Intertwining operator in thermal CFT_d*

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It has long been known that two-point functions of conformal field theory (CFT) are nothing but the integral kernels of intertwining operators for two equivalent representations of conformal algebra. Such intertwining operators are known to fulfill some operator identities — the intertwining relations — in the representation space of conformal algebra. Meanwhile, it has been known that the S-matrix operator in scattering theory is nothing but the intertwining operator between the Hilbert spaces of in- and out-particles. Inspired by this algebraic resemblance, in this paper, we develop a simple Lie-algebraic approach to momentum-space two-point functions of thermal CFT living on the hyperbolic space-time $\mathbb{H}^1 \times \mathbb{H}^{d-1}$ by exploiting the idea of Kerimov's intertwining operator approach to exact S-matrix. We show that in thermal CFT on $\mathbb{H}^1 \times \mathbb{H}^{d-1}$, the intertwining relations reduce to certain linear recurrence relations for two-point functions in the complex momentum space. By solving these recurrence relations, we obtain the momentum-space representations of advanced and retarded two-point functions as well as positive- and negative-frequency two-point Wightman functions for a scalar primary operator in arbitrary space-time dimension $d \geq 3$.

Keywords: Conformal field theory; finite temperature; correlation functions.

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1. Introduction and Summary

It is widely recognized that, up to a few numerical factors, two-point functions for primary operators of conformal field theory (CFT) are completely determined through the $SO(2, d)$ conformal symmetry in any space-time dimension $d \geq 1$. However, it is less recognized — or generally unappreciated except to experts¹⁻⁴ — that two-point functions in CFT have special representation-theoretic meanings:

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they are the integral kernels of intertwining operators for two equivalent representations of conformal algebra $\mathfrak{so}(2, d)$. Namely, once given a two-point function for a primary operator of scaling dimension Δ , we can construct an operator G_Δ which maps a primary state of scaling dimension $d - \Delta$ to another primary state of scaling dimension Δ .^a Such operator G_Δ — the intertwining operator — satisfies the following commutative diagram and operator identities (intertwining relations):

$$\begin{array}{ccc}
 \mathcal{V}_{d-\Delta} & \xrightarrow{J_{d-\Delta}^{ab}} & \mathcal{V}_{d-\Delta} \\
 G_\Delta \downarrow & & \downarrow G_\Delta \\
 \mathcal{V}_\Delta & \xrightarrow{J_\Delta^{ab}} & \mathcal{V}_\Delta
 \end{array}
 \quad J_\Delta^{ab} G_\Delta = G_\Delta J_{d-\Delta}^{ab}$$

where $\mathcal{V}_\alpha$ is a representation space of scaling dimension $\alpha \in \{\Delta, d - \Delta\}$ in which the quadratic Casimir operator $C_2[\mathfrak{so}(2, d)]$ takes a definite value and $J_\alpha^{ab} = -J_\alpha^{ba}$ are the $SO(2, d)$ generators that act on the representation space $\mathcal{V}_\alpha$. An important observation here is that the intertwining operator G_Δ is quite analogous to an S-matrix operator S in scattering theory which satisfies the following commutative diagram and operator identity:

$$\begin{array}{ccc}
 \mathcal{H}_{\text{out}} & \xrightarrow{A_{\text{out}}} & \mathcal{H}_{\text{out}} \\
 S \downarrow & & \downarrow S \\
 \mathcal{H}_{\text{in}} & \xrightarrow{A_{\text{in}}} & \mathcal{H}_{\text{in}}
 \end{array}
 \quad A_{\text{in}} S = S A_{\text{out}}$$

where $\mathcal{H}_{\text{in}}$ and $\mathcal{H}_{\text{out}}$ are the Hilbert spaces of in- and out-particles and A_{in} and A_{out} are some operators (typically creation and annihilation operators) that act on $\mathcal{H}_{\text{in}}$ and $\mathcal{H}_{\text{out}}$.

The purpose of this paper is to pursue this analogy between G_Δ and S . To be more concrete, in this paper, we shall develop a Lie-algebraic method to compute momentum-space two-point functions of $d(\geq 3)$ -dimensional CFT at finite temperature by exploiting the idea of Kerimov's intertwining operator approach to exact S-matrix⁶ (see also Refs. 7–9). In the late 1990s, Kerimov developed an elegant Lie-algebraic method to compute S-matrix elements of exactly-solvable quantum-mechanical models whose dynamics are governed by dynamical symmetry $SO(p, q)$, $p \geq q > 1$. He found that in such models, S-matrix operators are nothing but the intertwining operators for two equivalent representations of $SO(p, q)$ and showed

^aThis is essentially equivalent to the so-called “shadow operator formalism”⁵ (for more details, see Sec. 3).

that, by using the intertwining relations, S-matrix elements must fulfill certain linear recurrence relations. By solving those recurrence relations, he obtained a large class of quantum-mechanical exact S-matrices in a purely algebraic fashion.^b We shall apply his method to the problem of computing momentum-space correlation functions of finite-temperature CFT in space-time dimension $d \geq 3$.^c In general, it is a very hard task to calculate Fourier transforms of position-space conformal correlators. This is true even at zero temperature and the situation gets worse at finite temperature. In fact, to the best of our knowledge, momentum-space correlation functions of thermal CFT in generic dimension $d \geq 3$ have not been explicitly derived even for scalar two-point functions.^d We shall see that, at least for the case of two-point functions, the intertwining relations may well provide an alternative to Fourier transform.

In addition to the intertwining operator, there is another key component for the study of thermal conformal correlators: the Unruh effect. In order to thermalize d -dimensional CFT, we shall put CFT on the Rindler wedge $W_{R/L}$ — the causally connected region for an eternally uniformly accelerating observer — and identify the temporal coordinate with the $SO(1, 1) \subset SO(2, d)$ Lorentz boost parameter. According to Sewell's theorem,¹⁸ which follows from the celebrated Bisognano–Wichmann theorem,^{19,20} any such quantum field theory (not necessarily conformally-invariant) automatically satisfies the Kubo–Martin–Schwinger (KMS) thermal equilibrium condition²¹ for Wightman functions and hence gets thermalized geometrically.^e It should be noted that, though Sewell's theorem is proved axiomatically and hence can be applied to any relativistic quantum field theory with or without conformal symmetry, special things happen when the theory enjoys conformal invariance. The point is that the conformal group $SO(2, d)$ admits other geometrical realizations of the action of the one-parameter subgroup $SO(1, 1)$. One is the subgroup $SO(1, 1)$ generated by the dilatation and the other by the linear combination of the temporal components of momentum and special conformal transformation generators; see Fig. 1. Orbits of the actions of these two subgroups coincide with the worldlines for an observer in semi-eternal uniform motion and a uniformly

^bIt should be noted that, though the Kerimov's S-matrix theory is limited to quantum-mechanical scattering problems, it is also true in relativistic quantum field theory that the S-matrix operator is nothing but an intertwining operator; see, e.g. Chapter 6 of Strocchi's book.¹⁰ Note also that quantum-mechanical exact S-matrices (i.e. reflection and transmission amplitudes) are related to exact two-body S-matrix elements of nonrelativistic quantum field theory; see, e.g. Ref. 11.

^cThere are many parallels between the Kerimov's S-matrix theory and our Lie-algebraic approach to thermal two-point functions. There is, however, a crucial difference. The difference is the basis for the representation of $\mathfrak{so}(2, d)$: the Kerimov's method⁶ is based on the basis that diagonalizes the maximal compact subgroup $SO(2) \times SO(d) \subset SO(2, d)$, whereas our method is based on the basis that diagonalizes the noncompact subgroup $SO(1, 1) \times SO(1, d-1) \subset SO(2, d)$.

^dFor the case of two-dimensional thermal CFT, the momentum-space two- and three-point Wightman functions for a scalar primary operator were calculated in Refs. 12–15. See also our previous works on holographic approach to momentum-space thermal two-point functions of one- and two-dimensional CFT.^{16,17}

^eA nice exposition of Sewell's theorem can be found in Chapter 5 of Ref. 22.

