiknot system that is described by quantum field theory and a knot (the elementary volume-changing of two entangled vortex-membranes) becomes elementary fermionic particle.

5.3.1. Fermionic statistics

To derive the formulation of quantum field theory for knot-crystal, we consider the knot-crystal as a many-knot system. We may introduce N -wavefunction to describe the motions of N -knots, $\Psi(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N)$. The situation is just that in a many-body quantum mechanics. An important feature is the statistics. In quantum mechanics, an assumption is spin-statistics of fermions. An electron is a fermion with half spin. The wavefunction of a system of identical spin-1/2 particles changes sign when two particles exchange. Particles with wavefunctions antisymmetric under exchange are called fermions.

To distinguish the statistics for the knots, we consider two knots. We show that the wavefunctions antisymmetric by exchanging knots is due to π -phase changing nature. When we exchange two 1D knots, the angle in extra space (that is just the phase of wavefunction) is π . According to the definition of knots, we have two static unified knots and get knot-functions

$$\Psi(\mathbf{x}, \mathbf{x}') = \hat{U}_{\text{knot}}(\mathbf{x}') \cdot \hat{U}_{\text{knot}}(\mathbf{x}) \mathbf{Z}_{\text{knot-crystal}}(\mathbf{x}, t). \quad (298)$$

Due to

$$\hat{U}_{\text{knot}}^*(\mathbf{x}) \cdot \hat{U}_{\text{knot}}(\mathbf{x}') \cdot \hat{U}_{\text{knot}}(\mathbf{x}) = -\hat{U}_{\text{knot}}(\mathbf{x}'), \quad (299)$$

after exchanging two knots, we get

$$\begin{aligned} \Psi(\mathbf{x}', \mathbf{x}) &= [\hat{U}_{\text{knot}}(\mathbf{x}') \cdot \hat{U}_{\text{knot}}(\mathbf{x})] \mathbf{Z}_{\text{knot-crystal}}(\mathbf{x}, t) \rightarrow \Psi(\mathbf{x}, \mathbf{x}') \\ &= [\hat{U}_{\text{knot}}(\mathbf{x}) \cdot \hat{U}_{\text{knot}}(\mathbf{x}')] \mathbf{Z}_{\text{knot-crystal}}(\mathbf{x}, t) \\ &= [\hat{U}_{\text{knot}}(\mathbf{x}') \cdot \hat{U}_{\text{knot}}(\mathbf{x})] \mathbf{Z}'_{\text{knot-crystal}}(\mathbf{x}, t) \\ &= -\Psi(\mathbf{x}, \mathbf{x}'), \end{aligned} \quad (300)$$

where

$$\mathbf{Z}'_{\text{knot-crystal}}(\mathbf{x}, t) = \mathbf{Z}_{\text{knot-crystal}}(\mathbf{x}, t) e^{i\pi}. \quad (301)$$

The results are independent of the dimensions. As a result, the knot obeys fermionic statistics in different dimensions.

We then introduce second quantization representation for knots by defining fermionic operator $c_1^\dagger(\mathbf{x})$, $c_2^\dagger(\mathbf{x})$ as

$$\hat{U}_{\text{knot}}(\mathbf{x}) \Rightarrow c_1^\dagger(\mathbf{x}), \quad \hat{U}_{\text{knot}}(\mathbf{x}') \Rightarrow c_2^\dagger(\mathbf{x}). \quad (302)$$

The knot-crystal without extra knots corresponds to the vacuum state $|\text{vacuum}\rangle$

$$\mathbf{Z}_{\text{knot-crystal}}(\mathbf{x}, t) \Rightarrow |\text{vacuum}\rangle. \quad (303)$$

Then after exchanging two identical particles, the many-body wavefunction

$$|\Psi_{\text{initial}}(\mathbf{x}', \mathbf{x})\rangle = c^\dagger(\mathbf{x}) c^\dagger(\mathbf{x}') |\text{vacuum}\rangle \quad (304)$$

changes by a phase to be

$$|\Psi_{\text{initial}}(\mathbf{x}', \mathbf{x})\rangle \rightarrow \Psi_{\text{final}}(\mathbf{x}', \mathbf{x}) = e^{i\pi} c^\dagger(\mathbf{x}') c^\dagger(\mathbf{x}) |\text{vacuum}\rangle. \quad (305)$$

After considering the four internal degrees of freedom, we have

$$c(\mathbf{x}, t) = \begin{pmatrix} c_{\uparrow, A}(\mathbf{x}, t) \\ c_{\uparrow, B}(\mathbf{x}, t) \\ c_{\downarrow, A}(\mathbf{x}, t) \\ c_{\downarrow, B}(\mathbf{x}, t) \end{pmatrix}. \quad (306)$$

We identify $c_{\uparrow, A/B}^\dagger(\mathbf{x}, t)$ ($c_{\uparrow, A/B}(\mathbf{x}, t)$) the creation (or annihilation) operator of a spin- $\uparrow$ knot on vortex-membrane-A/B; $c_{\downarrow, A/B}^\dagger(\mathbf{x}, t)$ ($c_{\downarrow, A/B}(\mathbf{x}, t)$) the creation (or annihilation) operator of a spin- $\downarrow$ knot on vortex-membrane-A/B. According to the fermionic statistics, there exists anti-commutation relation

$$\{c(\mathbf{x}, t), c(\mathbf{x}', t)\} = 0. \quad (307)$$

In addition, we can introduce the field representation via Grassmann number

$$c(\mathbf{x}, t) \implies \psi(\mathbf{x}, t) = \begin{pmatrix} \psi_{\uparrow, A}(\mathbf{x}, t) \\ \psi_{\uparrow, B}(\mathbf{x}, t) \\ \psi_{\downarrow, A}(\mathbf{x}, t) \\ \psi_{\downarrow, B}(\mathbf{x}, t) \end{pmatrix}. \quad (308)$$

5.3.2. Path integral formulation for quantum mechanics

Path integral formulation for quantum mechanics is another formulation describing the dynamic evolution of the distribution of knot-pieces. By considering information scaling the effective Planck constant of knot-crystal to that of knots, the probability amplitude $K_{\text{knot}}(\mathbf{x}', t_f; \mathbf{x}, t_i)$ for a knot (a knot-piece) from an initial position $\mathbf{x}$ at time $t = t_i$ (that is described by a state $|t_i, \mathbf{x}\rangle$) to position $\mathbf{x}'$ at a later time $t = t_f$ ($|t_f, \mathbf{x}'\rangle$) is obtained as

$$\begin{aligned} K_{\text{knot}}(\mathbf{x}', t_f; \mathbf{x}, t_i) &= \langle t_f, \mathbf{x}' | t_i, \mathbf{x} \rangle = \sum_n e^{iS_n/\hbar_{\text{knot}}} \\ &= \int \mathcal{D}\mathbf{p}_{\text{knot}}(t) \mathcal{D}\mathbf{x}(t) e^{iS/\hbar_{\text{knot}}}, \end{aligned} \quad (309)$$

where

$$S = \int \left[\mathbf{p}_{\text{knot}} \frac{d\mathbf{x}}{dt} - \hat{H}_{\text{knot}}(\mathbf{p}_{\text{knot}}, \mathbf{x}) \right] dt. \quad (310)$$

Each knot-piece's path contributes $e^{iS_n/\hbar_{\text{knot}}}$, where S_n is the n th classical action i.e.,

$$\sum_n e^{iS_n/\hbar_{\text{knot}}}. \quad (311)$$

With the help of particle operators, we define the many-body Hamiltonian of the system with multiknot as

$$\hat{\mathcal{H}} = \int d^3x [\psi^\dagger(\mathbf{x}) \hat{H}_{\text{knot}} \psi(\mathbf{x})], \quad (312)$$

where $\hat{H}_{\text{knot}}$ is the Hamiltonian operator of single knot and $\psi(\mathbf{x}) = \frac{1}{\sqrt{V}} \sum_p c_p \exp(i\mathbf{p} \cdot \mathbf{x}/\hbar_{\text{knot}})$ ($c_p = \begin{pmatrix} c_{\uparrow, p} \\ c_{\downarrow, p} \end{pmatrix}$). As a result, we have

$$\hat{\mathcal{H}} = \sum_k \hat{H}_{\text{knot}}(\mathbf{p}) c_p^\dagger c_p = \sum_k \hat{H}_{\text{knot}}(\mathbf{p}) \hat{n}_p, \quad (313)$$

where $\hat{n}_p = c_p^\dagger c_p$. The definition of total Hamiltonian $\hat{\mathcal{H}}$ means that for a given Kelvin wave there are n_p knots.

Now, we consider the path integral formulation of multiknot. The probability amplitude

$$K(\mathbf{x}'_M, \dots, \mathbf{x}'_2, \mathbf{x}'_1, t_f; \mathbf{x}_M, \dots, \mathbf{x}_2, \mathbf{x}_1, t_i) = \langle t_f, \mathbf{x}'_M, \dots, \mathbf{x}'_2, \mathbf{x}'_1 | t_i, \mathbf{x}_M, \dots, \mathbf{x}_2, \mathbf{x}_1 \rangle \quad (314)$$

becomes a multivariable function, where x'_j and x_j denote the final position and initial position of j th knot, respectively. For a multiknot system, quantum processes are described by

$$\begin{aligned} K(\mathbf{x}'_M, \dots, \mathbf{x}'_2, \mathbf{x}'_1, t_f; \mathbf{x}_M, \dots, \mathbf{x}_2, \mathbf{x}_1, t_i) &= \langle t_f, \mathbf{x}'_M, \dots, \mathbf{x}'_2, \mathbf{x}'_1 | t_i, \mathbf{x}_M, \dots, \mathbf{x}_2, \mathbf{x}_1 \rangle \\ &= \prod_j \sum_n e^{i\Delta\phi_{j,n}} = \prod_j \sum_n e^{iS_{j,n}/\hbar_{\text{knot}}} = \sum_n e^{i\sum_j S_{j,n}/\hbar_{\text{knot}}} \\ &= \prod_p \psi_p^\dagger(\mathbf{x}, t) \psi_p(\mathbf{x}, t) e^{iS_p/\hbar_{\text{knot}}} = \int \mathcal{D}\psi^\dagger(\mathbf{x}, t) \mathcal{D}\psi(\mathbf{x}, t) e^{iS/\hbar_{\text{knot}}}, \end{aligned} \quad (315)$$

where

$$S = \sum_{\omega, \mathbf{p}} S_{\omega, \mathbf{p}} = \int \mathcal{L} dt d^3x \quad (316)$$

with

$$S_{\omega, \mathbf{p}} = \psi_{\mathbf{p}}^\dagger (i\hbar\omega(\mathbf{p}) - \hat{H}_{\text{knot}}(\mathbf{p})) \psi_{\mathbf{p}} \quad (317)$$

and

$$\mathcal{L} = i\psi^\dagger \partial_t \psi - \hat{\mathcal{H}}. \quad (318)$$

The symbol $\sum_n$ denotes the summation of different paths and the symbol $\prod_j$ denotes the product of different volume changing with different zeros.

