Time-dependent interaction between a two-level atom and a $su(1, 1)$ Lie algebra quantum system

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The problem of the interaction between a two-level atom and a two-mode field in the parametric amplifier-type is considered. A similar problem appears in an ion trapped in a two-dimensional trap. The problem is transformed into an interaction governed by $su(1, 1)$ Lie algebraic operators with phase and coupling parameter depending on time. Under an integrability condition, that relates phase and coupling, a solution to the wavefunction is obtained using the Schrödinger equation. The effects of the functional dependence of the coupling and the initial state of the two-level atom on atomic inversion, the degree of entanglement, the fidelity and the Glauber second-order correlation function are investigated. It is shown that the acceleration term plays an important role in controlling the function behavior of the considered quantities.

Keywords: Interaction between two quantum systems; wavefunction; entanglement; correlation function.

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1. Introduction

Since the middle of last century, field-field, atom-field and atom-atom interactions have been regarded as fundamental problems in nonlinear optical phenomena.

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Furthermore, the interaction between three electromagnetic fields also represents an important nonlinear parametric interaction in quantum optics.^{1–10} This type of interaction plays a significant role in many relevant physical phenomena, such as stimulated and spontaneous emission of radiation and coherent Raman and Brillouin scattering.^{1–3} In the field of optics, two different interactions occur: frequency conversion and parametric amplification. In a recent experiment, Bodani *et al.*¹¹ produced an interaction between five modes in a nonlinear crystal, which is suitable for generating a three-mode entangled state. Concurrently, researchers generated squeezed light using nondegenerate parametric down conversion to excite a two-photon transition in atomic cesium.^{12–14} Theoretically, to obtain these types of processes, we have to introduce a Hamiltonian model consisting of a three-wave interaction. In the field of quantum optics, it is well known that the susceptibility $\chi^{(2)}$ leads to cubic nonlinearity, which is responsible for three-wave mixing processes in parametric devices, as well as second harmonic and subharmonic generation.¹⁵ Note that the original coupling scheme between matter (two-level atoms) and radiation is the Rabi coupling which reduces to Jayns–Cumming coupling when the appropriate resonance condition is assumed.^{16,17} In general, solving Hamiltonian becomes possible in the absence of resonance case, see for example Ref. 18. Also, it is interesting to refer to the work by authors of Ref. 19 where they introduced a generalized model for (JC) closely related to the present work in the absence of the time dependence.¹⁷

The most relevant model that describes the above phenomena in terms of boson operators is the trilinear Hamiltonian^{20–23}

$$\frac{\hat{H}}{\hbar} = \sum_{i=1}^3 \omega_i \hat{a}_i^\dagger \hat{a}_i + \lambda (\hat{a}_1^\dagger \hat{a}_2 \hat{a}_3 + \hat{a}_1 \hat{a}_2^\dagger \hat{a}_3^\dagger), \quad (1)$$

where $\hat{a}_i^\dagger$ and $\hat{a}_i$ are the boson creation and annihilation operators of the i th mode with angular frequency ω_i . These operators satisfy the commutation relations

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}, \quad i, j = 1, 2, 3, \quad (2)$$

where δ_{ij} is the Kronecker delta with the properties $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ otherwise and λ is the nonlinear coupling parameter that generally depends on the susceptibility $\chi^{(2)}$. This has applications in connection with phenomena related to optical phase conjugation.^{24,25} Furthermore, it is relevant to the stimulated Raman and Brillouin scattering as well as parametric coupling of Stokes and anti-Stokes radiation. For example, in Brillouin scattering, an intense monochromatic laser source induces parametric coupling between two scattered electromagnetic fields and the acoustical phonons in the scattering medium.²⁶ In Raman scattering, similar coupling occurs between the scattered Stokes and anti-Stokes waves and the optical phonons of the Raman-active medium.^{3,4,27–30} Therefore, it is difficult to determine the time-dependent dynamical operators or the solution of the Schrödinger wavefunction for the trilinear Hamiltonian (1), so the operator $\hat{a}_1$ can be approximated

with its time-dependent c -number. In this case, we have the Hamiltonian (1) in the form

$$\frac{\hat{H}(t)}{\hbar} = \sum_{i=1}^2 \omega_i \hat{a}_i^\dagger \hat{a}_i + \lambda_1(t) (\hat{a}_2 \hat{a}_3 \exp(i\phi(t)) + \hat{a}_2^\dagger \hat{a}_3^\dagger \exp(-i\phi(t))), \quad (3)$$

where $\phi(t)$ and $\lambda_1(t)$ are the time-dependent phase and the coupling parameter, respectively. In this context, we may refer to the work by Louisell *et al.*, whom established that, in order to obtain a time-dependent coupling parameter between two fields within cavity, we have to use a time-dependent permeability, see Ref. 26.