Fig. 1. (Color online) Flows of $SO(1,1)$ conformal Killing vectors projected onto the (x^0, x^1) -plane. Blue curves are the orbits of the action of the one-parameter subgroup $SO(1,1) \subset SO(2,d)$ on a space-time point $x^\mu \in W_{R/L}, V_\pm, D$, where the subgroup $SO(1,1)$ is generated by (a) the Lorentz boost generator $M^{10} = J^{10}$, (b) the dilatation generator $D = J^{d,d+1}$ and (c) the combination $\frac{1}{2}(\ell P^0 - \frac{1}{\ell} K^0) = J^{d0}$; see Subsecs. 2.2–2.4 for more details.

accelerating observer with finite lifetime, each of whose causally connected regions are the future (past) light-cone $V_\pm$ and diamond D , respectively.^f As discussed first by Buchholz for $V_\pm$ (Ref. 24) and later by Hislop and Longo for D and $V_\pm$ (Ref. 25) in the context of modular theory in operator algebra, CFTs restricted on these regions are shown to be thermal as well under the identifications of temporal coordinates with these $SO(1,1)$ group parameters. In this way, CFT in any space-time dimension d easily gets thermalized by just putting it on the Rindler wedge $W_{R/L}$, light-cone $V_\pm$, or diamond D with suitably-chosen temporal coordinates. The price to pay, however, is that thus obtained thermal CFTs are *not* conformal to those on the flat Minkowski space-time: they all describe finite-temperature quantum field theory living on the hyperbolic space-time $\mathbb{H}^1 \times \mathbb{H}^{d-1}$ rather than $\mathbb{R}^{1,d-1}$.^g Though its physical applications are less obvious,^h for the sake of simplicity, this paper studies finite-temperature CFT living on the hyperbolic space-time by restricting zero-temperature CFT on the Rindler wedge, light-cone or diamond.

The rest of the paper is organized as follows. The next two sections are devoted to preliminary materials. Section 2 is a description of d -dimensional CFT on the Rindler wedge, light-cone and diamond in terms of the embedding space formalism. The embedding space formalism is quite an old idea for describing conformal symmetry²⁷ but has recently been revived thanks to its usefulness for studying correlation functions and conformal blocks; see, e.g. Refs. 28–30. After

^fNote, however, that proper times of these observers do *not* coincide with the $SO(1,1)$ group parameters: they are related by certain coordinate transformations²³ (see also Sec. 2).

^g $\mathbb{H}^1 \times \mathbb{H}^{d-1}$ is often written as $\mathbb{R} \times \mathbb{H}^{d-1}$ in the literature. (Note that $\mathbb{H}^1$ is isometric to $\mathbb{R}$.)

^hThermal CFT on $\mathbb{H}^1 \times \mathbb{H}^{d-1}$ (or its Euclidean version on $\mathbb{S}^1 \times \mathbb{H}^{d-1}$) has recently attracted much attention in the context of (holographic) entanglement entropy; see, e.g. Ref. 26. Unfortunately, however, immediate applications of our results are less obvious.

reviewing everything we need in Subsec. 2.1, we introduce in Subsecs. 2.2–2.4, the d -dimensional Rindler wedge, light-cone and diamond as particular subspaces of $(d+1)$ -dimensional cone and discuss thermal correlation functions in the position space. We shall see that the two-point function for a scalar primary operator of scaling dimension Δ living on the hyperbolic space–time $\mathbb{H}^1 \times \mathbb{H}^{d-1}$ takes the following form:

$$\left[\frac{2\pi^2 T^2}{-\cosh(2\pi T(t-t')) - H \cdot H'} \right]^\Delta, \quad (1.1)$$

where $(t, H), (t', H') \in \mathbb{H}^1 \times \mathbb{H}^{d-1}$ and T is the Unruh temperature. We then introduce the intertwining operator for a scalar primary operator in Sec. 3. We first review the basics of intertwining operator and then show that, for the case of zero-temperature CFT, the intertwining relations just reduce to the well-known conformal Ward–Takahashi identities for two-point functions. Section 4 is the main part of the present paper and discusses implications of intertwining relations for thermal two-point functions. In Subsec. 4.1, we construct the representation of conformal algebra $\mathfrak{so}(2, d)$ in which the $SO(1, 1)$ Lorentz boost generator becomes diagonal. This is quite unconventional yet the most important part of this paper, because, in geometrically thermalized CFT, the time-translation generator generates the noncompact subgroup $SO(1, 1)$ rather than, say, the compact subgroup $SO(2)$. By using this representation, we show in Subsec. 4.2 that, for the case of finite-temperature CFT on the hyperbolic space–time $\mathbb{H}^1 \times \mathbb{H}^{d-1}$, the intertwining relations reduce to the following linear recurrence relations for two-point function $\tilde{G}_\Delta(\omega, k)$ in the complex momentum space:

$$\tilde{G}_\Delta(\omega \pm i2\pi T, k \pm i2\pi T) = \frac{\Delta - (d-2)/2 \mp i(\omega+k)/2\pi T}{\tilde{\Delta} - (d-2)/2 \mp i(\omega+k)/2\pi T} \tilde{G}_\Delta(\omega, k), \quad (1.2a)$$

$$\tilde{G}_\Delta(\omega \pm i2\pi T, k \mp i2\pi T) = \frac{\Delta - (d-2)/2 \mp i(\omega-k)/2\pi T}{\tilde{\Delta} - (d-2)/2 \mp i(\omega-k)/2\pi T} \tilde{G}_\Delta(\omega, k), \quad (1.2b)$$

where ω is the frequency conjugate to the $SO(1, 1)$ boost parameter — the proper time for an eternally uniformly accelerating observer — and $k(>0)$ is the modulus of spatial momentum which is related to the eigenvalue of the quadratic Casimir operator of the subalgebra $\mathfrak{so}(1, d-1)$. Here $\tilde{\Delta} = d - \Delta$ is the scaling dimension of the so-called “shadow operator.”⁵ The goal of this paper is to derive and solve these recurrence relations. We shall see that the retarded, advanced and positive- as well as negative-frequency two-point Wightman functions are all obtained by solving these recurrence relations. We conclude in Sec. 5. Appendix A presents computational details for the Fourier transforms of positive- and negative-frequency two-point Wightman functions, which are given by just replacing $t - t'$ with $t - t' \mp i\epsilon$ in (1.1). Making use of the harmonic analysis on the hyperbolic space–time $\mathbb{H}^1 \times \mathbb{H}^{d-1}$, we show that the Fourier transforms of Wightman functions — very hard d -dimensional integrals that involve the modified Bessel functions — indeed coincide with two of the solutions to the recurrence relations (1.2a) and (1.2b).

2. Embedding Space Formalism

Conformal transformations are nonlinear transformations of space-time coordinates. However, they can be linearly realized if we embed the d -dimensional Minkowski space-time into the $(d+2)$ -dimensional ambient space $\mathbb{R}^{2,d}$. In this section, we first review the basics of embedding space formalism for zero-temperature CFT. We then introduce the Rindler wedge, light-cone and diamond as particular subspaces of $(d+1)$ -dimensional cone which can be embedded into $\mathbb{R}^{2,d}$. We shall see that these subspaces are all conformal to the hyperbolic space-time $\mathbb{H}^1 \times \mathbb{H}^{d-1}$ and the Wightman functions on $\mathbb{H}^1 \times \mathbb{H}^{d-1}$ indeed satisfy the KMS thermal equilibrium condition under the identification of temporal coordinates with the $SO(1,1)$ group parameters.

We emphasize that this section is mostly a condensation of known results, yet contains some small new results in the construction of thermal CFTs and thermal correlation functions. We have collected all the necessary ingredients in order to make the paper self-contained. Note that a similar construction of Euclidean thermal CFT on $\mathbb{S}^1 \times \mathbb{H}^{d-1}$ can be found in Refs. 31 and 32.

Throughout this section, the space-time dimension is $d \geq 2$. It is also briefly commented on the case $d = 1$.