5.3.3. Discrete spatial translation symmetry for knots on knot-crystal

Symmetry and symmetry-breaking play profound roles in particle physics for quantum field theory and quantum many-body physics for solid physics.

In solid physics, a crystal is the system of periodically arranged atoms with spontaneous breaking of continuous spatial translation symmetry to discrete spatial translation symmetry. On a crystal, due to spontaneous breaking of continuous spatial translation symmetry to discrete spatial translation symmetry, the eigenstates are quantum states $|\mathbf{k}\rangle$. The Bloch vector $|\mathbf{k}\rangle$ is defined as a linear superposition of the localized states $|\mathbf{R}, \mathbf{a}_j\rangle$ in the unit cell $\mathbf{R}$ at position $\mathbf{a}_j$,

$$|\mathbf{k}\rangle = \sum_{\mathbf{R}, j} c_j(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}} |\mathbf{R}, \mathbf{a}_j\rangle, \quad (319)$$

that depends on Bloch momentum $\mathbf{k}$ and type of orbital j in the unit cell. One cannot do a translation operation $\mathcal{T}(\Delta\mathbf{x})$ with $\Delta\mathbf{x} = \alpha\mathbf{R}$, where α is a noninteger

real number. The Bloch theorem is very useful in description of a quasi-particle in crystals that states that the basis vector $|\mathbf{k}\rangle$ translated for the Bravais vector $\mathbf{R}$ changes as

$$\mathcal{T}(\mathbf{R})|\mathbf{k}\rangle = e^{i\mathbf{k}\cdot\mathbf{R}}|\mathbf{k}\rangle, \quad (320)$$

where $T(\mathbf{R})$ is translation operation. Due to the existence of Brillouin zone (BZ), we have $|\mathbf{k}\rangle = |\mathbf{k} + \mathbf{Q}\rangle$, where $\mathbf{Q}$ is reciprocal lattice vector.

For knot-crystal, the eigenstates of an arbitrary quantum state in pseudo-quantum mechanics have generalized spatial translation symmetry $|\mathbf{k}\rangle$, where $\mathbf{k}$ is wave-vector. By defining a translation operation $T(\Delta\mathbf{x})$, we have

$$\mathcal{T}(\Delta\mathbf{x})|\mathbf{k}\rangle = e^{i\mathbf{k}\cdot\Delta\mathbf{x}}|\mathbf{k}\rangle, \quad (321)$$

where $\Delta\mathbf{x}$ denotes the distance from generalized spatial translation operation.

However, for emergent quantum mechanics, the situation changes. If we consider the vortex-piece with a zero as the elementary physics object, there exists a generalization of the Bloch theorem that incorporates generalized translational symmetries for quantum states of knots on a knot-crystal in emergent quantum mechanics.

For a plane wave state $|\mathbf{k}_{L/R}\rangle$ of zeros on L/R-sublattice along the direction $\mathbf{k}$, we do a translation operation $\mathcal{T}(\Delta\mathbf{x})$ (that corresponds to $T(\Delta\mathbf{x})$ on knot-crystal) and get

$$\mathcal{T}(\Delta\mathbf{x})|\mathbf{k}_{L/R}\rangle = |\mathbf{k}_{L/R}\rangle, \quad (322)$$

where $(\Delta\mathbf{x}) = 2a$.

For the case of $(\Delta\mathbf{x}) = a$, we have

$$\hat{T}(\Delta\mathbf{x})|\mathbf{k}_{L/R}\rangle = |\mathbf{k}_{R/L}\rangle \quad (323)$$

that is reduced to the results in solid state physics. And we have a generalized BZ along a given direction $\mathbf{e}$ as

$$|\mathbf{k}_{L/R}\rangle = |\mathbf{k}_{R/L} + \mathbf{k}_0\rangle, \quad (324)$$

where $\mathbf{k}_0 = \frac{\pi}{a}\mathbf{e}$ is reciprocal lattice vector along the given direction $\mathbf{e}$. The generalized Bloch theorem constrains the formulation of the Hamiltonian which becomes manifestly invariant under additional spatial rotating symmetry.

For the case of $(\Delta\mathbf{x}) \bmod a \neq 0$, we can recover the continuous spatial translation symmetry by changing projection angle

$$\theta \rightarrow \theta' = \theta \mp k_0 \cdot [a[(\Delta\mathbf{x}) \bmod a]] = \theta \mp \pi[(\Delta\mathbf{x}) \bmod a]. \quad (325)$$

5.3.4. Quantum field theory for knot-crystal

In emergent quantum mechanics, a knot-crystal becomes multiknot system described by a tensor-network state, of which the effective theory becomes a Dirac model in quantum field theory.

In emergent quantum mechanics, the Hamiltonian for a 3D SOC knot-crystal has two terms — the kinetic term and the mass term from leapfrogging motion. In Fig. 11(a), we consider a knot-crystal to be a “two-sublattice” model with discrete spatial translation symmetry. We then label the knots by Wannier state $|i, L\rangle$, $|i + 1, R\rangle$, $|i + 2, L\rangle$, $\dots$. According to the definition of Nambu states $|L\rangle$ and $|R\rangle$, the phase region of Wannier states $c_{L,i}^\dagger|\text{vacuum}\rangle$ and $c_{R,j}^\dagger|\text{vacuum}\rangle$ are $(0, \pi)$ and $(-\pi, 0)$, respectively.

We use the tight-binding formula to characterize the Hamiltonian that couples two nearest neighbor states $|i, L\rangle$ and $|i + e_I, R\rangle$ ($I = X, Y, Z$),

$$-J(|i, L\rangle\langle i + e_I, R| + |i + e_I, R\rangle\langle i, L|) = -J(|i, L\rangle\langle i + e_I, R| + \text{h.c.}), \quad (326)$$

where i denotes a given knot. The total kinetic term is obtained as

$$-J \sum_{\langle i, j \rangle} (|i, L\rangle\langle j, R| + \text{h.c.}), \quad (327)$$

where $\langle i, j \rangle$ denotes the nearest neighbor knots. With help of $\hat{T}(a)$ and $\iota^\pm$, we have

$$|i + e_I, L\rangle = \hat{T}(a) \cdot \iota^\dagger |i, R\rangle, \quad (328)$$

where $\hat{T}(a)$ is the translation operator $\hat{T}(a)|i\rangle = |i + e_I\rangle$ that is

$$\hat{T}(a) = e^{ia(\hat{k} \cdot \sigma)} \quad (329)$$

Fig. 11. (Color online) An illustration of two “sublattices” of knot-crystal: $|L\rangle$ and $|R\rangle$ denotes states on different sublattices; A and B denote the indices of vortex-membranes; (b) An illustration of the periodic structure along time-axis of knot states $|L\rangle$ and $|R\rangle$ for leapfrogging knot-crystal. $\Upsilon(x, t) = \eta_{A, \theta}(x, t) - \eta_{B, \theta}(x, t)$ is function to characterize the oscillating effect from leapfrogging motion. We define the states of $\text{sgn}[\Upsilon] = 1$ by $|L\rangle$ and the states $\text{sgn}[\Upsilon] = -1$ by $|R\rangle$, respectively.

and $\iota^\pm$ is an operator denoting switching of the chiral (sublattice) degrees of freedom as $\iota^- = (\text{L} \rightarrow \text{R}) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $\iota^+ = (\text{R} \rightarrow \text{L}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. The kinetic term can also be described by a familiar formulation

$$\hat{\mathcal{H}}_{\text{coupling}} = -J \sum_{\langle i,j \rangle} c_i^\dagger c_j + \text{h.c.}, \quad (330)$$

where $\hat{c}_i^\dagger$ is the creation operator of knots at the site i .

We then use path-integral formulation to characterize the effective Hamiltonian for a knot-crystal as

$$\int \mathcal{D}\psi^\dagger(t, \mathbf{x}) \mathcal{D}\psi(t, \mathbf{x}) e^{i\mathcal{S}/\hbar}, \quad (331)$$

where $\mathcal{S} = \int \mathcal{L} dt$ and $\mathcal{L} = i \sum_i \psi_i^\dagger \partial_t \psi_i - \mathcal{H}_{\text{coupling}}$. In emergent quantum mechanics, $\psi^\dagger(t, \mathbf{x})$ is an eight-component fermion field as

$$\psi^\dagger(t, \mathbf{x}) = (\psi_{\uparrow\text{L}}^\dagger(t, \mathbf{x}), \psi_{\uparrow\text{R}}^\dagger(t, \mathbf{x}), \psi_{\downarrow\text{L}}^\dagger(t, \mathbf{x}), \psi_{\downarrow\text{R}}^\dagger(t, \mathbf{x})), \quad (332)$$

where A, B labels vortex degrees of freedom, $\uparrow, \downarrow$ labels two spin degrees of freedom that denote the two possible winding directions, L, R denotes the chiral-degrees of freedom. In the continuum limit, we have

$$\begin{aligned} \mathcal{H}_{\text{coupling}} &= -J \sum_{\langle i,j \rangle} \psi_i^\dagger \psi_j + \text{h.c.} \\ &= -J \sum_i \psi_i^\dagger (\hat{\mathcal{T}}(a) \otimes \iota^-) \psi_i + \text{h.c.} \\ &= -J \sum_{k>0} \psi_k^\dagger (e^{ia(\mathbf{k} \cdot \boldsymbol{\sigma})} \otimes \iota^-) \psi_k + \text{h.c.} \\ &= -J \sum_k \psi_k^\dagger [\cos(a(k \cdot \boldsymbol{\sigma})) \otimes \iota_x] \psi_k \\ &\quad - J \sum_k \psi_k^\dagger [\sin(a(\mathbf{k} \cdot \boldsymbol{\sigma})) \otimes \iota_y] \psi_k. \end{aligned} \quad (333)$$

where $\mathbf{k}$ is the wave vector along the direction $\mathbf{e}$ and $\iota_y = i\iota^- - i\iota^+$. In continuum limit $\mathbf{k} \rightarrow \Delta\mathbf{k} \rightarrow 0$, the effective Hamiltonian is reduced into,

$$\mathcal{H}_{\text{coupling}} \simeq -J \sum_k \psi_k^\dagger (\mathbf{1} \otimes \iota_x) \psi_k - aJ \sum_k \psi_k^\dagger [\Delta\mathbf{k} \cdot \boldsymbol{\sigma} \otimes \iota_y] \psi_k. \quad (334)$$

Thus, the reduced BZ $|\mathbf{k}_{\text{L/R}}\rangle$ is given $[-\frac{\pi}{2}, \frac{\pi}{2}) \otimes [-\frac{\pi}{2}, \frac{\pi}{2}) \otimes [-\frac{\pi}{2}, \frac{\pi}{2})$. There exists a node at $(0, 0, 0)$ and the excitation modes have chiral properties. This dispersion is physical consequence of “two-sublattice”.