It should be mentioned that $\text{su}(1,1)$ Lie algebra can be realized in terms of boson annihilation and creation operators, and it is isomorphic to the Lie algebra of the noncompact $\text{su}(1,1)$ group. Therefore, the above Hamiltonian can be written in terms of $\text{su}(1,1)$ Lie algebra generators $\hat{K}_-$, $\hat{K}_+$ and $\hat{K}_z$ that satisfy the commutation relations

$$[\hat{K}_-, \hat{K}_+] = 2\hat{K}_z, \quad [\hat{K}_z, \hat{K}_\pm] = \pm \hat{K}_\pm, \quad (4)$$

where

$$\hat{K}_- = \hat{a}_2 \hat{a}_3, \quad \hat{K}_+ = \hat{a}_2^\dagger \hat{a}_3^\dagger, \quad \hat{K}_z = \frac{1}{2}(\hat{n}_2 + \hat{n}_3 + 1) \quad (5)$$

with the corresponding Casimir operator $\hat{K}$ given by

$$\hat{K}^2 = \hat{K}_z^2 - \frac{1}{2}(\hat{K}_- \hat{K}_+ + \hat{K}_+ \hat{K}_-), \quad (6)$$

which has the eigenvalue $k(k-1)$, where k is the Bargmann index and $k = \frac{1}{2}(n_2 - n_3 + 1)$ in this case. The other operators have the following action on the eigenstate $|m; k\rangle$ of the two operators $\hat{K}^2$ and $\hat{K}_z$:

$$\begin{aligned} \hat{K}_z |m; k\rangle &= (m+k) |m; k\rangle, \quad \hat{K}^2 |m; k\rangle = k(k-1) |m; k\rangle, \\ \hat{K}_+ |m; k\rangle &= \sqrt{(m+1)(m+2k)} |m+1; k\rangle, \\ \hat{K}_- |m; k\rangle &= \sqrt{m(m+2k-1)} |m-1; k\rangle. \end{aligned} \quad (7)$$

Note that $\hat{K}_- |0; k\rangle = 0$, whereas m is any nonnegative integer. In this case, Eq. (3) for $\omega_2 + \omega_3 = \omega/2$ can be written in the form

$$\frac{\hat{H}(t)}{\hbar} = \frac{\omega}{2} \hat{K}_z + \lambda_1(t) (\hat{K}_- \exp(i\phi(t)) + \hat{K}_+ \exp(-i\phi(t))). \quad (8)$$

Note that we have ignored a term of the form $\frac{1}{2}(\omega_1 - \omega_2)(n_2 - n_3)$ which is related to the invariant k and hence commutes with all terms of Eq. (8). Suppose that we inject a two-level atom into a cavity containing a medium described by the above Hamiltonian. In terms of the rotating wave approximation, the Hamiltonian for such a system is cast in the following form:

$$\frac{\hat{H}(t)}{\hbar} = \frac{\omega}{2} \hat{K}_z + \frac{\omega_0}{2} \hat{\sigma}_z + \lambda_1(t) (\hat{K}_- \hat{\sigma}_+ \exp(i\phi(t)) + \hat{K}_+ \hat{\sigma}_- \exp(-i\phi(t))), \quad (9)$$

where $\hat{\sigma}_+$ ($\hat{\sigma}_-$) is the raising (lowering) operator, $\hat{\sigma}_z$ is the Pauli operator, while $\phi(t)$ is time-dependent phase pump. The frequency ω_0 is the energy difference between the two levels of the atom. Note that the operators $\hat{\sigma}_\pm$ and $\hat{\sigma}_z$ obey the commutation relations

$$[\hat{\sigma}_+, \hat{\sigma}_-] = \hat{\sigma}_z, \quad [\hat{\sigma}_z, \hat{\sigma}_\pm] = \pm 2\hat{\sigma}_\pm. \quad (10)$$