2.1. Embedding the Minkowski space-time into the cone

To begin with, let us first consider the $(d+1)$ -dimensional null cone, Cone_{d+1} , which can be embedded into the $(d+2)$ -dimensional ambient space $\mathbb{R}^{2,d}$ as follows:

$$\text{Cone}_{d+1} = \{X^a = (X^0, \dots, X^{d+1}) : X \cdot X = 0 \text{ \& } X^a \neq 0\}, \quad (2.1)$$

where $X \cdot X = \eta_{ab} X^a X^b$ with $\eta_{ab} = \eta^{ab} = \text{diag}(-1, +1, \dots, +1, -1)$ being the ambient space metric. Thus defined cone is obviously invariant under the following transformations:

$$X^a \mapsto g^a_b X^b \quad \text{and} \quad X^a \mapsto \lambda X^a, \quad (2.2)$$

where g is an element of $O(2, d)$ and λ is a nonvanishing real. As usual, in this paper, we shall focus on the identity component of indefinite orthogonal group $O(2, d)$ which, with a slight abuse of notation, we just denote by $SO(2, d)$. The cone (2.1) can be parametrized as follows:

$$X^\mu = \ell \frac{x^\mu}{z}, \quad X^d = \frac{\ell^2 - x \cdot x}{2z}, \quad X^{d+1} = \frac{\ell^2 + x \cdot x}{2z}, \quad (2.3)$$

or, conversely,

$$z = \frac{\ell^2}{X^d + X^{d+1}} \quad \text{and} \quad x^\mu = \ell \frac{X^\mu}{X^d + X^{d+1}}, \quad (2.4)$$

where $z \in \mathbb{R}_\times = \{z \in \mathbb{R} : z \neq 0\}$, $x^\mu = (x^0, \dots, x^{d-1}) \in \mathbb{R}^{1, d-1}$ and $x \cdot x = \eta_{\mu\nu} x^\mu x^\nu$. Here $\ell > 0$ is an arbitrary reference length scale which is just introduced to adjust the length dimension of X^a and (z, x^μ) . We call the coordinate system (2.4) the

Poincaré coordinate system. The d -dimensional Minkowski space-time can then be identified with the following section of the cone:

$$\text{Poincaré}_d = \{X^a \in \text{Cone}_{d+1} : X^d + X^{d+1} = \ell\}. \quad (2.5)$$

This section — the Poincaré section — is the $z = \ell$ hypersurface of the cone and hence can be parametrized as follows:

$$X^\mu = x^\mu, \quad X^d = \frac{\ell^2 - x \cdot x}{2\ell}, \quad X^{d+1} = \frac{\ell^2 + x \cdot x}{2\ell}. \quad (2.6)$$

It is easy to check that the induced metric on this section indeed becomes the Minkowski metric in the Cartesian coordinate system:

$$\begin{aligned} ds_{\text{Poincaré}_d}^2 &= -(dX^0)^2 + (dX^1)^2 + \cdots + (dX^d)^2 - (dX^{d+1})^2 \Big|_{X \in \text{Poincaré}_d} \\ &= \eta_{\mu\nu} dx^\mu dx^\nu. \end{aligned} \quad (2.7)$$

Hence the Poincaré section (2.5) is isometric to the Minkowski space-time.

Let us next move on to the $SO(2, d)$ transformations on the cone (2.1). In terms of the ambient space coordinates X^a , the $SO(2, d)$ transformations are linearly realized as $g : X^a \mapsto X_g^a = g^a_b X^b$ with $g \in SO(2, d)$. In terms of the Poincaré coordinates (z, x^μ) , on the other hand, the $SO(2, d)$ transformations are nonlinearly realized as $g : (z, x^\mu) \mapsto (z_g, x_g^\mu) = \left(\frac{\ell^2}{X_g^d + X_g^{d+1}}, \ell \frac{X_g^\mu}{X_g^d + X_g^{d+1}} \right)$. There are four physically distinct subgroups in $SO(2, d)$:

- *Dilatation.* The one-dimensional subgroup

$$g = \begin{pmatrix} \mathbf{1}_d & & \\ & \cosh \varphi & \sinh \varphi \\ & \sinh \varphi & \cosh \varphi \end{pmatrix} \in SO(1, 1) \quad (2.8)$$

induces the dilatation $g : (z, x^\mu) \mapsto (z_g, x_g^\mu)$, where

$$z_g = e^{-\varphi} z \quad \text{and} \quad x_g^\mu = e^{-\varphi} x^\mu. \quad (2.9)$$

- *Translation.* The d -dimensional subgroupⁱ

$$g = \begin{pmatrix} \mathbf{1}_d & \frac{a^\mu}{\ell} & \frac{a^\mu}{\ell} \\ -\frac{a_\nu}{\ell} & 1 - \frac{1}{2} \frac{a \cdot a}{\ell^2} & -\frac{1}{2} \frac{a \cdot a}{\ell^2} \\ \frac{a_\nu}{\ell} & \frac{1}{2} \frac{a \cdot a}{\ell^2} & 1 + \frac{1}{2} \frac{a \cdot a}{\ell^2} \end{pmatrix} \in E(1)^d \quad (2.10)$$

induces the translation $g : (z, x^\mu) \mapsto (z_g, x_g^\mu)$, where

$$z_g = z \quad \text{and} \quad x_g^\mu = x^\mu + a^\mu. \quad (2.11)$$

ⁱ $E(1)$ stands for the one-dimensional Euclidean group. (In general, the Euclidean group consists of translations and rotations. In one dimension, however, there is no rotation such that $E(1)$ is just the translation group.)

- *Special conformal transformation.* The d -dimensional subgroup

$$g = \begin{pmatrix} \mathbf{1}_d & \ell b^\mu & -\ell b^\mu \\ -\ell b_\nu & 1 - \frac{\ell^2}{2} b \cdot b & \frac{\ell^2}{2} b \cdot b \\ -\ell b_\nu & -\frac{\ell^2}{2} b \cdot b & 1 + \frac{\ell^2}{2} b \cdot b \end{pmatrix} \in E(1)^d \quad (2.12)$$

induces the special conformal transformation $g : (z, x^\mu) \mapsto (z_g, x_g^\mu)$, where

$$z_g = \frac{z}{1 - 2(b \cdot x) + (b \cdot b)(x \cdot x)} \quad \text{and} \quad x_g^\mu = \frac{x^\mu - b^\mu(x \cdot x)}{1 - 2(b \cdot x) + (b \cdot b)(x \cdot x)}. \quad (2.13)$$

- *Lorentz transformation.* The $d(d-1)/2$ -dimensional subgroup

$$g = \begin{pmatrix} \Lambda & \\ & \mathbf{1}_2 \end{pmatrix} \in SO(1, d-1) \quad (2.14)$$

induces the Lorentz transformation $g : (z, x^\mu) \mapsto (z_g, x_g^\mu)$, where

$$z_g = z \quad \text{and} \quad x_g^\mu = \Lambda^\mu{}_\nu x^\nu. \quad (2.15)$$

It should be noted that the translation (2.11) and Lorentz transformation (2.15) are isometries of the Poincaré section; that is, they leave the coordinate $z = \ell^2/(X^d + X^{d+1})$ unchanged. The dilatation (2.9) and special conformal transformation (2.13), on the other hand, change the coordinate z and hence map a point on the section to a point outside the section. But such a point outside the section can be pulled back to a point on the section by (coordinate-dependent) rescaling $X_g^a \mapsto \lambda(X_g) X_g^a$ with $\lambda(X_g) = (X^d + X^{d+1})/(X_g^d + X_g^{d+1})$, without making any change in the ratio $\ell X_g^\mu/(X_g^d + X_g^{d+1}) = x_g^\mu$. This combination of $SO(2, d)$ rotations and rescaling furnishes the conformal transformations on the Poincaré section which, as mentioned before, can be identified with the d -dimensional Minkowski space-time.