Next, we consider the term from global rotating motion and leapfrogging motion, of which the angular frequency ω^* . For leapfrogging motion obtained by Ref. 27, the function of the two entangled vortex-membranes is simplified by

$$\begin{pmatrix} z_A(\mathbf{x} = 0, t) \\ z_B(\mathbf{x} = 0, t) \end{pmatrix} = \frac{r_0}{2} \begin{pmatrix} 1 + e^{i\omega^* t} \\ 1 - e^{i\omega^* t} \end{pmatrix} e^{-i\omega_0 t}. \quad (335)$$

At $t = 0$, we have knot state $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$; At $t = \frac{\pi}{\omega^*}$, we have knot state $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Thus, the leapfrogging knot-crystal leads to periodic varied knot states, i.e., at $t = 0$, we have $\text{sgn}[\Upsilon(t)] > 0$ that is denoted by $|L\rangle$; at $t = \frac{\pi}{\omega^*}$ we have $\text{sgn}[\Upsilon(t)] < 0$ that is denoted by $|R\rangle$. See the illustration in Fig. 11(b). From the global rotating motion denoted $e^{i\omega_0 t}$, the projected states also change periodically. As a result, we use the following formulation to characterize the global rotating process and the leapfrogging process:

$$\psi_i \rightarrow \psi'_i(t) = e^{(\mathbf{1} \otimes \iota_x)(2i(\omega_0 + \omega^*)t)} \cdot \psi_i. \quad (336)$$

After considering the energy from the global rotating process and the leapfrogging process, a corresponding term is given by

$$\begin{aligned} i\hbar_{\text{knot}} \sum_i \psi_i^\dagger \partial_t \psi'_i &= i\hbar_{\text{knot}} \sum_i (\psi_i^\dagger e^{(\mathbf{1} \otimes \iota_x)(-2i(\omega_0 + \omega^*)t)}) \partial_t (e^{(\mathbf{1} \otimes \iota_x)(2i(\omega_0 + \omega^*)t)} \psi_i) \\ &= i\hbar_{\text{knot}} \sum_i (\psi_i^\dagger \partial_t \psi_i + 2i\psi_i^\dagger (\omega_0 + \omega^*) (\mathbf{1} \otimes \iota_x) \psi_i). \end{aligned}$$

We write down the total Lagrangian of a 3D leapfrogging knot-crystal in tight-binding formula as

$$\mathcal{L} = i\hbar_{\text{knot}} \sum_i \psi_i^\dagger \partial_t \psi'_i - \mathcal{H}_{\text{coupling}} = i\hbar_{\text{knot}} \int \psi^\dagger \partial_t \psi d^3x - \mathcal{H}_{\text{knot}}, \quad (337)$$

where

$$\begin{aligned} \mathcal{H}_{\text{knot}} &= \mathcal{H}_{\text{coupling}} + 2\hbar_{\text{knot}}(\omega_0 + \omega^*) \int \psi^\dagger (\mathbf{1} \otimes \iota_x) \psi d^3x \\ &= -J \sum_k \psi_k^\dagger (\mathbf{1} \otimes \iota_x) \psi_k - aJ \sum_k \psi_k^\dagger [\Delta \mathbf{k} \cdot \boldsymbol{\sigma} \otimes \iota_y] \psi_k \\ &\quad + 2\hbar_{\text{knot}}(\omega_0 + \omega^*) \sum_k \psi_k^\dagger (\mathbf{1} \otimes \iota_x) \psi_k. \end{aligned} \quad (338)$$

From the fact that the energy of vortex-membranes is equal to the global rotating energy of the two entangled vortex-membranes. In long wavelength limit, $k \rightarrow 0$ we obtain the low energy effective Hamiltonian as

$$\begin{aligned} \mathcal{H}_{\text{knot}} &\simeq -aJ \sum_k \psi_k^\dagger [((\Delta \mathbf{k} \cdot \boldsymbol{\sigma}) \otimes \iota_y)] \psi_k + 2\hbar_{\text{knot}} \omega^* \sum_k \psi_k^\dagger (\mathbf{1} \otimes \iota_x) \psi_k \\ &= -c_{\text{eff}} \int \psi^\dagger [(\boldsymbol{\sigma} \otimes \iota_y) \cdot \mathbf{p}_{\text{knot}}] \psi d^3x + m_{\text{knot}} c_{\text{eff}}^2 \int \psi^\dagger (\mathbf{1} \otimes \iota_x) \psi d^3x. \end{aligned} \quad (339)$$

where $c_{\text{eff}} = \frac{aJ}{\hbar_{\text{knot}}} = 2a\omega_0$ play the role of light speed and $m_{\text{knot}} c_{\text{eff}}^2 = 2\hbar_{\text{knot}} \omega^*$ plays role of the mass of knots. $\mathbf{p}_{\text{knot}} = \hbar_{\text{knot}} \mathbf{k}$ is the momentum operator. We then re-write the effective Hamiltonian to be

$$\mathcal{H}_{\text{knot}} = \int (\psi^\dagger \hat{\mathbf{H}}_{\text{knot}} \psi) d^3x \quad (340)$$

and

$$\hat{H}_{\text{knot}} = -c_{\text{eff}} \mathbf{\Gamma} \cdot \mathbf{p}_{\text{knot}} + m_{\text{knot}} c_{\text{eff}}^2 \Gamma^5, \quad (341)$$

where $\Gamma^5 = \mathbf{1} \otimes \iota_x$, $\mathbf{\Gamma} = (\Gamma^1, \Gamma^2, \Gamma^3)$ and $\Gamma^1 = \sigma^x \otimes \iota_y$, $\Gamma^2 = \sigma^y \otimes \iota_y$, $\Gamma^3 = \sigma^z \otimes \iota_y$.

It is obvious that the total Hamiltonian $\mathcal{H}$ has translation symmetry and rotation symmetry

$$\left[\hat{\mathcal{T}}(\Delta \mathbf{x}), \hat{H}_{\text{knot}} \right] = 0 \quad (342)$$

and

$$\left[\hat{R}, \hat{H}_{\text{knot}} \right] = 0, \quad (343)$$

respectively. Here, $\hat{\mathcal{T}}(\Delta \mathbf{x}) = e^{ia(\hat{\mathbf{k}} \cdot \tilde{\mathbf{r}})} \cdot \hat{S}$ is translation operator. That means we have generalized translation symmetry. Another important property of the Dirac model for knot-crystal is spin-orbital coupling. The global spatial rotation operator is defined by

$$\hat{R} = \hat{R}_{\text{spin}} \cdot \hat{R}_{\text{space}}, \quad (344)$$

where $\hat{R}_{\text{spin}}$ is SO(3) spin rotation operator $\hat{R}_{\text{spin}} \tilde{\mathbf{\Gamma}} \hat{R}_{\text{spin}}^{-1} = \tilde{\mathbf{\Gamma}}'$, and $\hat{R}_{\text{space}}$ is SO(3) spatial rotation operator, $\hat{R}_{\text{space}} \mathbf{p} \hat{R}_{\text{space}}^{-1} = \mathbf{p}'$, $\hat{R}_{\text{space}} \mathbf{x} \hat{R}_{\text{space}}^{-1} = \mathbf{x}'$. After doing a global spatial rotation operation, the motion direction changes from $\mathbf{e}$ to $\mathbf{e}'$. That means, we have generalized rotation symmetry.

The ground state of knot-crystal is a filled state of knots, of which the total number of knots is N_{knot} . The excitation is a knot, of which the energy dispersion is

$$E_p = \pm \sqrt{\hbar_{\text{knot}}^2 k^2 c_{\text{eff}}^2 + m_{\text{knot}}^2 c_{\text{eff}}^4}, \quad (345)$$

where $k^2 = k_x^2 + k_y^2 + k_z^2$ and m_{knot} is the mass of knot. There are four energy bands for Dirac states. For a particle-like excitation, the energy is $\sqrt{\hbar_{\text{knot}}^2 k^2 c_{\text{eff}}^2 + m_{\text{knot}}^2 c_{\text{eff}}^4}$; for a hole-like excitation, the energy is $-\sqrt{\hbar_{\text{knot}}^2 k^2 c_{\text{eff}}^2 + m_{\text{knot}}^2 c_{\text{eff}}^4}$. For a particle-like excitation, an extra knot with a zero is put on the knot-crystal and the total volume of the deformed knot-crystal becomes $(N_{\text{knot}} + 1)V_{\text{knot}}$; for a hole-like excitation, a knot with a zero is taken off from the knot-crystal and the total volume of the deformed knot-crystal is $(N_{\text{knot}} - 1)V_{\text{knot}}$. The situation is very similar to the quasi-particles in solid physics. From $\hat{H}_{3D}$, for a particle-like knot, we can easily obtain the Schrödinger equation in low velocity limit,

$$i\hbar_{\text{knot}} \frac{d\psi(\mathbf{x}, t)}{dt} = \hat{H}_{\text{knot}} \psi(\mathbf{x}, t), \quad (346)$$

where

$$\hat{H}_{\text{knot}} = \frac{\hat{p}_{\text{knot}}^2}{2m_{\text{knot}}}. \quad (347)$$

Finally, the low energy effective Lagrangian of 3D SOC knot-crystal is derived as

$$\mathcal{L}_{3D} = i\psi^\dagger \partial_t \psi - \mathcal{H}_{\text{knot}} = \bar{\psi}(i\gamma^\mu \hat{\partial}_\mu - m_{\text{knot}})\psi, \quad (348)$$

where $\bar{\psi} = \psi^\dagger \gamma^0$, γ^μ are the reduced Gamma matrices, $\gamma^1 = \gamma^0 \Gamma^1$, $\gamma^2 = \gamma^0 \Gamma^2$, $\gamma^3 = \gamma^0 \Gamma^3$, and $\gamma^0 = \Gamma^5 = \mathbf{1} \otimes \iota_x$, $\gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3$. This is Lagrangian for massive Dirac fermions with emergent SO(3,1) Lorentz-invariance. Since the velocity c_{eff} only depends on the microscopic parameter k_0 and κ , we may regard c_{eff} to be “light-velocity” and the invariance of light-velocity becomes a fundamental principle for the knot physics.