The Hamiltonian (9) can describe a two-level ion contained within a two-dimensional trap that is approximated as harmonic oscillators of frequencies ω_1 and ω_2 . The position of the center-of-mass of the trapped ion is given by $(\hat{x}, \hat{y})$ quantized by the operators $\hat{a}$, $\hat{a}^\dagger$ and $\hat{b}$, $\hat{b}^\dagger$ that annihilate and create quanta of the vibrational motion of the center-of-mass of the ion. A laser of frequency ω_c is directed along a direction in the xy -plane of the trap. The Hamiltonian of such system is given by³¹

$$\frac{\hat{H}}{\hbar} = \omega_1 \hat{a}^\dagger \hat{a} + \omega_2 \hat{b}^\dagger \hat{b} + \frac{\omega_0}{2} \hat{\sigma}_z + \mu \varepsilon(t) (\hat{\sigma}_+ \exp i(k_1 \hat{x} + k_2 \hat{y} - \omega_c t) + \text{h.c.}) \quad (11)$$

with μ the dipole moment of the transition between the two states of the ion and $\varepsilon(t)$ is amplitude and frequency modulation of the laser field

$$\hat{x} = \sqrt{\frac{\hbar}{2M\omega_1}} (\hat{a} + \hat{a}^\dagger) \quad \text{and} \quad \hat{y} = \sqrt{\frac{\hbar}{2M\omega_1}} (\hat{b} + \hat{b}^\dagger), \quad (12)$$

expanding the exponential terms in terms of powers of the operators $\hat{a}$, $\hat{a}^\dagger$, $\hat{b}$, $\hat{b}^\dagger$, and tuning the laser frequency ω_c to the frequency $\omega_0 + \omega_1 + \omega_2$, one reaches the following form^{27,28}

$$\frac{\hat{H}}{\hbar} = \omega_1 \hat{a}^\dagger \hat{a} + \omega_2 \hat{b}^\dagger \hat{b} + \frac{\omega_0}{2} \hat{\sigma}_z - \eta_1 \eta_2 \mu \varepsilon(t) \hat{a} \hat{b} \hat{\sigma}_+ + \text{h.c.}, \quad (13)$$

where $\eta_i = k_i (\hbar/2\omega_i M)^{\frac{1}{2}}$ is the Lamb-Dicke parameter. Taking $\omega_1 + \omega_2 = \omega/2$ and $\lambda_1 = -\eta_1 \eta_2 \mu |\varepsilon(t)|$ with the obvious definitions of $\hat{K}_-$ and $\hat{K}_+$ in terms of $\hat{a}$, $\hat{a}^\dagger$, $\hat{b}$, $\hat{b}^\dagger$ brings the above Hamiltonian to the form of Eq. (9). In order to find the wavefunction, we use the Heisenberg equation of motion to obtain constants of motion from which we are able to reach our goal.

The Heisenberg equation of motion for any dynamical operator $\hat{Q}$ is given by

$$\frac{d\hat{Q}}{dt} = \frac{1}{i\hbar} [\hat{Q}, \hat{H}] + \frac{\partial \hat{Q}}{\partial t}. \quad (14)$$

Therefore, the Heisenberg equations of motion for the operators appearing in Eq. (9) lead to the following equations:

$$\begin{aligned} \frac{d\hat{\sigma}_z}{dt} &= 2i\lambda_1(t) (\hat{K}_+ \hat{\sigma}_- \exp(-i\phi(t)) - \hat{K}_- \hat{\sigma}_+ \exp(i\phi(t))), \\ \frac{d\hat{K}_z}{dt} &= i\lambda_1(t) (\hat{K}_+ \hat{\sigma}_- \exp(-i\phi(t)) - \hat{K}_- \hat{\sigma}_+ \exp(i\phi(t))). \end{aligned} \quad (15)$$

Thus, we deduce that the operator $\hat{N} = \hat{K}_z - \hat{\sigma}_z/2$ is a constant of motion. The Hamiltonian [Eq. (9)] then becomes

$$\frac{\hat{H}(t)}{\hbar} = \frac{\omega}{2}\hat{N} + \frac{\Delta}{2}\hat{\sigma}_z + \lambda_1(t)(\hat{K}_-\hat{\sigma}_+ \exp(i\phi(t)) + \hat{K}_+\hat{\sigma}_- \exp(-i\phi(t))), \quad (16)$$

where $\Delta = \omega/2 + \omega_0$.

The main aim of the present work is to investigate some statistical properties that enable the behavior of some physical quantities to be investigated. The paper is organized as follows: Section 2 deals with the derivation of the wavefunction. The atomic inversion is dealt with in Sec. 3. Section 4 is devoted to a discussion of entanglement while Fidelity and correlation function are examined in Secs. 5 and 6. Finally, concluding remarks are drawn in Sec. 7.