Now, let us next move on to infinitesimal conformal transformations. Let $g^a{}_b = \delta^a{}_b + \epsilon^a{}_b$ be a generic element of $SO(2, d)$ with $\epsilon_{ab} = -\epsilon_{ba}$ being antisymmetric infinitesimal parameters. Under the identification

$$\begin{pmatrix} \epsilon^\mu{}_\nu & \epsilon^\mu{}_d & \epsilon^\mu{}_{d+1} \\ \epsilon^d{}_\nu & 0 & \epsilon^d{}_{d+1} \\ \epsilon^{d+1}{}_\nu & \epsilon^{d+1}{}_d & 0 \end{pmatrix} = \begin{pmatrix} \omega^\mu{}_\nu & \frac{a^\mu}{\ell} + \ell b^\mu & \frac{a^\mu}{\ell} - \ell b^\mu \\ -\frac{a_\nu}{\ell} - \ell b_\nu & 0 & \varphi \\ \frac{a_\nu}{\ell} - \ell b_\nu & \varphi & 0 \end{pmatrix}, \quad (2.16)$$

where $\omega_{\mu\nu} = -\omega_{\nu\mu}$ is related to Λ in (2.14) by $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$, the infinitesimal conformal transformation $x^\mu \mapsto x_{1+\epsilon}^\mu$ associated with $g = 1 + \epsilon$ is just given by the sum of infinitesimal forms of (2.9), (2.11), (2.13) and (2.15):

$$x_{1+\epsilon}^\mu = x^\mu - \varphi x^\mu + a^\mu + b_\nu(-\eta^{\mu\nu} x \cdot x + 2x^\mu x^\nu) + \omega^\mu{}_\nu x^\nu. \quad (2.17)$$

Substituting the parametrizations

$$\varphi = \epsilon^d_{d+1}, \quad a^\mu = \frac{\ell}{2}(\epsilon^\mu_d + \epsilon^\mu_{d+1}), \quad b^\mu = \frac{1}{2\ell}(\epsilon^\mu_d - \epsilon^\mu_{d+1}), \quad \omega^\mu_\nu = \epsilon^\mu_\nu, \quad (2.18)$$

one immediately sees that the infinitesimal conformal transformation (2.17) can be put into the following form:

$$x^\mu_{1+\epsilon} = x^\mu + \frac{1}{2}\epsilon_{ab}k^{\mu ab}(x), \quad (2.19)$$

where $k^{\mu ab}(x) = -k^{\mu ba}(x)$ are conformal Killing vectors given by

$$k^{\mu\nu\lambda} = \eta^{\mu\nu}x^\lambda - \eta^{\mu\lambda}x^\nu, \quad (2.20a)$$

$$k^{\mu\nu d} = \frac{\ell^2 - x \cdot x}{2\ell}\eta^{\mu\nu} + \frac{x^\mu x^\nu}{\ell}, \quad (2.20b)$$

$$k^{\mu\nu, d+1} = \frac{\ell^2 + x \cdot x}{2\ell}\eta^{\mu\nu} - \frac{x^\mu x^\nu}{\ell}, \quad (2.20c)$$

$$k^{\mu d, d+1} = -x^\mu. \quad (2.20d)$$

It is easy to check that these vectors indeed satisfy the conformal Killing equations in the flat space-time background, $\partial_\mu k_\nu^{ab} + \partial_\nu k_\mu^{ab} = \frac{2}{d}\eta_{\mu\nu}\partial_\rho k^{\rho ab}$. We shall repeatedly use these conformal Killing vectors in the subsequent sections.

Before closing this section, let us finally recall the construction of conformal correlation functions in the embedding space formalism. A basic ingredient for us is a $(d+2)$ -dimensional homogeneous scalar field $\mathcal{O}_\Delta(X)$ that satisfies the following transformation laws:

$$\mathcal{O}_\Delta(gX) = \mathcal{O}_\Delta(X) \quad \text{and} \quad \mathcal{O}_\Delta(\lambda X) = \lambda^{-\Delta}\mathcal{O}_\Delta(X), \quad (2.21)$$

where $g \in SO(2, d)$ and λ is a nonvanishing real. The two-point function for $\mathcal{O}_\Delta(X)$ that satisfies $G_\Delta(gX, gX') = G_\Delta(X, X')$ and $G_\Delta(\lambda X, \lambda X') = \lambda^{-2\Delta}G_\Delta(X, X')$ is uniquely determined (up to the overall normalization and the possible $i\epsilon$ -prescription) and given by

$$G_\Delta(X, X') = \frac{1}{(-2X \cdot X')^\Delta} = \frac{1}{[(X - X')^2]^\Delta}, \quad (2.22)$$

where the second equality follows from $X \cdot X = X' \cdot X' = 0$ and we have set the normalization constant to be unity. It then follows from the parametrization (2.6) that the two-point function projected onto the Poincaré section gives the well-known two-point function for a scalar primary operator of zero-temperature CFT in d dimensions:

$$G_\Delta(X, X')|_{X, X' \in \text{Poincaré}_d} = \frac{1}{[(x - x')^2]^\Delta} = \frac{1}{[-(x^0 - x'^0)^2 + |\mathbf{x} - \mathbf{x}'|^2]^\Delta}. \quad (2.23)$$

Now, if we worked in Euclidean signature, this would be the end of the story for the construction of conformal two-point function. In Lorentzian signature, however, this is not the end of the story because we have to specify the $i\epsilon$ -prescription

in order to avoid including the branch points at $x^0 = x^{0'} \pm |\mathbf{x} - \mathbf{x}'|$ which lie exactly on the real x^0 -axis. There are several basic two-point functions in Lorentzian quantum field theory. Among them are the positive- and negative-frequency two-point Wightman functions $G_{\Delta}^{\pm}(x, x')$, which are the vacuum expectation values of primary operators $\langle \Omega | \mathcal{O}_{\Delta}(x) \mathcal{O}_{\Delta}(x') | \Omega \rangle$ and $\langle \Omega | \mathcal{O}_{\Delta}(x') \mathcal{O}_{\Delta}(x) | \Omega \rangle$ and given by the following $i\epsilon$ -prescriptions:

$$\begin{aligned} G_{\Delta}^{\pm}(x, x') &= \frac{1}{[-(x^0 - x^{0'} \mp i\epsilon)^2 + |\mathbf{x} - \mathbf{x}'|^2]^{\Delta}} \\ &= \frac{1}{[-(x^0 - x^{0'})^2 + |\mathbf{x} - \mathbf{x}'|^2 \pm i\epsilon \operatorname{sgn}(x^0 - x^{0'})]^{\Delta}}, \end{aligned} \quad (2.24)$$

where ϵ is a positive infinitesimal and $\operatorname{sgn}(x) = x/|x|$ is the sign function. All the other two-point functions are given in terms of these Wightman functions. For example, the time-ordered two-point function $G_{\Delta}^F(x, x') = \langle \Omega | T \{ \mathcal{O}_{\Delta}(x) \mathcal{O}_{\Delta}(x') \} | \Omega \rangle$ is given by

$$\begin{aligned} G_{\Delta}^F(x, x') &= \theta(x^0 - x^{0'}) G_{\Delta}^{+}(x, x') + \theta(x^{0'} - x^0) G_{\Delta}^{-}(x, x') \\ &= \frac{1}{[-(x^0 - x^{0'})^2 + |\mathbf{x} - \mathbf{x}'|^2 + i\epsilon]^{\Delta}}. \end{aligned} \quad (2.25)$$

Similarly, the retarded and advanced correlators $G_{\Delta}^R(x, x') = \theta(x^0 - x^{0'}) \langle \Omega | [\mathcal{O}_{\Delta}(x), \mathcal{O}_{\Delta}(x')] | \Omega \rangle$ and $G_{\Delta}^A(x, x') = \theta(x^{0'} - x^0) \langle \Omega | [\mathcal{O}_{\Delta}(x'), \mathcal{O}_{\Delta}(x)] | \Omega \rangle$ are given by

$$\begin{aligned} G_{\Delta}^R(x, x') &= \theta(x^0 - x^{0'}) (G_{\Delta}^{+}(x, x') - G_{\Delta}^{-}(x, x')) \\ &= G_{\Delta}^F(x, x') - G_{\Delta}^{-}(x, x'), \end{aligned} \quad (2.26a)$$

$$\begin{aligned} G_{\Delta}^A(x, x') &= \theta(x^{0'} - x^0) (G_{\Delta}^{-}(x, x') - G_{\Delta}^{+}(x, x')) \\ &= G_{\Delta}^F(x, x') - G_{\Delta}^{+}(x, x'). \end{aligned} \quad (2.26b)$$

In this way, one can as well study two-point functions for generic primary tensors or higher-point functions by the embedding space formalism; see, e.g. Refs. 28–30. In this paper, however, we will just focus on the scalar two-point function for simplicity.