As a result, we answer the question — *how to characterize the evolution of the object with a zero?* The pseudo-quantum mechanics describes the entanglement-fluctuation of two entangled vortex-membrane with fixed volume in the 5D fluid; the emergent quantum field theory describes the volume-changing entanglement-fluctuation of two entangled vortex-membrane in the 5D fluid. From the point of view of information, the elementary volume-changing with a zero is a knot. We consider a given pattern of knot-pieces $\psi(\mathbf{x})$ as the initial condition. We expand $\psi(\mathbf{x})$ by the eigenstates of plane waves as $\psi(\mathbf{x}) = \sum_k c_k e^{i\mathbf{p}_{\text{knot}} \cdot \mathbf{x} / \hbar_{\text{knot}}}$. Thus, under time evolution, the function of vortex-membranes becomes $\psi(\mathbf{x}, t) = \sum_k c_k e^{i\mathbf{p}_{\text{knot}} \cdot \mathbf{x} / \hbar_{\text{knot}}} e^{-iE_{\text{knot}} \cdot t / \hbar_{\text{knot}}}$. Finally, the density of knot-pieces is obtained as

$$\rho_{\text{knot}}(\mathbf{x}, t) = \psi^\dagger(\mathbf{x}, t) \psi(\mathbf{x}, t). \quad (349)$$

We point out that under time evolution, the energy $H(\mathbf{p}_{\text{knot}})$, the momentum $\mathbf{p}_{\text{knot}}$ (the Lamb impulse) and the (Lamb impulse) angular momentum (the effective Planck constant $\hbar_{\text{knot}}$) are all conserved. And owing to the conservation conditions of the volume of the knot in 5D space, the shape of the knot is never changed and the corresponding Kelvin waves of knots cannot evolve smoothly. Instead, the knot can only split and knot-pieces evolve following the equation of motion of Biot–Savart equation (Schrödinger equation). This is the fundamental principle of quantum mechanics.

So for 3D SOC knot, there exist two types of Hamiltonian — one type on flat vortex-membranes with continuum spatial translation symmetry, the other on knot-crystal with lattice translation symmetry. For, a knot on flat vortex-membranes, the equation of motion is determined by the Schrödinger equation with the Hamiltonian

$$\hat{H}_{\text{knot}} = \sum_I \left[\frac{(\hat{p}_{\text{knot}}^I)^2}{2m_{\text{pseudo}}} \right] + \frac{\hbar_{\text{knot}} \omega^*}{2} (\mathbf{1} \otimes (\tau_x - \mathbf{1})). \quad (350)$$

There does not exist Lorentz invariance; for the knots on 3D SOC knot-crystal, the equation of motion is determined by the Schrödinger equation with the Hamiltonian

$$\hat{H}_{\text{knot}} = -c_{\text{eff}} \mathbf{\Gamma} \cdot \mathbf{p}_{\text{knot}} + m_{\text{knot}} c_{\text{eff}}^2 \Gamma^5. \quad (351)$$

5.3.5. Possible physical realization

We address the issue of the physical realization of a 1D knot-crystal based on entangled vortex-lines in ^4He superfluid.

We consider two rectilinear vortex-lines in SF that are stretched between opposite points on the system. Then we rotate one vortex-line around the other with a rotating velocity ω_0 . According to the theory of Kelvin waves, the winding vortex-line becomes a helical one that is described by $r_0 e^{ik_0 \cdot x - i\omega_0 t + i\phi_0}$ where $\omega_0 \simeq (\frac{\kappa}{4\pi} \ln \frac{1}{k_0 a_0}) k_0^2$. As a result, in principle, we realize a knot-crystal.

We estimate the two physical constants, $\hbar_{\text{knot}}$ and c_{eff} . The emergent Planck constant $\hbar_{\text{knot}}$ is defined by $\hbar_{\text{knot}} = \frac{\pi r_0^2 \rho_0 \kappa}{2k_0}$. We set the length of the half pitch of the windings $a = \frac{\pi}{k_0}$ to be 10^{-5} cm, and the distance between two vortex-lines r_0 as 10^{-6} cm. For ^4He superfluid, κ is about 10^{-3} cm²/s, then the effective Planck constant is estimated to be $\hbar_{\text{knot}} = N_{\text{He4}} \cdot \hbar \sim 10^5 \hbar$ where $N_{\text{He4}} = \pi r_0^2 a \cdot \rho^*$ is the atom's number inside a knot. ρ^* is the superfluid density. Another important parameter is the effective light speed c_{eff} , which is defined by $c_{\text{eff}} = 2a\omega_0$. The effective light speed is estimated to be $c_{\text{eff}} \sim 12$ m/s.

5.4. Measurement theory

Measurement is an important issue of quantum mechanics. P. A. M. Dirac (1958) in “*The Principles of Quantum Mechanics said, ‘A measurement always causes the system to jump into an eigenstate of the dynamical variable that is being measured, the eigenvalue this eigenstate belongs to being equal to the result of the measurement’.*” According to the Copenhagen interpretation, the state of a system is assumed to “collapse” into an eigenstate of the operator during measurement. That is the so-called measurement process of “wavefunction collapse”. The wavefunction collapse is random and indeterministic and the predicted value of the measurement is described by a probability distribution. The wavefunction collapse raises “the measurement problem”, as well as questions of determinism and locality, as demonstrated in the Einstein–Podolsky–Rosen (EPR) paradox.³⁰ In this section, we show measurement theory in emergent quantum mechanics.

5.4.1. Information measurement and observers

In physics, people try to know the information of patterns for perturbative Kelvin wave of knots described by $\mathbf{Z}_{\text{knot}}(\mathbf{x}, t)$, i.e.,

$$\text{Information} = \mathbf{Z}_{\text{knot}}(\mathbf{x}, t). \quad (352)$$

In mathematics, the information is the distribution of zeros between projected vortex-membranes that is determined by knot density for a given quantum state described by $|\psi(\mathbf{x}, t)\rangle$, i.e.,

$$\text{Information} = |\psi(\mathbf{x}, t)\rangle. \quad (353)$$

That means to identify the entanglement pattern of vortex-membranes $\mathbf{Z}_{\text{knot}}(\mathbf{x}, t)$, people detect the distribution function of knot-pieces $\rho_{\text{knot}}(\mathbf{x}, t)$, i.e.,

$$\text{Measurement} = \text{to detect } \mathbf{Z}_{\text{knot}}(\mathbf{x}, t) \rightarrow \text{to detect } \rho_{\text{knot}}(\mathbf{x}, t). \quad (354)$$

To do measurement, there exist observers. To detect the distribution of knot-pieces, there are two types of observers: one type is the inner observers that are special multi-knot systems on vortex-membranes, the other is outer observers that are objects out of vortex-membranes (or even out of the inviscid incompressible fluid). For the outer observers, the rulers and clocks are independent on the physical properties of the knot-crystal. In principle, the outer observers have the ability to detect the every detail of the perturbative Kelvin waves on the knot-crystal. As a result, the theory for outer observers is Biot-Savart mechanics. However, for the inner observers, the rulers and clocks depend on the physical properties of the knot-crystal. In general, the inner observers have very slow clocks and much larger rules and have no the ability to detect the every detail of the perturbative Kelvin waves on the knot-crystal. By using a huge ruler, the inner observers obtain a coarse-graining picture of a given system of multi-knots; by using a slow clock, the inner observers obtain average results of a dynamic system of multiknots via time. So the inner observers will never obtain the complete information. In particular, the information of absolute phase factor cannot be observed for inner observers owing to a slow clock and a huge rule with very large scale-distance. In this paper, we consider the measurement theory for inner observers by considering slow clock. The situation for huge rule with very large scale-distance is similar.

5.4.2. Dynamic projection

In this part, we show the concept of dynamic projection that is a process to detect $\rho_{\text{knot}}(\mathbf{x}, t)$.

Knot is a topological object that leads to π -phase changing ($\Delta\phi = \pi$) and knot-piece is tiny phase-changing ($\Delta\phi \rightarrow 0$) with fixed phase angle ($\phi = \phi_0$). To find a knot (or a knot-piece), people need to detect the phase-changing $\Delta\phi$ that changes the density of zeros between two vortex-membranes, $\rho_{\text{zero}} \rightarrow \rho_{\text{zero}}(x, t)$. So we may project the vortex-membranes and solve the knot-function to look for extra solutions owing to the knot-piece $\Delta\phi$. In experiments, people identify phase-changing $\Delta\phi$ via dynamic projection

$$\hat{P}_{\text{dynamic}} = \text{to capture a knot-piece } \Delta\phi \text{ at } t = t_0. \quad (355)$$

To find a fragmentized knot, we need to do a projection

$$\hat{P}_{\text{dynamic}}(t_0, \theta)_{\text{ZB}}(\mathbf{x}, t) = \hat{P}_{\text{dynamic}}(t_0, \theta)_{\text{ZA}}(\mathbf{x}_0, t_0). \quad (356)$$

Under a projection, a fragmentized (or randomized) knot is fixed at x_0 and time t_0 .