2. Wavefunction

In this section, we determine the wavefunction so that we can construct the density matrix, from which we can calculate the expectation values of some dynamical operators. To accomplish this, we have to solve the Schrödinger equation

$$\frac{\hat{H}}{\hbar}|\psi\rangle = i\frac{\partial}{\partial t}|\psi\rangle. \quad (17)$$

For this reason, we define the initial wavefunction

$$|\psi(0)\rangle = (c_0|e\rangle + c_1|g\rangle) \sum_{m=0}^{\infty} q_{m,k}|m;k\rangle, \quad (18)$$

$$q_{m,k} = \sqrt{\left(\frac{|\eta|^{2k-1}}{I_{2k-1}(2\eta)}\right)} \frac{\eta^m}{\sqrt{m!\Gamma(2k+m)}},$$

where $|c_0|^2 + |c_1|^2 = 1$, $\sum_{m=0}^{\infty} q_{m,k}|m;k\rangle$ is the Barut–Girardello coherent state,³² where $I_n(2\eta)$ is the n th-order modified Bessel function.³³ Note that the Barut–Girardello coherent state is a pure state, while $c_0|e\rangle + c_1|g\rangle$ is the initial atomic state. The wavefunction at any time $t > 0$ can be written as

$$|\psi(t)\rangle = \sum_{m=0}^{\infty} A_m(t)|e\rangle|m;k\rangle + B_{m+1}(t)|g\rangle|m+1;k\rangle. \quad (19)$$

From Eqs. (17) and (19) together with Eq. (9), we have³⁴

$$i\frac{dA_m}{dt} = \frac{\Delta}{2}A_m + \lambda_1(t)F(m;k)\exp(i\phi(t))B_{m+1},$$

$$i\frac{dB_{m+1}}{dt} = -\frac{\Delta}{2}B_{m+1} + \lambda_1(t)F(m;k)\exp(-i\phi(t))A_m, \quad (20)$$

where

$$F(m;k) = \sqrt{(m+1)(m+2k)}. \quad (21)$$

If we define

$$\chi(t) = \phi(t) - \Delta t, \quad X_m(t) = A_m \exp\left(\frac{i\Delta}{2}t\right), \quad Y_{m+1} = B_{m+1} \exp\left(\frac{-i\Delta}{2}t\right), \quad (22)$$

then Eq. (20) has the form

$$\frac{dX_m}{dt} = -iF \lambda_1(t) \exp(i\chi) Y, \quad \frac{dY_{m+1}}{dt} = -iF \lambda_1(t) \exp(-i\chi) X. \quad (23)$$

The above two differential equations can be compared with that given in Ref. 35.

To obtain the general solution of these equations, we have to introduce the integrability condition, that is, $\dot{\chi}(t) = C\lambda_1(t)$. Note that the integrability condition can be obtained from the linear invariant, for more details see Refs. 35–39. In this case, we have

$$\begin{aligned} A_m(t) &= \exp\left[-\frac{i}{2}(\Delta t - \chi(t))\right] \left[A_m(0) \cos \sqrt{\left(\frac{F^2}{C^2} + \frac{1}{4}\right)} \chi(t) \right. \\ &\quad \left. - i \left(\frac{F^2}{C^2} + \frac{1}{4}\right)^{-\frac{1}{2}} \left(B_{m+1}(0) \frac{F}{C} + \frac{A_m(0)}{2} \right) \sin \sqrt{\left(\frac{F^2}{C^2} + \frac{1}{4}\right)} \chi(t) \right], \\ B_{m+1}(t) &= \exp\left[\frac{i}{2}(\Delta t - \chi(t))\right] \left[B_m(0) \cos \sqrt{\left(\frac{F^2}{C^2} + \frac{1}{4}\right)} \chi(t) \right. \\ &\quad \left. + i \left(\frac{F^2}{C^2} + \frac{1}{4}\right)^{-\frac{1}{2}} \left(\frac{B_{m+1}(0)}{2} - \frac{F}{C} A_m(0) \right) \sin \sqrt{\left(\frac{F^2}{C^2} + \frac{1}{4}\right)} \chi(t) \right]. \end{aligned} \quad (24)$$

It is to be remarked that this solution is a general one and generalizes earlier results³⁵; especially the dependence of the sinusoidal arguments on the function $\chi(t)$.

Having obtained the solution of the wavefunction, we can therefore consider some statistical properties of the system. To do this, we use Eqs. (19) and (24) to construct the density operator $\hat{\rho} = |\psi(t)\rangle\langle\psi(t)|$, from which we can calculate the expectation values for discussion of some physical properties. In the following section, we will discuss atomic inversion, where we can discuss the collapse and revival phenomenon.