2.2. Embedding the Rindler wedge into the cone

Now let us turn to the problem of constructing finite-temperature CFT in d dimensions. To this end, let us first introduce the following wedge regions of the cone:

$$\bar{W}_R = \{X^a \in \text{Cone}_{d+1} : X^1 \pm X^0 > 0 \text{ \& } X^{d+1} > 0\}, \quad (2.27a)$$

$$\bar{W}_L = \{X^a \in \text{Cone}_{d+1} : X^1 \pm X^0 < 0 \text{ \& } X^{d+1} > 0\}. \quad (2.27b)$$

We first wish to specify these wedge regions in terms of the Poincaré coordinate system (z, x^{μ}) given in (2.3). To do this, let us first note that the conditions

$X^1 \pm X^0 > 0$ together with $X \cdot X = 0$ imply $(X^{d+1})^2 - (X^d)^2 = (X^1 + X^0)(X^1 - X^0) + \dots + (X^{d-1})^2 > 0$, which, together with $X^{d+1} > 0$, implies the conditions $X^{d+1} \pm X^d > 0$. Hence $z = \ell^2 / (X^d + X^{d+1})$ must be positive-definite on $\bar{W}_{R/L}$. With this in mind, we immediately see that the conditions $X^1 \pm X^0 > 0$ and $X^{d+1} > 0$ are translated into $x^1 \pm x^0 > 0$ and $\ell^2 + (x^1 + x^0)(x^1 - x^0) + (x^2)^2 + \dots + (x^{d-1})^2 > 0$ in the Poincaré coordinate system, the latter is automatically satisfied if the former is fulfilled. Hence these wedge regions can also be specified as follows:

$$\bar{W}_R = \{(z, x^\mu) : z > 0 \text{ \& } x^1 \pm x^0 > 0\}, \quad (2.28a)$$

$$\bar{W}_L = \{(z, x^\mu) : z > 0 \text{ \& } x^1 \pm x^0 < 0\}. \quad (2.28b)$$

Now it is obvious that the intersections of these wedges and the Poincaré section give the following d -dimensional hypersurfaces of the cone:

$$\begin{aligned} W_{R/L} &= \{X^a \in \bar{W}_{R/L} : X^d + X^{d+1} = \ell\} \\ &= \begin{cases} \{x^\mu : x^1 \pm x^0 > 0\}, \\ \{x^\mu : x^1 \pm x^0 < 0\}, \end{cases} \end{aligned} \quad (2.29)$$

which are nothing but the standard right and left Rindler wedges, respectively.^j Hence, after fixing the coordinate z , the wedge regions (2.27a) and (2.27b) indeed describe the Rindler wedges of d -dimensional Minkowski space-time.

Let us next consider the isometries of the wedge $\bar{W}_{R/L}$. The most important isometry of the wedge $\bar{W}_{R/L}$ is the following hyperbolic rotation on the (X^0, X^1) -plane:

$$g(\theta) = \begin{pmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \\ & & \mathbf{1}_d \end{pmatrix} \in SO(1, 1), \quad (2.30)$$

which maps the wedge into itself; that is, the one-parameter subgroup (2.30) maps a point X^a on the wedge to another point $X^a(\theta) = g(\theta)^a_b X^b$ on the wedge and hence gives a globally well-defined transformation on $\bar{W}_{R/L}$. Substituting $X^a(\theta) = (X^0 \cosh \theta + X^1 \sinh \theta, X^0 \sinh \theta + X^1 \cosh \theta, X^2, \dots, X^{d+1})$ into (2.4), one immediately sees that the subgroup (2.30) induces the following one-parameter family of boost flow on the wedge $\bar{W}_{R/L}$:

$$z(\theta) = z, \quad (2.31a)$$

$$x^0(\theta) = x^0 \cosh \theta + x^1 \sinh \theta, \quad (2.31b)$$

$$x^1(\theta) = x^0 \sinh \theta + x^1 \cosh \theta, \quad (2.31c)$$

$$\mathbf{x}_\perp(\theta) = \mathbf{x}_\perp, \quad (2.31d)$$

^jThe conditions $x^1 \pm x^0 > 0$ and $x^1 \pm x^0 < 0$ are often written as $x^1 > |x^0|$ and $x^1 < |x^0|$ in the literature.

where $\mathbf{x}_\perp = (x^2, \dots, x^{d-1})$. Note that Eqs. (2.31b)–(2.31d) can also be obtained as solutions to the following flow equation:

$$\frac{\partial x^\mu}{\partial \epsilon_{10}} = k^{\mu 10}(x), \quad (2.32)$$

where $k^{\mu 10}(x)$ is the conformal Killing vector given in (2.20a) and $\epsilon^1_0 = \theta$. In other words, Eqs. (2.31b)–(2.31d) are the flow of conformal Killing vector $k^{\mu 10}$ under the initial condition $x^\mu(\theta = 0) = (x^0, x^1, \mathbf{x}_\perp)$. We also note that Eqs. (2.31b)–(2.31d) coincide with the worldline for an eternally uniformly accelerating observer in the d -dimensional Minkowski space–time. In fact, if we choose the initial point as $x^\mu(0) = (0, \pm 1/a, \mathbf{x}_\perp)$, Eqs. (2.31b)–(2.31d) reduce to the well-known worldlines for uniformly accelerating observers of constant proper acceleration $a > 0$ moving on the right and left Rindler wedges:

$$x^0(\tau) = \pm \frac{1}{a} \sinh(a\tau), \quad x^1(\tau) = \pm \frac{1}{a} \cosh(a\tau), \quad \mathbf{x}_\perp(\tau) = \mathbf{x}_\perp. \quad (2.33)$$

Here the proper time τ and group parameter θ are related as $\theta = \pm a\tau$, where “+” for an observer in the right Rindler wedge and “–” for an observer in the left Rindler wedge. Note that the proportional coefficient a gives the Unruh temperature $T = a/(2\pi)$. The conformal Killing vector flow associated with the one-parameter subgroup (2.30) is depicted in Fig. 1(a).

Let us next move on to the parametrization of Rindler wedge $W_{R/L}$. We wish to find a coordinate patch for $W_{R/L}$ where the temporal coordinate is identified with the $SO(1, 1)$ group parameter (up to an overall dimensionful parameter that gives the Unruh temperature $2\pi T$). Such coordinate system is constructed as follows. First, let us pick up a reference point $Y_0^a = (Y_0^0, \dots, Y_0^{d+1})$ on the Rindler wedge $W_{R/L}$:

$$Y_0^a = (0, \pm \ell, 0, \dots, 0, \ell) \in W_{R/L}. \quad (2.34)$$

Second, we map this point Y_0^a on the Rindler wedge $W_{R/L}$ to a point Y^a on the wedge $\bar{W}_{R/L}$ by applying the following transformation matrix:

$$g = \begin{pmatrix} \cosh(t/\ell) & \sinh(t/\ell) \\ \sinh(t/\ell) & \cosh(t/\ell) \\ & & \Lambda \end{pmatrix} \in SO(1, 1) \times SO(1, d-1), \quad (2.35)$$

where $\Lambda \in SO(1, d-1)$ is a $d \times d$ matrix that satisfies the conditions $\eta_{ab} \Lambda^a_c \Lambda^b_d = \eta_{cd}$ ($a, b, c, d \in \{2, \dots, d+1\}$), $\det \Lambda = 1$ and $\Lambda^{d+1}_{d+1} \geq 1$. The resultant point $Y^a = g^a_b Y_0^b$ takes the following form:

$$Y^a = (\pm \ell \sinh(t/\ell), \pm \ell \cosh(t/\ell), \ell H^a) \in \bar{W}_{R/L}, \quad (2.36)$$

where $H^a = (H^2, \dots, H^{d+1}) \equiv (\Lambda^2_{d+1}, \dots, \Lambda^{d+1}_{d+1})$ is the rightmost column vector of $\Lambda \in SO(1, d-1)$ and parametrizes the $(d-1)$ -dimensional hyperbolic space $\mathbb{H}^{d-1}$:

$$H \cdot H \equiv (H^2)^2 + \dots + (H^d)^2 - (H^{d+1})^2 = -1 \quad \text{and} \quad H^{d+1} \geq 1. \quad (2.37)$$