Because a knot is really a sharp phase-changing that could interact with the measuring devices,

$$\begin{aligned}\tilde{\mathbf{Z}}'_{\text{fragment}}(\mathbf{x}, t) &= \hat{\mathbf{U}}_{\text{fragment}} \mathbf{Z}_{\text{knot-crystal}}(\mathbf{x}, t) \\ &= \prod_{i=1, I}^{N \rightarrow \infty} \hat{\mathbf{U}}_{\text{fragment}}^I(\Delta\phi_{\frac{1}{N}\text{-knot}}(X_0)_i) \mathbf{Z}_{\text{knot-crystal}}(\mathbf{x}, t).\end{aligned}\quad (357)$$

Thus, according to the different effects, we can decompose the dynamic operator into three parts:

$$\hat{P}_{\text{dynamic}}(t_0, \theta) = \hat{P}_{\text{fragment}} \hat{P}_{\text{Kelvin-wave}} \hat{P}(t_0), \quad (358)$$

where $\hat{P}(t_0)$ denotes the projection at time t_0 , $\hat{P}_{\text{Kelvin-wave}}$ denotes the projection on a given eigenstate of perturbative Kelvin waves

$$e^{-i\Delta\omega \cdot t_0 + i\Delta\mathbf{k} \cdot \mathbf{x}_0} \begin{pmatrix} u_{\mathbf{k}_0, A} \\ u_{\mathbf{k}_0, B} \\ u_{-\mathbf{k}_0, A} \\ u_{-\mathbf{k}_0, B} \end{pmatrix}$$

with fixed spin and vortex degrees of freedom, $\hat{P}_{\text{fragment}}$ denotes the projection on a given knot-piece for the projected perturbative Kelvin waves. As a result, the dynamic projection operator is defined by

$$\begin{aligned}\hat{P}_{\text{dynamic}}(t_0, \theta) z(\mathbf{x}, t_0) \\ = \hat{P}_{\text{dynamic}}(t_0, \theta) \prod_{i=1, I}^{N \rightarrow \infty} \hat{\mathbf{U}}_{\text{fragment}}^I(\Delta\phi_{\frac{1}{N}\text{-knot}}(X_0)_i)\end{aligned}\quad (359)$$

$$\begin{aligned}&\cdot \left[\sum_{\Delta k} a_{\Delta k} e^{\pm i\Delta\omega t_0 \pm i\Delta\mathbf{k} \cdot \mathbf{x}_0} \mathbf{Z}_{\text{knot-crystal}}(\mathbf{x}, t) \right] \\ &= \hat{\mathbf{U}}_{\text{fragment}}^I(\Delta\phi_{\frac{1}{N}\text{-knot}}(X_0)_i) \\ &\cdot [a_{\Delta k} e^{-i\Delta\omega t_0 + i\Delta\mathbf{k} \cdot \mathbf{x}_0} \mathbf{Z}_{\text{knot-crystal}}(\mathbf{x}_0, t_0)] \Rightarrow [z(\phi(x_0))_{\text{fragment}}] \\ &\cdot [e^{-i\Delta\omega t_0 + i\Delta\mathbf{k} \cdot \mathbf{x}_0} \mathbf{Z}_{\text{knot-crystal}}(\mathbf{x}_0, t_0)].\end{aligned}\quad (360)$$

After repeating the measurement processes $\hat{P}_{\text{dynamic}}$, lots of knot-pieces are observed and people know their distribution $\rho_{\text{knot}}(\mathbf{x}, t)$. Finally, the information of perturbative Kelvin waves on vortex-membrane $\mathbf{Z}_{\text{knot}}(\mathbf{x}, t)$ is explored. During measurement processes, the phase angle of knot-pieces is fixed to be $\phi_0(t_0)$ that depends on the time of measurement. This gives an explanation of “wavefunction collapse” in quantum mechanics.

Fig. 12. (Color online) An illustration of dynamic projection: (a) shows phase ϕ via x and (b) shows phase changing $\frac{d\phi}{dx}$ via x . There are two measurement processes at time $t = t_1$ and $t = t_2$. The green spots in (a) and the purple spots in (b) denote the position of knot-pieces under projection. In (a), people can detect the phase changing and the information for a knot can be fully observed; In (b), people can only detect the phase changing and the information for a knot cannot be fully observed.

5.4.3. Fast-clock effect

We point out that the probability of emergent quantum mechanics comes from dynamically projecting entangled vortex-membranes via stochastic projected time t_0 that leads to stochastic projected angle.

It is known that the knots are phase-changing of knot-crystal. For a knot crystal, $\mathbf{Z}_{\text{knot-crystal}}(\mathbf{x}, t) \sim e^{-i(\omega_0 + \Delta\omega)t + \phi_0}$, the period T_{clock} of the phase angle changing 2π is $T_{\text{clock}} = \frac{2\pi}{\omega_0 + \Delta\omega}$. So we have an intrinsic clock for knots, of which the phase is just the phase of the wavefunction. The clock for inner observers T_{inner} must be much slower than the intrinsic clock, $T_{\text{inner}} \gg T_{\text{clock}}$. The inner observers can do measurement at the scheduled time $\Delta t = nT_{\text{inner}}$ (n is an arbitrary integer number), but cannot accurately do measurement at the scheduled time $\Delta t = nT_{\text{clock}}$. In the limit of $\frac{T_{\text{inner}}}{T_{\text{clock}}} \rightarrow \infty$, $[\frac{\Delta t}{T_{\text{clock}}} - \text{mod}(\frac{\Delta t}{T_{\text{clock}}})]$ is a random value and the projection angles $[\frac{\Delta t}{T_{\text{clock}}} - \text{mod}(\frac{\Delta t}{T_{\text{clock}}})]\omega_0$ become stochastic for the different projection processes. As a result, owing to the stochastic projecting time $\hat{P}(t_0)$, the projecting angle θ is a random value, i.e.,

$$\theta \rightarrow \theta + \left[\frac{\Delta t}{T_{\text{clock}}} - \text{mod} \left(\frac{\Delta t}{T_{\text{clock}}} \right) \right] \omega_0 = a \text{ random value.} \quad (361)$$

We call it “fast-clock” effect.

As shown in Fig. 12(b), the detecting process can be exactly demonstrated by pulse sample figures — people can detect the phase-changing from knots but cannot detect the absolute phase angle.

For stochastic projection process due to “fast-clock” effect, the probability for a knot-piece is defined by the average of the projection angle. If there exists a knot solution at $\bar{x}(\bar{\theta})$ for a given projection angle $\bar{\theta}$, we call $\bar{\theta}$ the permitted projection angle; If there does not exist a knot solution for a given projected angle $\tilde{\theta}$, we call $\tilde{\theta}$ the unpermitted projection angle. Thus, the probability of a knot-piece is

obtained as $\frac{1}{2\pi} \int d\bar{\theta} = \frac{1}{N}$ where $\int d\bar{\theta} = \frac{2\pi}{N}$ denotes the summation of all permitted projected angles $\bar{\theta}$. For a system with a knot-piece, the probability for finding a knot (a zero between two projected vortex-membranes) is just its knot number, i.e., $\frac{1}{2\pi} \int d\bar{\theta} = \langle \hat{K} \rangle = \frac{1}{N}$. As a result, if we ignore the contribution from background (the knot crystal), the probability density for finding a knot $n_{\text{knot}}(\mathbf{x}, t) = \frac{1}{\Delta V_p} \cdot \frac{1}{2\pi} \int d\bar{\theta}$ is obtained as

$$n_{\text{knot}}(\mathbf{x}, t) = \left\langle \frac{\Delta \hat{K}}{\Delta V_p} \right\rangle \equiv \rho_{\text{knot}}(\mathbf{x}, t). \quad (362)$$

This result indicates that the probability density for finding a knot is equal to the knot density $\rho_{\text{knot}}(\mathbf{x}, t)$.

5.4.4. Probability interpretation for wavefunctions

We point out that the function of a Kelvin wave with a fragmentized knot describes the distribution of the knot-pieces and plays the role of the wavefunction in quantum mechanics as

$$\psi(\mathbf{x}, t) \Longleftrightarrow \sqrt{\rho_{\text{knot}}(\mathbf{x}, t)} e^{i\Delta\phi(\mathbf{x}, t)}, \quad (363)$$

and

$$\rho_{\text{knot}}(\mathbf{x}, t) \Longleftrightarrow n_{\text{knot}}(\mathbf{x}, t), \quad (364)$$

where the function of Kelvin wave with a fragmentized knot becomes the wavefunction $\psi(\mathbf{x}, t)$ in emergent quantum mechanics, the angle $\Delta\phi_B(x, t)$ becomes the quantum phase angle of wavefunction, the knot density $\rho_{\text{knot}} = \langle \frac{\Delta \hat{K}}{\Delta V_p} \rangle$ becomes the probability density of knots $n_{\text{knot}}(\mathbf{x})$. Thus, the measurement is to find the zero between two projected vortex-membranes that occurs at certain fragmentized knot with probability in a given region, $\psi^*(\mathbf{x}, t)\psi(\mathbf{x}, t)\Delta V_p$.

In particular, we emphasize knot has holographic property. Although a knot-piece has a probability of $\rho_{\text{knot}} = \frac{1}{N}$, to find a knot, not only its tensor state is same as that of a knot, but also its mass is m_{knot} rather than m_{knot}/N .

5.4.5. Complementarity principle

In emergent quantum mechanics, complementarity principle comes from complementarity property of knots. On the one hand, a knot piece is phase-changing — a sharp, time-independent, topological phase-changing, $\Delta\phi \neq 0$. To count the knot number, we had introduced knot-number operator $\hat{K} = -i\frac{d}{d\phi}$; on the other hand, a knot-piece has a fixed phase angle, ϕ_0 that is determined by perturbative Kelvin waves on the vortex-membranes. One cannot exactly determine the phase angle of a knot-piece by observing its phase-changing. We call this property to be complementarity principle in emergent quantum mechanics that leads to uncertainty principle. Based on the complementarity principle, we discuss wave-particle duality and uncertainty principle.