3. Atomic Inversion

It is well known that atomic inversion is the difference between the probability of finding the atom in its excited and ground state:

$$W(t) = \langle \hat{\sigma}_z(t) \rangle = \sum_{m=0}^{\infty} (|A_m(t)|^2 - |B_{m+1}(t)|^2). \quad (25)$$

Fig. 1. (Color online) The time evolution of the atomic inversion against the scaled time λt for a two-level atom system initially in the superposition state interacting with a $su(1,1)$ quantum system that is initially in the Barut–Girardello coherent state for fixed values of $\eta = 25$, $\phi = \pi/4$, $k = 1/4$, $\delta/\lambda = 1$ and $c_0 = 1/\sqrt{2}$. (a) $\gamma = 0$ and $\Delta = 0$, (b) $\gamma = 0$ and $\Delta = 1$ (c) $\gamma/\lambda^2 = 0.01$ and $\Delta = 0$ and (d) $\gamma/\lambda^2 = 0.01$ and $\Delta = 1$.

To discuss the behavior of atomic inversion in the squall, we considered $\lambda_1 = \lambda \sin(\delta t + \gamma t^2 + \varphi)$, where δ, φ and γ are arbitrary constants, this means that the atom is moving in the direction of polarization with velocity and acceleration. Figure 1 shows the function behavior for fixed values of $\eta = 25$, $\varphi = \pi/4$, $c_0 = c_1 = 1/\sqrt{2}$ and $k = 1/4$. In Fig. 1(a), we plot the function for $\delta/\lambda = 1$, $\gamma = 0$, and $\Delta = 0$. In this case, the function shows a short period of revival after the onset of the interaction, which is followed by a long period of collapse. Two successive periods of revival occur for a short period of time followed by a long period of collapse. This phenomenon is repeated for the time period considered.

The function shows different behavior when detuning occurs and $\Delta = 1$. In this case, the function shows a weak interaction between the $su(1,1)$ Lie group and the two-level atom, where the strong interaction occurs two times after a considerable period of the time. This behavior is observed during the revival periods where interference occurs between the pattern and the phenomenon of the superstructure is observed. Periods of collapse as well as irregular fluctuations are also observed

between the two condensed revivals [see Fig. 1(b)]. When we considered the case in which $\gamma/\lambda^2 = 0.01$, and $\Delta = 0$, the function shows different behavior, although after a first period of collapse, where there is a period of revival. The sequence of this behavior occurs with an increase in the function amplitude. However, as the time increases, the amplitude decreases, which means that the interaction becomes weaker [see Fig. 1(c)]. However, the revival periods which become very narrow when we consider $\Delta = 1$, and $\gamma/\lambda^2 = 0.01$, where collapses and revivals occur. Furthermore, revivals occur between 0.2 and -0.2 either in the positive or negative range, or in both ranges. This means that the energy fluctuates between the atom and $\text{su}(1)$, as one can see the amount of exchanged energy is small compared with the other cases [see Fig. 1(d)].

4. Degree of Entanglement

In this section, we consider an aspect of the quantum information, namely, the degree of entanglement. The degree of entanglement can be quantified using the von Neumann entropy for systems that start from pure states. The von Neumann entanglement entropy S is defined as the von Neumann entropy of either of its reduced states, and the result is independent of which one is chosen. That is, for a pure state $\hat{\rho} = |\psi\rangle\langle\psi|_{ab}$, it is given by

$$S(\rho_a) = -\text{Tr}(\rho_a \log \rho_a) = -\text{Tr}(\rho_b \log \rho_b) = S(\rho_b), \quad (26)$$

where $\rho_a = \text{Tr}_b(\rho_{ab})$ and $\rho_b = \text{Tr}_a(\rho_{ab})$ are the reduced density matrices for each partition. Here, we use

$$S_{a(b)} = -\text{Tr}_{b(a)}\{\rho_{a(b)} \ln \rho_{a(b)}\}, \quad (27)$$

provided that we treat both separately. Because the trace is invariant under a similarity transformation, we can use a basis in which the density matrix of the field is diagonal and then express the field entropy $S_f(t)$ in terms of the eigenvalue $\lambda_k^\pm(t)$ for the reduced field density operator. However, the simplest way to calculate the various field eigenstates is to apply the relation^{28,40}

$$\begin{aligned} \lambda_k^\pm(t) &= \langle X(t)|X(t)\rangle \pm \exp[\mp\xi]|\langle X(t)|Y(t)\rangle| \\ &= \langle Y(t)|Y(t)\rangle \pm \exp[\pm\xi]|\langle X(t)|Y(t)\rangle|, \end{aligned} \quad (28)$$

from which we can obtain the eigenvalues for the reduced density operator, where

$$\xi = \sinh^{-1} \left(\frac{\langle X(t)|X(t)\rangle - \langle Y(t)|Y(t)\rangle}{2|\langle X(t)|Y(t)\rangle|} \right), \quad (29)$$

where

$$X(t) = \sum_m X_m \exp(-i\Delta t/2)|m, k\rangle, \quad Y(t) = \sum_m Y_m \exp(i\Delta t/2)|m, k\rangle \quad (30)$$

with X_m and Y_{m+1} given by Eq. (22).