Note that Eq. (2.36) parametrizes the foliation $\mathbb{H}^1 \times \mathbb{H}^{d-1} \subset \bar{W}_{R/L}$ of the cone. Finally, we bring back this point Y^a on the wedge $\bar{W}_{R/L}$ to a point X^a on the Rindler wedge $W_{R/L}$ by multiplying the coordinate-dependent scaling factor:

$$X^a = \lambda(Y)Y^a \quad \text{with} \quad \lambda(Y) = \frac{\ell}{Y^d + Y^{d+1}} = \frac{1}{H^d + H^{d+1}} > 0. \quad (2.38)$$

One can easily check that thus defined parametrization (2.38) indeed satisfies the conditions $X \cdot X = 0$, $X^1 \pm X^0 > 0$, $X^{d+1} > 0$ and $X^d + X^{d+1} = \ell$ and hence covers the whole Rindler wedge $W_{R/L}$. By construction, it is obvious that in this coordinate system, the time translation $t \rightarrow t + \alpha$ is given by the $SO(1, 1)$ Lorentz boost transformation (2.30) with $\theta = \alpha/\ell$. A straightforward calculation shows that the induced metric on the Rindler wedge $W_{R/L}$ takes the following form:

$$ds_{W_{R/L}}^2 = \lambda^2(Y) dY \cdot dY = \frac{-dt^2 + \ell^2 dH \cdot dH}{(H^d + H^{d+1})^2}, \quad (2.39)$$

which is manifestly conformal to the hyperbolic space-time $\mathbb{H}^1 \times \mathbb{H}^{d-1}$. As we will see shortly, the metric (2.39) indeed reduces to the well-known Rindler metric if we choose an appropriate coordinate system for $\mathbb{H}^{d-1}$.

Now, let us finally move on to the study of thermal correlation functions of d -dimensional CFT. As mentioned in Sec. 1, any quantum field theory restricted on the Rindler wedge becomes thermal in the sense of the KMS condition for Wightman functions. To see this, let us first consider the two-point function on the Rindler wedge, which is obtained by just restricting $G_\Delta(X, X') = (-2X \cdot X')^{-\Delta}$ on $W_{R/L}$:

$$G_\Delta(X, X')|_{X, X' \in W_{R/L}} = \lambda^{-\Delta}(Y) \lambda^{-\Delta}(Y') \left[\frac{1/(2\ell^2)}{-\cosh(\frac{t-t'}{\ell}) - H \cdot H'} \right]^\Delta. \quad (2.40)$$

The overall factors are irrelevant for us and can be removed by a conformal transformation. Rescaling back to the hyperbolic space-time, $X^a \mapsto Y^a = \lambda^{-1}(Y)X^a$, we get the following two-point function on $\mathbb{H}^1 \times \mathbb{H}^{d-1}$:

$$G_\Delta(Y, Y')|_{Y, Y' \in \mathbb{H}^1 \times \mathbb{H}^{d-1}} = \frac{1}{(-2Y \cdot Y')^\Delta} = \left[\frac{2\pi^2 T^2}{-\cosh(2\pi T(t-t')) - H \cdot H'} \right]^\Delta, \quad (2.41)$$

where $T = 1/(2\pi\ell)$ is the Unruh temperature. This is the two-point function for a scalar primary operator of scaling dimension Δ in finite-temperature CFT on the hyperbolic space-time $\mathbb{H}^1 \times \mathbb{H}^{d-1}$.

Several comments are in order at this stage:

- *KMS condition.* Thermal equilibrium of quantum field theory is characterized by the KMS condition, which is the boundary condition for positive- and negative-frequency two-point Wightman functions in the complex time plane; see, e.g.

Chapter V of Haag's book.³³ In general, in finite-temperature quantum field theory in Lorentzian signature, the positive-frequency two-point Wightman function $G_{\Delta}^{+}(t) = \langle \mathcal{O}_{\Delta}(t) \mathcal{O}_{\Delta}(0) \rangle$ is analytic in the strip $-\beta < \text{Im } t < 0$ as a complex function of time t , whereas the negative-frequency two-point Wightman function $G_{\Delta}^{-}(t) = \langle \mathcal{O}_{\Delta}(0) \mathcal{O}_{\Delta}(t) \rangle$ is analytic in the strip $0 < \text{Im } t < \beta$,^k where $\beta = 1/T$ is the inverse temperature and we have suppressed the spatial coordinates for simplicity. The KMS condition is the relation between the boundary values of $G_{\Delta}^{+}(t)$ and $G_{\Delta}^{-}(t)$ in these strips and is expressed as the following equality:¹

$$G_{\Delta}^{\pm}(t) = G_{\Delta}^{\mp}(t \pm i\beta), \quad t \in \mathbb{R}. \quad (2.42)$$

In our problem, the positive- and negative-frequency two-point Wightman functions $G_{\Delta}^{+}(t)$ and $G_{\Delta}^{-}(t)$ correspond to $G_{\Delta}(Y, Y')$ and $G_{\Delta}(Y', Y)$. Naively, these are the same functions because $G_{\Delta}(Y, Y')$ is a symmetric function of Y and Y' . One might therefore think that the KMS condition (2.42) is trivially satisfied for (2.41) since the hyperbolic cosine $\cosh(2\pi T t)$ is periodic in the imaginary time direction with period $\beta = 1/T$. There is, however, a subtle point because there are branch points lying exactly on the boundaries of the strips $-\beta < \text{Im } t < 0$ and $0 < \text{Im } t < \beta$. As we will see shortly in (2.46), $G_{\Delta}(Y, Y')$ possesses infinitely many branch points at $t = \pm r + in\beta$ ($n \in \mathbb{Z}$) in the complex time plane. One way to avoid including such branch points is to deform the boundaries of the strips as depicted in Fig. 2. An alternative (yet equivalent) way to avoid including the branch points is the following $i\epsilon$ -prescriptions:

$$G_{\Delta}^{\pm}(t) \equiv \begin{cases} \left[\frac{2\pi^2 T^2}{-\cosh(2\pi T(t \mp i\epsilon)) - H \cdot H'} \right]^{\Delta} & \text{for } t \in \mathbb{R}, \\ \left[\frac{2\pi^2 T^2}{-\cosh(2\pi T(t \pm i\epsilon)) - H \cdot H'} \right]^{\Delta} & \text{for } t \in \mathbb{R} \mp i\beta, \end{cases} \quad (2.43)$$

where ϵ is a positive infinitesimal. It is now almost trivial to show that the Wightman functions satisfy the KMS condition (2.42) under the $i\epsilon$ -prescriptions (2.43). It should be pointed out here that, in terms of the Fourier transform

^kThese analyticity properties stem from statistical mechanics. The positive-frequency two-point Wightman function $G_{\Delta}^{+}(t) = \langle \mathcal{O}_{\Delta}(t) \mathcal{O}_{\Delta}(0) \rangle$ corresponds to the statistical average $G_{\Delta}^{+}(t) = \frac{1}{Z} \text{Tr}(e^{-\beta H} \mathcal{O}_{\Delta}(t) \mathcal{O}_{\Delta}(0)) = \frac{1}{Z} \sum_{E, E' > 0} \langle E | e^{-\beta H} e^{iHt} \mathcal{O}_{\Delta}(0) e^{-iHt} | E' \rangle \langle E' | \mathcal{O}_{\Delta}(0) | E \rangle = \frac{1}{Z} \sum_{E, E' > 0} e^{iE(t+i\beta) - iE't} |\langle E | \mathcal{O}_{\Delta}(0) | E' \rangle|^2$, where $Z = \text{Tr } e^{-\beta H}$ and $|E\rangle$ is an eigenstate of Hamiltonian H which is assumed to be positive-definite. This infinite series expansion does not converge unless t is in the strip $-\beta < \text{Im } t < 0$, which is the analytic domain of $G_{\Delta}^{+}(t)$. Likewise, $G_{\Delta}^{-}(t) = \langle \mathcal{O}_{\Delta}(0) \mathcal{O}_{\Delta}(t) \rangle$ is analytic in the domain $0 < \text{Im } t < \beta$.