Wave-particle duality is the fact that elementary particles exhibit both particle-like behavior and wave-like behavior. As Einstein wrote: “*It seems as though we must use sometimes the one theory and sometimes the other, while at times we may use either. We are faced with a new kind of difficulty. We have two contradictory pictures of reality; separately neither of them fully explains the phenomena of light, but together they do*”. Here, we point out that wave-particle duality of quantum particles is really “information-motion duality” that is the property of complementarity principle. On the one hand, a knot is information unit that is a sharp, fragmentized, topological phase-changing in physics and would become a “point” after projection in mathematics. Thus, it shows particle-like behavior; on the other hand, the dynamic, smooth, nontopological phase-changing from perturbative Kelvin waves shows wave-like behavior that is characterized by wave-functions. In other words, the Kelvin waves look like pilot-waves,¹³ on which knot-pieces stay.

For emergent quantum mechanics, the uncertainty principle is related to the information nature of knots. From the point of view of information, the momentum denotes the spatial distribution of knot-pieces (or information), the energy denotes the temporal distribution of knot-pieces (or information). On the one hand, a uniform distribution of knot-pieces generated by $\hat{U}_{\text{fragment}}$ on knot-crystal with an excited Kelvin wave $\psi(x, t) = e^{-i\omega t + i\mathbf{k} \cdot \mathbf{x}}$ is described by a wavefunction of a plane wave with a fixed projected momentum $\mathbf{p}_{\text{knot}} = \hbar_{\text{knot}} \mathbf{k}$. For this case, we know momentum information of the knot but we do not know its position information; on the other hand, a unified knot generated by $\hat{U}_{\text{Unified-knot}}$ is really a special distribution of knot-pieces and can be regarded as a superposition state of $\psi(x, t) \sim \sum_{\mathbf{k}} e^{-i\omega t + i\mathbf{k} \cdot \mathbf{x}}$. For this case, we know the position information of the knot but we do not know its momentum information.

5.4.6. Applications: the Schrödinger’s cat paradox and the double-slit experiment

From the above discussion, we show the answer to Schrödinger’s cat paradox. The stochastic and uncertainty come from dynamic projection and fragmentized knot. A cat is a classical object with slow inner clock. The time scale of a cat is the same order of the detecting objects. There is no dynamic projection process and all information of a cat can be obtained. So there is no Schrödinger’s cat paradox at all.

We then give an explanation on the double-slit experiment. Before measurement in double-slit experiment, the knot can be regarded as fragmentized knot on a perturbative Kelvin wave, of which the probability distribution is described by the wavefunction. It has no classical path. There exists particular interference pattern on the screen that agrees to the prediction from quantum mechanics. However, after measurement, the phase angle of knot-pieces becomes randomized. As a result, the phase coherence is destructed and the interference disappears.

5.5. Quantum entanglement theory

At the end of the section, we address the issue of quantum entanglement. Quantum entanglement is a physical phenomenon for two knots such that the quantum state of each knot cannot be described independently of the others, even when the particles are separated by a large distance. An entangled state for n -body quantum system comes from a new type of knots — *composite knot* with n zeros. We call it (composite) $m\pi$ -knots that change the volume of vortex-membranes via $\pm mV_{\text{knot}}$. Here, m may be not equal to n . For a fragmentized $m\pi$ -knot, the distribution of its pieces is described by its “wavefunctions”. In general, the quantum states for a fragmentized $m\pi$ -knot cannot be reduced into a product state of the wavefunction for n knots.

Firstly, we study entangled states for two knots.

We consider two spin degrees of freedom for a knot $\begin{pmatrix} |\uparrow\rangle_{\text{knot}} \\ |\downarrow\rangle_{\text{knot}} \end{pmatrix}$. On the basis of

$$\begin{pmatrix} |\uparrow\rangle_{\text{knot},1} \otimes |\uparrow\rangle_{\text{knot},2} \\ |\uparrow\rangle_{\text{knot},1} \otimes |\downarrow\rangle_{\text{knot},2} \\ |\downarrow\rangle_{\text{knot},1} \otimes |\uparrow\rangle_{\text{knot},2} \\ |\downarrow\rangle_{\text{knot},1} \otimes |\downarrow\rangle_{\text{knot},2} \end{pmatrix}, \quad (365)$$

an arbitrary entangled state for two knots is given by

$$\begin{pmatrix} b_1^{1,2} \\ b_2^{1,2} \\ b_3^{1,2} \\ b_4^{1,2} \end{pmatrix}, \quad (366)$$

where $b_i^{1,2}$ ($i = 1, 2, 3, 4$) denotes the weights for different elements. For example, Bell states are defined as

$$\frac{1}{\sqrt{2}}(|\uparrow\rangle_{\text{knot},1} \otimes |\uparrow\rangle_{\text{knot},2} \pm |\downarrow\rangle_{\text{knot},1} \otimes |\downarrow\rangle_{\text{knot},2}), \quad (367)$$

or

$$\frac{1}{\sqrt{2}}(|\uparrow\rangle_{\text{knot},1} \otimes |\downarrow\rangle_{\text{knot},2} \pm |\downarrow\rangle_{\text{knot},1} \otimes |\uparrow\rangle_{\text{knot},2}). \quad (368)$$

We then define the composite knots for each state of the basis $|\uparrow\rangle_{\text{knot},1} \otimes |\uparrow\rangle_{\text{knot},2}$, $|\uparrow\rangle_{\text{knot},1} \otimes |\downarrow\rangle_{\text{knot},2}$, $|\downarrow\rangle_{\text{knot},1} \otimes |\uparrow\rangle_{\text{knot},2}$, $|\downarrow\rangle_{\text{knot},1} \otimes |\downarrow\rangle_{\text{knot},2}$:

(1) The function for unified $|\uparrow\rangle_{\text{knot},1} \otimes |\uparrow\rangle_{\text{knot},2}$ 2π -knot is given by

$$Z_{\uparrow\uparrow,2\pi\text{-knot}} = r(x) \exp[i\phi_{\uparrow\uparrow,2\pi\text{-knot}}(x, t)], \quad (369)$$

with

$$r(x) \rightarrow \begin{cases} 0, & x \in (-\infty, x_0] \\ r_0, & x \in (x_0, x_0 + 2a] \\ 0, & x \in (x_0 + 2a, \infty) \end{cases} \quad (370)$$

Fig. 13. (Color online) (a) A unified knot with 1 zero; (b) A unified 2π -knot with 2 correlated zeros; (c) A unified 3π -knot with correlated 3 zeros. Each crossing between two vortex-lines corresponds to a zero.

and

$$\phi_{\uparrow\uparrow, 2\pi\text{-knot}}(x) = \begin{cases} -\phi_0, & x \in (-\infty, x_0] \\ -\phi_0 + k_0(x - x_0), & x \in (x_0, x_0 + 2a] \\ -\phi_0 - 2\pi, & x \in (x_0 + 2a, \infty) \end{cases}, \quad (371)$$

where $k_0 = \frac{\pi}{a}$ and ϕ_0 is constant. We denote this state by $|\uparrow\uparrow\rangle_{2\pi\text{-knot}}$. See the illustration of a 2π -knot in Fig. 13(b).

(2) The function for unified $|\downarrow\rangle_{\text{knot},1} \otimes |\downarrow\rangle_{\text{knot},2}$ 2π -knot is given by

$$z_{\downarrow\downarrow, 2\pi\text{-knot}} = r(x) \exp[i\phi_{\downarrow\downarrow, 2\pi\text{-knot}}(x, t)], \quad (372)$$

with

$$r(x) \rightarrow \begin{cases} 0, & x \in (-\infty, x_0] \\ r_0, & x \in (x_0, x_0 + 2a] \\ 0, & x \in (x_0 + 2a, \infty) \end{cases} \quad (373)$$

and

$$\phi_{\downarrow\downarrow, 2\pi\text{-knot}}(x) = \begin{cases} \phi_0, & x \in (-\infty, x_0] \\ \phi_0 - k_0(x - x_0), & x \in (x_0, x_0 + 2a] \\ \phi_0 - 2\pi, & x \in (x_0 + 2a, \infty) \end{cases}. \quad (374)$$

We denote this state by $|\downarrow\downarrow\rangle_{2\pi\text{-knot}}$.

(3) The function for unified $|\uparrow\rangle_{\text{knot},1} \otimes |\downarrow\rangle_{\text{knot},2}$ 0-knot is given by

$$z_{\uparrow\downarrow, 0\text{-knot}} = r(x) \exp[i\phi_{\uparrow\downarrow, 0\text{-knot}}(x, t)], \quad (375)$$

with

$$r(x) \rightarrow \begin{cases} 0, & x \in (-\infty, x_0] \\ r_0, & x \in (x_0, x_0 + 2a] \\ 0, & x \in (x_0 + 2a, \infty) \end{cases} \quad (376)$$

and

$$\phi_{\uparrow\downarrow,0\text{-knot}}(x) = \begin{cases} -\phi_0, & x \in (-\infty, x_0] \\ -\phi_0 + k_0(x - x_0), & x \in (x_0, x_0 + a] \\ -\phi_0 + \pi - k_0(x - x_0), & x \in (x_0 + a, x_0 + 2a] \\ -\phi_0, & x \in (x_0 + 2a, \infty) \end{cases}. \quad (377)$$

We denote this state by $|\uparrow\downarrow\rangle_{0\text{-knot}}$.

(4) The function for unified $|\downarrow\rangle_{\text{knot},1} \otimes |\uparrow\rangle_{\text{knot},2}$ 0-knot is given by

$$z_{\downarrow\uparrow,0\text{-knot}} = r(x) \exp[i\phi_{\downarrow\uparrow,0\text{-knot}}(x, t)], \quad (378)$$

with

$$r(x) \rightarrow \begin{cases} 0, & x \in (-\infty, x_0] \\ r_0, & x \in (x_0, x_0 + 2a] \\ 0, & x \in (x_0 + 2a, \infty) \end{cases} \quad (379)$$

and

$$\phi_{\downarrow\uparrow,0\text{-knot}}(x) = \begin{cases} \phi_0, & x \in (-\infty, x_0] \\ \phi_0 - k_0(x - x_0), & x \in (x_0, x_0 + a] \\ \phi_0 - \pi + k_0(x - x_0 - a), & x \in (x_0 + a, x_0 + 2a] \\ \phi_0, & x \in (x_0 + 2a, \infty) \end{cases}. \quad (380)$$

We denote this state by $|\downarrow\uparrow\rangle_{0\text{-knot}}$.