Fig. 2. (Color online) Time evolution of the degree of entanglement against the scaled time λt for a two-level atom system initially in the superposition state interacting with a $\text{su}(1, 1)$ quantum system that is initially in the Barut–Girardello coherent state for fixed values of $\eta = 25$, $\phi = \pi/4$, $k = 1/4$, $\delta/\lambda = 1$ and $c_0 = 1/\sqrt{2}$. (a) $\gamma = 0$ and $\Delta = 0$, (b) $\gamma = 0$ and $\Delta = 1$, (c) $\gamma/\lambda^2 = 0.01$ and $\Delta = 0$ and (d) $\gamma/\lambda^2 = 0.01$ and $\Delta = 1$.

The quantum system entropy $S_k(t)$ can be expressed in terms of the eigenvalues $\lambda_k^\pm(t)$ for the reduced quantum system density operator as

$$S_k(t) = -[\lambda_k^+(t) \ln \lambda_k^+(t) + \lambda_k^-(t) \ln \lambda_k^-(t)]. \quad (31)$$

From a quantum information point of view, the von Neumann entropy would aid in discussion of the phenomenon of entanglement. In Fig. 2, we plot the function using the same values of the involved parameters as in the previous section. For example, we choose $\eta = 25$, $\phi = \pi/4$, $k = 1/4$, $c_0 = 1/\sqrt{2}$ and $\delta/\lambda = 1$ as fixed parameters, and consider $\gamma = 0$ and $\Delta = 0$ in Fig. 2(a). In this case, the function shows partial entanglement with a maximum value around 0.7, while the minimum value approaches 0. Minimum values occur three times in the considered interval of time at the periods of revivals in Fig. 1(a). The function also regularly repeats itself as the time increases. When the detuning parameter occurs ($\Delta = 1$), both the maximum and minimum values of the function increase, although the regularity is

broken, but the minimum values occur during revival periods as shown in Fig. 2(b). Different behavior is observed when we consider $\gamma/\lambda^2 = 0.01$ and $\Delta = 0$: the function shows regular fluctuations within the partial entanglement region. However, as the time increases, the range of the fluctuation decreases and consequently the period of the partial entanglement decreases and the disentanglement never occurs [see Fig. 2(c)]. For the case with $\gamma/\lambda^2 = 0.01$ and $\Delta = 1$, the function shows irregular fluctuations during the period of partial entanglement, but with interference between the patterns at its maximum value, but once again disentanglement never occurs [see Fig. 2(d)]. In all cases, minima of the degree of entanglement occur during the revival periods of the atomic inversion.

5. Fidelity

The role of the fidelity, a basic notion of quantum information science, in characterizing quantum phase transitions (QPTs) is well known. As a distance measure, the fidelity describes the closeness of two given quantum states. Therefore, it is natural

Fig. 3. (Color online) The time evolution of the fidelity against the scaled time λt for a two-level atom system initially in the mixed state interacting with a $su(1, 1)$ quantum system that is initially in the Barut–Girardello coherent state for fixed values of $\eta = 25$, $\phi = \pi/4$, $k = 1/4$, $c_0 = 1/\sqrt{2}$ for $\delta/\lambda = 1$. (a) $\gamma = \Delta = 0$, (b) $\gamma = 0$, $\Delta = 1$, (c) $\gamma/\lambda^2 = 0.08$, $\Delta = 0$ and (d) $\gamma/\lambda^2 = \Delta = 0.08$.

to expect that the fidelity can be used to characterize drastic changes in quantum states when systems undergo QPTs, regardless of what type of internal order is present in the quantum many-body states. In this section, we consider the fidelity of the present system defined by

$$F(t) = \sqrt{\langle \psi(0) | \hat{\rho}(t) | \psi(0) \rangle} = |\langle \psi(0) | \psi(t) \rangle|. \quad (32)$$

Using the above equation, we can examine the fidelity for fixed values of the parameters $\eta = 25$, $\phi = \pi/4$, $k = 1/4$, $c_0 = 1/\sqrt{2}$, and $\delta/\lambda = 1$, and for different values of the other parameters. For example, we examine the case in which $\gamma = \Delta = 0$ and in this case, the function shows periodical behavior and fluctuates between 0 and 1 after a certain period of *zero* value. The fidelity suffers periods of sudden death and rebirth. The periods of fidelity death occur during the collapse periods, while the maxima occur during the revival periods of the atomic inversion see Fig. 3(a). When we increase the detuning parameter to $\Delta = 1$, we note that the fidelity death periods are short, but it reaches perfect match $F = 1$ at points during the revival periods of the atomic inversion [see Fig. 3(b)]. The picture is changed when we take $\gamma/\lambda^2 = 0.08$ and $\Delta = 0$ as in Fig. 3(c) where the fidelity deteriorates and the maxima are decreased considerably as time increases. Finally, we consider the case in which $\gamma/\lambda^2 = \Delta = 0.08$, where the function reaches the maxima as well as some death periods. As the time increases, the function decreases its value and tends to complete death after long period of time, see [Fig. 3(d)]. Thus, we may conclude that the fidelity sudden death and rebirth occur in all cases that we have considered. The effect of the γ parameter is to deteriorate the fidelity.