¹The KMS condition stems from the cyclic property of the trace in statistical average. Noting the identity $e^{-\beta H} \mathcal{O}_{\Delta}(t) e^{\beta H} = \mathcal{O}_{\Delta}(t + i\beta)$, one easily gets $G_{\Delta}^{+}(t) = \frac{1}{Z} \text{Tr}(e^{-\beta H} \mathcal{O}_{\Delta}(t) \mathcal{O}_{\Delta}(0)) = \frac{1}{Z} \text{Tr}(\mathcal{O}_{\Delta}(t + i\beta) e^{-\beta H} \mathcal{O}_{\Delta}(0)) = \frac{1}{Z} \text{Tr}(e^{-\beta H} \mathcal{O}_{\Delta}(0) \mathcal{O}_{\Delta}(t + i\beta)) = G_{\Delta}^{-}(t + i\beta)$. Similarly, it follows from the trace cyclicity that $G_{\Delta}^{-}(t) = G_{\Delta}^{+}(t - i\beta)$.

Fig. 2. (Color online) Analytic domains of the positive- and negative-frequency two-point Wightman functions. Black crosses represent the branch points at $t = \pm r + in\beta$, $n \in \mathbb{Z}$. The KMS conditions $G_{\Delta}^{\pm}(t) = G_{\Delta}^{\mp}(t \pm i\beta)$ are the equalities between the boundary values of G_{Δ}^{+} and G_{Δ}^{-} in the blue and red strips.

$\tilde{G}_{\Delta}^{\pm}(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} G_{\Delta}^{\pm}(t)$, the KMS condition is simply expressed as follows:^m

$$\tilde{G}_{\Delta}^{+}(\omega) = e^{\beta\omega} \tilde{G}_{\Delta}^{-}(\omega). \quad (2.44)$$

In Subsec. 4.2 and App. A.2, we shall see whether the momentum-space representations of (2.43) indeed satisfy (2.44).

- *Zero-temperature limit.* The thermal two-point function (2.41) correctly reduces to the zero-temperature correlator (2.23) in the zero-temperature limit $T \rightarrow 0$. To see this, let us first note that $H \cdot H'$ just depends on the hyperbolic angle between the points H^a and $H^{a'}$. In fact, without any loss of generality, we can choose the point $Y^{a'} \in \mathbb{H}^1 \times \mathbb{H}^{d-1}$ as the reference point $Y^{a'} = (0, \pm\ell, 0, \dots, 0, \ell)$, which corresponds to $t' = 0$, $H^{2'} = \dots = H^{d'} = 0$ and $H^{d+1'} = \ell$ in (2.41). Then, by parametrizing $H^a = (H^2, \dots, H^{d+1})$ into the hyperbolic spherical coordinates

$$H^i = \sinh(r/\ell)\Omega^i, \quad H^{d+1} = \cosh(r/\ell), \quad (2.45)$$

^mHere is the proof: $\tilde{G}_{\Delta}^{+}(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} G_{\Delta}^{+}(t) = \int_{-\infty-i\beta}^{\infty-i\beta} dt e^{i\omega t} G_{\Delta}^{+}(t) = e^{\beta\omega} \int_{-\infty}^{\infty} dt e^{i\omega t} G_{\Delta}^{+}(t - i\beta) = e^{\beta\omega} \int_{-\infty}^{\infty} dt e^{i\omega t} G_{\Delta}^{-}(t) = e^{\beta\omega} \tilde{G}_{\Delta}^{-}(\omega)$, where the second equality follows from the deformation of integration contour to the lower boundary of the strip $-\beta < \text{Im } t < 0$. In the third equality, we have changed the integration variable as $t \rightarrow t - i\beta$ and in the fourth equality, we have used the KMS condition (2.42).

where $i \in \{2, \dots, d\}$ and $(\Omega^2)^2 + \dots + (\Omega^d)^2 = 1$, we get the following expression for the thermal two-point function:

$$\begin{aligned} G_\Delta(Y, Y')|_{Y, Y' \in \mathbb{H}^1 \times \mathbb{H}^{d-1}} &= \left[\frac{2\pi^2 T^2}{-\cosh(2\pi T t) + \cosh(2\pi T r)} \right]^\Delta \\ &= \left[\frac{\pi^2 T^2}{-\sinh(\pi T(t-r)) \sinh(\pi T(t+r))} \right]^\Delta, \end{aligned} \quad (2.46)$$

which is quite similar to the well-known thermal two-point function of two-dimensional CFT that can be factorized into the left- and right-moving sectors. (Note, however, that r ranges from 0 to ∞ for $d \geq 3$. For $d = 2$, on the other hand, r runs through $(-\infty, \infty)$ in (2.45).) Now it is obvious that (2.46) reduces to the zero-temperature correlator in the zero-temperature limit, $G_\Delta(Y, Y') \rightarrow 1/(-t^2 + r^2)^\Delta$ as $T \rightarrow 0$, where r plays the role of spatial radial coordinate for $d \geq 3$.

- *Rindler coordinates.* We have seen that the two-point function is remarkably simplified in the hyperbolic spherical coordinates in (2.45). For practical calculations, however, it is more convenient to work in the following coordinate patch for $\mathbb{H}^{d-1}$ ($d \geq 3$):

$$H^i = \frac{x_\perp^i}{x}, \quad H^d = \frac{1}{\ell} \frac{\ell^2 - x^2 - \mathbf{x}_\perp^2}{2x}, \quad H^{d+1} = \frac{1}{\ell} \frac{\ell^2 + x^2 + \mathbf{x}_\perp^2}{2x}, \quad (2.47)$$

where $i \in \{2, \dots, d-1\}$, $x \in (0, \infty)$ and $\mathbf{x}_\perp \in \mathbb{R}^{d-2}$. In this parametrization, $X^\mu (= x^\mu)$ in (2.38) just becomes the standard Rindler coordinates

$$x^\mu = (\pm x \sinh(t/\ell), \pm x \cosh(t/\ell), \mathbf{x}_\perp) \quad (2.48)$$

and the induced metric on the Rindler wedge $W_{R/L}$ becomes the well-known Rindler metric $ds_{W_{R/L}}^2 = -(x/\ell)^2 dt^2 + dx^2 + d\mathbf{x}_\perp^2$. A straightforward calculation shows that in this coordinate system, the two-point function takes the following form:

$$G_\Delta(Y, Y')|_{Y, Y' \in \mathbb{H}^1 \times \mathbb{H}^{d-1}} = \left[\frac{2\pi^2 T^2}{-\cosh(2\pi T(t-t')) + \frac{x^2 + x'^2 + |\mathbf{x}_\perp - \mathbf{x}'_\perp|^2}{2xx'}} \right]^\Delta. \quad (2.49)$$

Though it looks more complicated than (2.46), Eq. (2.49) is the best for calculating its Fourier transform (see App. A). The reason for this is that the complete orthonormal basis on $\mathbb{H}^{d-1}$ is most simply expressed in the coordinate system (2.47). We will also use the Rindler coordinates in Sec. 4.

To summarize, CFT restricted on the Rindler wedge describes thermal CFT on the hyperbolic space-time $\mathbb{H}^1 \times \mathbb{H}^{d-1}$ if we identify the temporal coordinate with the $SO(1, 1)$ Lorentz boost parameter (up to the dimensionful parameter $2\pi T$). It should be noted that, though we have just focused on the scalar two-point function for simplicity, one can also study thermal two-point functions for generic primary tensors and/or thermal n -point correlation functions for $n \geq 3$. For example, the

thermal two-point function for energy-momentum tensor in Euclidean CFT on $\mathbb{S}^1 \times \mathbb{H}^{d-1}$ was given in Ref. 34. Instead of studying those thermal correlation functions, however, let us proceed to construct thermal CFTs in which the temporal flows are identified with other conformal Killing vector flows associated with the one-parameter subgroup $SO(1, 1) \subset SO(2, d)$.