We can see that the entangled knots can be regarded as a unified object with two correlated zeros and fixed volume of vortex-membranes $\pm 2V_{\text{knot}}$. For example, the Bell state $|\uparrow\rangle_1 \otimes |\uparrow\rangle_2 + |\downarrow\rangle_1 \otimes |\downarrow\rangle_2$ is just $\frac{1}{\sqrt{2}}(|\uparrow\rangle_{2\pi\text{-knot}} + |\downarrow\rangle_{2\pi\text{-knot}})$. So after each dynamic projection process, there exist two zeros. When the properties of a knot (a zero) are detected, the other is exactly known. The nonlocality of quantum entangled states comes from the nonlocality of a perturbative Kelvin wave and the fixed phase difference between the knot-pieces of two knots. We point out that there exist different choices for an entangled state for two knots by considering different types of composite knots.

The concept of composite knot can be generalized to the entangled states for n knots. For entangled states of n knots, on the basis of

$$\begin{pmatrix} |\uparrow\rangle_{\text{knot},1} \otimes |\uparrow\rangle_{\text{knot},2} \otimes |\uparrow\rangle_{\text{knot},3} \cdots \otimes |\uparrow\rangle_{\text{knot},n} \\ |\downarrow\rangle_{\text{knot},1} \otimes |\uparrow\rangle_{\text{knot},2} \otimes |\uparrow\rangle_{\text{knot},3} \cdots \otimes |\uparrow\rangle_{\text{knot},n} \\ |\uparrow\rangle_{\text{knot},1} \otimes |\downarrow\rangle_{\text{knot},2} \otimes |\uparrow\rangle_{\text{knot},3} \cdots \otimes |\uparrow\rangle_{\text{knot},n} \\ \vdots \\ |\downarrow\rangle_{\text{knot},1} \otimes |\downarrow\rangle_{\text{knot},2} \otimes |\downarrow\rangle_{\text{knot},3} \cdots \otimes |\downarrow\rangle_{\text{knot},n} \end{pmatrix}, \quad (381)$$

an arbitrary entangled state for n knots is given by

$$\begin{pmatrix} b_1^{1,n} \\ b_2^{1,n} \\ b_3^{1,n} \\ \dots \\ b_{2^n}^{1,n} \end{pmatrix}, \quad (382)$$

where $b_i^{1,n}$ ($i = 1, 2, 3, \dots, 2^n$) denotes the weights for different elements. We could define 2^n different types of composite knots to characterize an arbitrary entangled state of n knot.

We take the states $|\uparrow\rangle_{\text{knot},1} \otimes |\uparrow\rangle_{\text{knot},2} \otimes |\uparrow\rangle_{\text{knot},3} \dots \otimes |\uparrow\rangle_{\text{knot},n}$ and $|\downarrow\rangle_{\text{knot},1} \otimes |\downarrow\rangle_{\text{knot},2} \otimes |\downarrow\rangle_{\text{knot},3} \dots \otimes |\downarrow\rangle_{\text{knot},n}$ as examples: The function for the unified $|\uparrow\rangle_{\text{knot},1} \otimes |\uparrow\rangle_{\text{knot},2} \otimes |\uparrow\rangle_{\text{knot},3} \dots \otimes |\uparrow\rangle_{\text{knot},n}$ $n\pi$ -knot is given by

$$Z_{\uparrow\uparrow\dots\uparrow,n\pi\text{-knot}} = r_0 \exp[i\phi_{\uparrow\uparrow\dots\uparrow,n\pi\text{-knot}}(x, t)], \quad (383)$$

with

$$r(x) \rightarrow \begin{cases} 0, & x \in (-\infty, x_0] \\ r_0, & x \in (x_0, x_0 + na] \\ 0, & x \in (x_0 + na, \infty) \end{cases} \quad (384)$$

and

$$\phi_{\uparrow\uparrow\dots\uparrow,n\pi\text{-knot}}(x) = \begin{cases} -\phi_0, & x \in (-\infty, x_0] \\ -\phi_0 + k_0(x - x_0), & x \in (x_0, x_0 + na] \\ -\phi_0 + n\pi, & x \in (x_0 + na, \infty) \end{cases}. \quad (385)$$

We denote this state by $|\uparrow\uparrow\dots\uparrow\rangle_{n\pi\text{-knot}}$, the function for the unified $|\downarrow\rangle_{\text{knot},1} \otimes |\downarrow\rangle_{\text{knot},2} \otimes |\downarrow\rangle_{\text{knot},3} \dots \otimes |\downarrow\rangle_{\text{knot},n}$ $n\pi$ -knot is given by

$$Z_{\downarrow\downarrow\dots\downarrow,n\pi\text{-knot}} = r_0 \exp[i\phi_{\downarrow\downarrow\dots\downarrow,n\pi\text{-knot}}(x, t)], \quad (386)$$

with

$$r(x) \rightarrow \begin{cases} 0, & x \in (-\infty, x_0] \\ r_0, & x \in (x_0, x_0 + na] \\ 0, & x \in (x_0 + na, \infty) \end{cases} \quad (387)$$

and

$$\phi_{\downarrow\downarrow\dots\downarrow,n\pi\text{-knot}}(x) = \begin{cases} \phi_0, & x \in (-\infty, x_0] \\ \phi_0 - k_0(x - x_0), & x \in (x_0, x_0 + na] \\ \phi_0 - n\pi, & x \in (x_0 + na, \infty) \end{cases}. \quad (388)$$

We denote this state by $|\downarrow\downarrow\dots\downarrow\rangle_{n\pi\text{-knot}}$.

The GHZ state for n knots

$$\frac{1}{\sqrt{2}}(|\uparrow\rangle_{\text{knot},1} \otimes |\uparrow\rangle_{\text{knot},2} \otimes |\uparrow\rangle_{\text{knot},3} \cdots \otimes |\uparrow\rangle_{\text{knot},n} \pm |\downarrow\rangle_{\text{knot},1} \otimes |\downarrow\rangle_{\text{knot},2} \otimes |\downarrow\rangle_{\text{knot},3} \cdots \otimes |\downarrow\rangle_{\text{knot},n}). \quad (389)$$

is just $\frac{1}{\sqrt{2}}(|\uparrow\uparrow\cdots\uparrow\rangle_{n\pi\text{-knot}} \pm |\downarrow\downarrow\cdots\downarrow\rangle_{n\pi\text{-knot}})$. From the point of view of entanglement, a 1D knot-crystal can be regarded as a big unified $(N_{\text{knot}}\pi)$ -knots with N_{knot} correlated zeros and fixed volume of vortex-membranes $\pm N_{\text{knot}}V_{\text{knot}}$, of which the entangled state can be $|\uparrow\uparrow\cdots\uparrow\rangle_{N_{\text{knot}}\pi\text{-knot}}$. This interpretation gives another explanation on information scaling from knot-crystal to knot.

6. Conclusion and Summary

In the end, we draw the conclusion. Figure 14 shows the correspondence between the Biot–Savart mechanics for knots and emergent quantum mechanics for particles. Owing to the conservation conditions of the volume of the knot in 5D space, the shape and the volume of knot are never changed and the knot can only split and knot-pieces evolve following the equation of motion of Schrödinger equation. Three-dimensional quantum Dirac model is obtained to effectively describe the deformed vortex-membranes: The elementary excitations are knots with information unit; the physics quality to describe local deformation is knot density that is defined by the density of zeros between two projected vortex-membranes; the Biot–Savart

Biot-Savart mechanics	Emergent quantum mechanics
Knot-crystal:	Massive Dirac model, $i\bar{\psi}\gamma_{\mu}\partial_{\mu}\psi + m\bar{\psi}\psi$
Knot with a zero,	Fermionic particle, $\{\psi, \psi^+\} = 1$
Biot-Savart equation: $i\frac{dZ_{\text{knot}}}{dt} = \frac{\delta H(z)}{\delta Z_{\text{knot}}^*}$	Schrödinger equation for Fermions particles: $i\hbar_{\text{knot}}\frac{d\psi(x,t)}{dt} = \hat{H}_{\text{knot}}\psi(x,t)$
Distribution function for knot-pieces: $Z_{\text{knot}}(x,t)$	Wave function for probability wave of knots: $\psi(x,t)$
Knot density (the density of knot-pieces), ρ_{knot}	Probability density of quantum particles (knots), $\bar{\psi}(x,t)\psi(x,t)$
Projected (Lamb impulse) momentum of a single knot	Momentum operator, $-i\hbar_{\text{knot}}\frac{d}{dx}$
Projected (Lamb impulse) energy of a single knot	Energy operator, $i\hbar_{\text{knot}}\frac{d}{dt}$
Projected (Lamb impulse) angular momentum of a single knot	Planck constant, $\hbar_{\text{knot}}$

Fig. 14. (Color online) Correspondence between Biot–Savart mechanics and emergent-quantum mechanics.

equation for Kelvin waves becomes Schrödinger equation for probability waves of knots; the classical functions for perturbative Kelvin waves become wave-functions for knots; the probability of quantum mechanics comes from dynamic projection for inner observers owing to “fast-clock” effect; the angular frequency for leapfrogging motion plays the role of the mass of knots, etc. This work would help people to understand the mystery of quantum mechanics.

Finally, there are several unsolved problems. In this paper, we only consider the perturbative effect on the Kelvin waves. According to Hasimoto, an important result after considering the nonlinear effect is the existence of soliton solutions by mapping the Biot–Savart equation to 1D nonlinear Schrödinger equation via Hasimoto representation.³¹ The generalization of Hasimoto representation to 3D knot-crystal may be interesting. In addition, after considering the dissipation effect, the Kelvin waves are subject to Kolmogorov-like turbulence (even in quantum fluid^{8,9}). The stability of 3D knot-crystals and the possible cascade of Kelvin waves to turbulence are also open issues. From the point of view of emergent quantum field theory, the emergence of gauge fields and gauge interaction on a knot-crystal is a critical problem. In the future, we will study these problems and develop a complete theory for emergent quantum physics.