6. Correlation Function

In this section, we turn our attention to the behavior of the Glauber second-order correlation function. The normalized second-order correlation function $g^{(2)}(t)$ is used to measure the classical and nonclassical behavior, as well as the coherence behavior. It is defined for the system as⁴¹

$$g^{(2)}(t) = \frac{\langle K_+^2 K_-^2 \rangle}{\langle \hat{K}_+ \hat{K}_- \rangle^2}, \quad (33)$$

where the function shows nonclassical behavior when $g^{(2)}(t) < 1$ and classical effect when $g^{(2)}(t) > 1$, while it shows coherence for $g^{(2)}(t) = 1$. For this reason, in Fig. 4, we plot the function for fixed parameters $\eta = 1$, $\phi = \pi/4$, $k = 1/4$, $c_0 = 1/\sqrt{2}$ and $\delta = 1$, but for different values of the other parameters. For the case in which $\gamma = 0 = \Delta$, the function shows regular fluctuations with periods of nonclassical behavior as well as classical behavior [see Fig. 4(a)]. Irregular fluctuations can be observed with a decrease in the function amplitude, which reflects a decrease in the periods of classical behavior and an increase in the periods of nonclassical behavior when we set $\gamma = 0$ and $\Delta = 1$ [see Fig. 4(b)]. When we take the acceleration term into account, $\gamma/\lambda^2 = 0.08$ and $\Delta = 0$, this increases the fluctuation periods

Fig. 4. (Color online) The time evolution of correlation function against the scaled time λt for a two-level atom system initially in the mixed state interacting with a $su(1,1)$ quantum system that is initially in the Barut–Girardello coherent state for fixed values of $\eta = 1$, $\phi = \pi/4$, $k = 1/4$, $\delta/\lambda = 1$ and $c_0 = 1/\sqrt{2}$. (a) $\gamma = \Delta = 0$, (b) $\gamma = 0$, $\Delta = 1$, (c) $\gamma/\lambda^2 = 0.08$, $\Delta = 0$ and (d) $\gamma/\lambda^2 = \Delta = 0.08$.

within the classical part ($g^{(2)}(t) > 1$). However, as the time increases, the periods of nonclassical behavior decrease [see Fig. 4(c)]. Finally, we consider the case in which $\gamma/\lambda^2 = 0.08 = \Delta$. In this case, there is an increase in the periods of nonclassical behavior with a decrease in the classical behavior [see Fig. 4(d)]. Thus, we conclude that the detuning parameter and the acceleration term play roles in controlling the behavior of the correlation function.

7. Conclusion

We have introduced the problem of the time-dependent interaction between two subsystem governed by $su(2)$ and the $su(1,1)$ Lie group as interaction between a two-level atom injected in a cavity with parametric amplifier, or through the interaction of a two-level trapped ion in a two-dimensional trap. The solution of the wavefunction in the Schrödinger picture is obtained under a certain integrability condition. The results have been used to discuss some statistical properties, where

we considered atomic inversion, entanglement, fidelity and the normalized second-order correlation function. For atomic inversion, the phenomenon of collapses and revivals is realized in all the cases considered, where the effect of γ leads to more periods of revival, while the detuning parameter decreases the function amplitude. For the degree of entanglement, the system shows partial entanglement during the considered time interval and reached almost disentanglement near the revival periods of atomic inversion. However, the function shows rapid fluctuations for $\gamma \neq 0$. In the fidelity case, the amplitude of the function decreases as the time increases when we take γ into consideration, even when $\Delta \neq 0$. Sudden death and rebirth of fidelity are shown to occur in this system during collapses and revivals of atomic inversion.

The correlation function shows nonclassical as well as classical behavior in all the cases considered, although the factor γ increases the period of nonclassical behavior. Finally, we may say that the study of the dynamics of the $su(1,1)$ -like Hamiltonian in the nonresonant case effected in Penna *et al.*⁴¹ and Singh *et al.*⁴² could be extended by performing the analysis of the entanglement and correlation properties along the same lines of the present paper.

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References

1. E. A. Mishkin and D. F. Walls, *Phys. Rev.* **185**, 1618 (1969).
2. G. P. Agrawal and C. L. Mehta, *J. Phys. A, Math. Gen.* **7**, 607 (1974).
3. J. Tucker and D. F. Walls, *Ann. Phys.* **52**, 1 (1969).
4. C. C. Gerry and H. Huang, *Physics* **45**, 8037 (1992).
5. C. L. Tang, *Phys. Rev.* **182**, 367 (1969).
6. J. Tucker and D. F. Walls, *Phys. Rev.* **178**, 20361 (1969).
7. D. F. Walls and R. Barakat, *Phys. Rev. A* **1**, 446 (1970).
8. M. S. Abdalla, M. M. A. Ahmed and S. Al-Homidan, *J. Phys. A, Math. Gen.* **31**, 3117 (1998).
9. V. Peřinová, *Opt. Acta* **28**, 747 (1981).
10. V. Peřinová and J. Peřina, *Opt. Acta* **28**, 769 (1981).
11. M. Bondani *et al.*, *Opt. Lett.* **29**, 180 (2004).
12. N. P. Georgiades *et al.*, *Phys. Rev. Lett.* **75**, 3426 (1995).
13. H. Lwasawa and K. Matsuo, *Opt. Commun.* **117**, 550 (1995).
14. K. M. Ng, C. F. Lo and K. L. Liu, *Eur. Phys. J. D* **6**, 119 (1999).
15. M. S. Abdalla, E. M. Khalil and A.-S. F. Obada, *J. Russ. Laser Res.* **38**, 37 (2017).
16. M. Tavis and F. W. Cummings, *Phys. Rev.* **170**, 379 (1968); *Phys. Rev.* **188**, 692 (1969).
17. Z. Lü, C. Zhao and H. Zheng, *J. Phys. A, Math. Theor.* **50**, 074002 (2017).
18. I. Travenese, *Phys. Rev. A* **85**, 043805 (2012).
19. S. Singh and P. A. Amrita, *Int. J. Theor. Phys.* **51**, 838 (2012).

20. M. S. Abdalla, *J. Phys. A, Math. Gen.* **29**, 1997 (1996).
21. M. S. Abdalla, *Physica A* **179**, 131 (1991).
22. F. A. A. El-Orany, J. Peřina and M. S. Abdalla, *Quantum Semiclass. Opt.* **3**, 67 (2001).
23. M. S. Abdalla, *Int. J. Mod. Phys. B* **16**, 2837 (2002).
24. I. A. Armstrong *et al.*, *Phys. Rev.* **127**, 1918 (1962).
25. A. Yariv, *J. Opt. Soc. Am.* **66**, 301 (1976).
26. W. H. Louisell, A. Yariv and A. E. Siggman, *Phys. A* **124**, 1646 (1961).
27. D. Liebfried *et al.*, *Rev. Mod. Phys.* **75**, 281 (2003).
28. C. C. Gerry and P. L. Knight, *Introductory Quantum Optics* (Cambridge University Press, Cambridge, 2005).
29. B. R. Mollow and R. J. Glauber, *Phys. Rev.* **160**, 1076 (1967).
30. B. R. Mollow, *Phys. Rev.* **162**, 1256 (1967).
31. A. S.-F. Obada, M. Abdel-Aty and M. S. Abdalla, *Int. J. Mod. Phys. B* **17**, 5925 (2003).
32. A. O. Barut and L. Girardello, *Commun. Math. Phys.* **21**, 41 (1971).
33. B. Spain and M. G. Smith, *Functions of Mathematical Physics* (Van Nostrand Reinhold, London, 1970).
34. C. Hamburger, *Ann. Mat. Pura Appl.* **169**, 32 (1995).
35. E. Y. C. Lu, *Phys. Rev. A* **8**, 1053 (1973).
36. V. V. Dodonov and V. I. Manko, *Invariant and Evolution of Nonstationary Quantum Systems* (Nova Science, Commack, 1989).
37. V. V. Dodonov and V. I. Manko, *Phys. Rev. A* **20**, 550 (1979).
38. I. A. Malkin, V. I. Manko and D. A. Trifonov, *J. Math. Phys.* **14**, 576 (1973).
39. R. K. Colegrave and M. S. Abdalla, *J. Phys. A, Math. Gen.* **16**, 3805 (1983).
40. F. A. A. El-Orany, *J. Mod. Opt.* **56**, 99 (2009).
41. V. Penna and F. A. Raffa, *Phys. Rev. A* **93**, 043814 (2016).
42. S. Singh, C. H. R. Ooi and P. A. Amrita, *Phys. Rev. A* **86**, 023810 (2012).