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This work is supported by NSFC Grant No. 11674026. Because it was Kelvin (Sir W. Thomson) who firstly proposed the two key concepts in this paper (dynamic knots for elementary particles and Kelvin waves). I wish to express my gratitude to this wonderful physicist who worked during the 19th century.

Appendix A. Theory of Kelvin Waves for 1D Vortex-Lines

Vortex-lines are one-dimensional topological objects in three-dimensional superfluid (SF). For SF with a vortex-line, a rotationless superfluid component flow $\nabla \times \mathbf{v} = 0$ is violated on one-dimensional singularities $\mathbf{s}(t)$, which depends on the variables — arc length and the time t . Away from the singularities, the velocity increases to infinity so that the circulation of the superfluid velocity remains constant,

$$\oint \mathbf{v} \cdot d\mathbf{l} = \kappa,$$

where $\mathbf{v} = \dot{\mathbf{s}}$. As a result, the superfluid vortex filament could be described by the Biot–Savart equation

$$\dot{\mathbf{r}} = \frac{\kappa}{4\pi} \int \frac{d\mathbf{s} \times (\mathbf{r} - \mathbf{s})}{|\mathbf{r} - \mathbf{s}|^3}, \quad (\text{A.1})$$

where $\kappa = h/m_4$ (for ^4He) or h/m_3 (for ^3He) is the discreteness of circulation. $\mathbf{r}$ is the vector that denotes the position of vortex filament and $\mathbf{s}$ denotes the integral variable. This means that the singularity of $\mathbf{v}(\mathbf{r})$ comes from integrating around the vicinity part of $\mathbf{s}_0$.

A.1. Local induction approximation

In Cartesian coordinate system, the position vector of vortex filament is $\mathbf{s} = (x, y, z)$. By using the natural parametrizations, it is convenient to express $\mathbf{s}$ in form of $\mathbf{s}(\xi)$, in which ξ is the (algebraic) arc length measured from the point $\mathbf{s}_0$. In the vicinity of $\mathbf{s}_0$, we can expand the function $\mathbf{s}(\xi)$ as

$$\mathbf{s}(\xi) = \mathbf{s}_0 + \mathbf{s}'\xi + \mathbf{s}''\xi^2/2 + \cdots \quad (\text{A.2})$$

and

$$d\mathbf{s} = \mathbf{s}'d\xi + \mathbf{s}''\xi d\xi + \cdots \quad (\text{A.3})$$

in which $\mathbf{s}' = \frac{\partial \mathbf{s}}{\partial \xi}$ and $\mathbf{s}'' = \frac{\partial^2 \mathbf{s}}{\partial \xi^2}$, respectively. From the geometrical point of view, $\mathbf{s}'$ is the unit vector tangent to vortex filament, $\mathbf{s}''$ is the vector normal to vortex filament.

When $\mathbf{r} \rightarrow \mathbf{s}_0$, we can separate the integral into two parts: One is in the vicinity of $\mathbf{s}_0$, the other is regular part without singularity, i.e.,

$$\begin{aligned} \mathbf{v}(\mathbf{r} \rightarrow \mathbf{s}_0) &= \frac{\kappa}{4\pi} \int_{-\xi_0}^{\xi_0} d\xi \frac{(\mathbf{s}''\xi + \mathbf{s}') \times (\mathbf{s}_0 - \mathbf{s})}{|\mathbf{s}_0 - \mathbf{s}|^3} + \text{regular} \\ &\approx \frac{\kappa}{4\pi} \int_{-\xi_0}^{\xi_0} d\xi \frac{(\mathbf{s}''\xi + \mathbf{s}') \times (\mathbf{s}_0 - \mathbf{s}_0 - \mathbf{s}'\xi - \mathbf{s}''\xi^2/2)}{|\mathbf{s}_0 - \mathbf{s}_0 - \mathbf{s}'\xi|^3} + \text{regular} \\ &= \frac{\kappa}{4\pi} \int_{-\xi_0}^{\xi_0} d\xi \frac{(\mathbf{s}''\xi + \mathbf{s}') \times (-\mathbf{s}'\xi - \mathbf{s}''\xi^2/2)}{|\mathbf{s}'\xi|^3} + \text{regular} \\ &= \frac{\kappa}{4\pi} \int_{-\xi_0}^{\xi_0} d\xi \frac{-\mathbf{s}'' \times \mathbf{s}'\xi^2 - \mathbf{s}' \times \mathbf{s}''\xi^2/2}{|\mathbf{s}'\xi|^3} + \text{regular} \\ &= \frac{\kappa}{8\pi|\mathbf{s}'|^3} \mathbf{s}' \times \mathbf{s}'' \int_{-\xi_0}^{\xi_0} \frac{d\xi}{|\xi|} + \text{regular}, \end{aligned} \quad (\text{A.4})$$

where ξ_0 is the length of the order of the curvature radius (or inter-vortex distance when the considered vortex filament is a part of a vortex tangle).

By using $\int_{-\xi_0}^{\xi_0} \frac{d\xi}{|\xi|} \rightarrow 2 \ln(\xi_0/a_0)$, we reduce the Biot-Savart equation by local induction approximation,

$$\begin{aligned} \mathbf{v}(\mathbf{r} \rightarrow \mathbf{s}_0) &\approx \frac{\kappa}{4\pi|\mathbf{s}'|^3} \mathbf{s}' \times \mathbf{s}'' \ln(\xi_0/a_0) + \text{regular} \\ &\approx \frac{\kappa}{4\pi} \mathbf{s}' \times \mathbf{s}'' \ln(\xi_0/a_0) \\ &= A\mathbf{s}' \times \mathbf{s}'' \end{aligned} \quad (\text{A.5})$$

or

$$\frac{\partial \mathbf{s}}{\partial t} = A\mathbf{s}' \times \mathbf{s}'', \quad (\text{A.6})$$

where

$$A = \frac{\kappa}{4\pi} \ln(\xi_0/a_0) \quad (\text{A.7})$$

and a_0 denotes the vortex filament radius that is much smaller than any other characteristic size in the system.

Appendix B. Physical Quantities of Kelvin Waves

Next, we discuss the physical quantities of the Kelvin waves along winding direction (z -direction), i.e., the projected momentum p_z and the projected angular momentum J_z . For a plane Kelvin wave of a lightly deformed straight vortex-line, the function is described by

$$x = a \cos(kz - \omega t), \quad y = a \sin(kz - \omega t). \quad (\text{B.1})$$

In SF with a vortex-line, the conventional momentum of the superfluid motion cannot be well defined. Instead, the hydrodynamic impulse (the Lamb impulse) plays the role of the effective momentum that denotes the total mechanical impulse of the nonconservative body force applied to a limited fluid volume to generate instantaneously from rest the given motion of the whole of the fluid at time t . In general, the (effective) momentum for a vortex-line from Lamb impulse density is defined by

$$\mathbf{P}_{\text{Lamb}} = \frac{1}{2} \rho_0 \int (\mathbf{s} \times \boldsymbol{\omega}) dV,$$

where $\boldsymbol{\omega} = \nabla \times \mathbf{v}$.

By using the definition of a vortex-line

$$\boldsymbol{\omega} = \kappa \int \mathbf{s}'(\xi) \delta(\mathbf{r} - \mathbf{s}(\xi, t)) d\xi,$$

we have

$$\begin{aligned} \mathbf{P}_{\text{Lamb}} &= \frac{\rho_0 \kappa}{2} \int \mathbf{s} \times d\mathbf{s} = \frac{\rho_0 \kappa}{2} \int \mathbf{s} \times \mathbf{s}' d\xi \\ &= \frac{\rho_0 \kappa}{2} \int [x(z, t) \mathbf{e}_x + y(z, t) \mathbf{e}_y + z(t) \mathbf{e}_z] \times (x_z \mathbf{e}_x + y_z \mathbf{e}_y + \mathbf{e}_z) z_\xi d\xi \\ &= \frac{\rho_0 \kappa}{2} \int [(y - zy_z) \mathbf{e}_x + (zx_z - x) \mathbf{e}_y + (xy_z - yx_z) \mathbf{e}_z] z_\xi d\xi. \end{aligned}$$

The projected (Lamb impulse) momentum along z -direction p_z of a vortex-line with a Kelvin wave is obtained as

$$p_z = \mathbf{P}_{\text{Lamb}} \cdot \mathbf{e}_z = \frac{\rho_0 \kappa}{2} \int (xy_z - yx_z) z_\xi d\xi = \frac{1}{2} \rho_0 \kappa l a^2 k, \quad (\text{B.2})$$

where l is length of the system along z -direction.

In general, the (Lamb impulse) angular momentum for a vortex-line is defined by

$$\begin{aligned}
 \mathbf{J}_{\text{Lamb}} &= \frac{\rho_0 \kappa}{2} \int \mathbf{s} \times (\mathbf{s} \times d\mathbf{s}) \\
 &= \frac{\rho_0 \kappa}{2} \int \mathbf{s} \times (\mathbf{s} \times \mathbf{s}') d\xi \\
 &= \frac{\rho_0 \kappa}{2} \int (xyy_z - y^2 x_z - z^2 x_z + zx) \mathbf{e}_x + (yz - z^2 y_z - x^2 y_z + xyx_z) \mathbf{e}_y \\
 &\quad + (xxz_z - x^2 - y^2 + yzy_z) \mathbf{e}_z dz,
 \end{aligned}$$

where ρ_0 is the superfluid mass density. Along z -direction, the projected (Lamb impulse) angular momentum of a vortex-line with a Kelvin wave is given by

$$\begin{aligned}
 J_z &= \mathbf{J}_{\text{Lamb}} \cdot \mathbf{e}_z \\
 &= \frac{\rho_0 \kappa}{2} \int (xxz_z - x^2 - y^2 + yzy_z) dz \\
 &= \frac{\rho_0 \kappa}{2} \int (-a^2 kz \cos(kz - \omega t) \sin(kz - \omega t) \\
 &\quad - a^2 + a^2 kz \cos(kz - \omega t) \sin(kz - \omega t)) dz \\
 &= -\frac{1}{2} \rho_0 \kappa l a^2.
 \end{aligned} \tag{B.3}$$

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