2.3. Embedding the light-cone into the cone

Let us next introduce the following wedge regions of the cone:

$$\bar{V}_+ = \{X^a \in \text{Cone}_{d+1} : X^d \pm X^{d+1} > 0 \ \& \ X^0 > 0\}, \quad (2.50a)$$

$$\bar{V}_- = \{X^a \in \text{Cone}_{d+1} : X^d \pm X^{d+1} > 0 \ \& \ X^0 < 0\}. \quad (2.50b)$$

Plugging the Poincaré coordinates (2.3) into these, we see that the conditions $X^d \pm X^{d+1} > 0$ and $X^0 > 0$ are translated into the conditions $z > 0$, $-x \cdot x > 0$ and $x^0 > 0$. Hence, in terms of the Poincaré coordinate system, these wedge regions are specified as follows:

$$\bar{V}_+ = \{(z, x^\mu) : z > 0 \ \& \ (x^0)^2 > \mathbf{x}^2 \ \& \ x^0 > 0\}, \quad (2.51a)$$

$$\bar{V}_- = \{(z, x^\mu) : z > 0 \ \& \ (x^0)^2 > \mathbf{x}^2 \ \& \ x^0 < 0\}. \quad (2.51b)$$

The intersections of $\bar{V}_\pm$ and the Poincaré section therefore give the standard future and past light-cones of d -dimensional Minkowski space-time:

$$\begin{aligned} V_\pm &= \{X^a \in \bar{V}_\pm : X^d + X^{d+1} = \ell\} \\ &= \{x^\mu : (x^0)^2 > \mathbf{x}^2 \ \& \ \pm x^0 > 0\}. \end{aligned} \quad (2.52)$$

Just as in the case of previous section, the wedge regions $\bar{V}_\pm$ are mapped to themselves under the following hyperbolic rotation on the (X^d, X^{d+1}) -plane:

$$g(\theta) = \begin{pmatrix} \mathbf{1}_d & & \\ & \cosh \theta & \sinh \theta \\ & \sinh \theta & \cosh \theta \end{pmatrix} \in SO(1, 1), \quad (2.53)$$

which, as we saw in Eq. (2.9), induces the following one-parameter family of dilatation flow on the wedge $\bar{V}_\pm$:

$$z(\theta) = ze^{-\theta}, \quad (2.54a)$$

$$x^\mu(\theta) = x^\mu e^{-\theta}. \quad (2.54b)$$

Note that Eq. (2.54b) can also be obtained as a solution to the following flow equation:

$$\frac{\partial x^\mu}{\partial \epsilon_{d,d+1}} = k^{\mu d, d+1}(x), \quad (2.55)$$

where $k^{\mu d, d+1}(x)$ is the conformal Killing vector given in (2.20d) and $\epsilon^d_{d+1} = \theta$. Note also that, though (2.53) is the isometry of the wedges $\bar{V}_\pm$, it is not the isometry

of the light-cones $V_{\pm}$ because the coordinate z does not remain intact under the transformation (2.53). As mentioned in Subsec. 2.1, however, any change in z can be removed by simultaneous rescaling $X^a \mapsto \lambda(X)X^a$ of the ambient space coordinates without making any change in the ratio $\ell X^\mu / (X^d + X^{d+1}) = x^\mu$.

Before proceeding further, it is wise to point out here a physical interpretation of the dilatation flow (2.54b). As discussed in Ref. 23, this dilatation flow coincides with the worldline for an observer in semi-eternal uniform motion:

$$x^\mu(\tau) = v^\mu \tau \quad \text{with} \quad v^\mu = \left(\frac{1}{\sqrt{1 - \mathbf{v}^2}}, \frac{\mathbf{v}}{\sqrt{1 - \mathbf{v}^2}} \right), \quad (2.56)$$

where τ is the proper time for the observer and $\mathbf{v}$ is the spatial velocity whose modulus is less than the speed of light, $|\mathbf{v}| < c = 1$. The proper time τ ranges from 0 to ∞ for an observer who is born at $x^\mu = 0$ and then goes away to the future infinity, whereas it lies between $-\infty$ and 0 for an observer who comes from the past infinity and then dies at $x^\mu = 0$. The causally connected regions for such observers are nothing but the future and past light-cones $V_{\pm}$. Note that the proper time τ does not exactly coincide with the dilatation flow parameter θ : they are related by the scale transformation $\tau = \pm \ell e^{-\theta}$, where “+” for the observer in V_+ and “−” for the observer in V_- . Here, at this stage, $\ell > 0$ is just an arbitrary reference length scale that must be introduced on the dimensional ground. The conformal Killing vector flow associated with the one-parameter subgroup (2.53) is depicted in Fig. 1(b).

Now, just as we did in the previous section, we wish to find a coordinate patch for $V_{\pm}$ where the temporal coordinate coincides with the $SO(1, 1)$ group parameter θ (but not with the proper time τ for an observer in semi-eternal uniform motion).ⁿ Since the construction of such coordinate system is parallel to the case of Rindler wedge, we here just present the result. The desired coordinate system is turned out to be of the following form:

$$X^a = \lambda(Y)Y^a \in V_{\pm}, \quad (2.57)$$

where $Y^a = (Y^0, \dots, Y^{d+1})$ parametrizes the foliation $\mathbb{H}^1 \times \mathbb{H}^{d-1} \subset \bar{V}_{\pm}$ of the cone

$$Y^a = (\pm \ell H^\mu, \ell \cosh(t/\ell), \ell \sinh(t/\ell)) \in \bar{V}_{\pm} \quad (2.58)$$

and $\lambda(Y)$ is the coordinate-dependent scaling factor that pulls back a point Y^a on $\bar{V}_{\pm}$ to a point X^a on $V_{\pm}$:

$$\lambda(Y) = \frac{\ell}{Y^d + Y^{d+1}} = e^{-t/\ell} > 0. \quad (2.59)$$

Here $H^\mu = (H^0, \dots, H^{d-1})$ parametrizes the $(d-1)$ -dimensional unit hyperbolic space $\mathbb{H}^{d-1}$ and satisfies the following conditions:

$$H \cdot H \equiv -(H^0)^2 + (H^1)^2 + \dots + (H^{d-1})^2 = -1 \quad \text{and} \quad H^0 \geq 1. \quad (2.60)$$

ⁿIf we work in a coordinate system in which the temporal coordinate is identified with the proper time, we have to deal with time-dependent temperature in the would-be thermal equilibrium system.²³

It is easy to check that thus defined parametrization indeed satisfies the conditions $X \cdot X = 0$, $X^d \pm X^{d+1} > 0$, $X^0 > 0$, and $X^d + X^{d+1} = \ell$ and covers the whole future (past) light-cones $V_{\pm}$. The induced metric on the light-cone $V_{\pm}$ takes the following form:

$$ds_{V_{\pm}}^2 = e^{-2t/\ell} (-dt^2 + \ell^2 dH \cdot dH), \quad (2.61)$$

which is again manifestly conformal to the hyperbolic space-time $\mathbb{H}^1 \times \mathbb{H}^{d-1}$. The two-point function projected out onto the light-cone $V_{\pm}$ is then given by

$$G_{\Delta}(X, X')|_{X, X' \in V_{\pm}} = \lambda^{-\Delta}(Y) \lambda^{-\Delta}(Y') \left[\frac{1/(2\ell^2)}{-\cosh\left(\frac{t-t'}{\ell}\right) - H \cdot H'} \right]^{\Delta}. \quad (2.62)$$

The overall scale factors can be removed by a conformal transformation and the resulting thermal two-point function on $\mathbb{H}^1 \times \mathbb{H}^{d-1}$ takes exactly the same form as (2.41). Remaining discussions are exactly the same as those in the previous section such that we shall not repeat it here anymore.

2.4. Embedding the diamond into the cone

Let us complete our construction of thermal CFT by considering the following wedge region of the cone:

$$\bar{D} = \{X^a \in \text{Cone}_{d+1} : X^d \pm X^0 > 0 \text{ \& } X^{d+1} > 0\}. \quad (2.63)$$

Let us first rewrite (2.63) in terms of the Poincaré coordinate system (z, x^{μ}) given in (2.3). To this end, we first note that the condition $X^d \pm X^0 > 0$ (i.e. $X^d > |X^0| \geq 0$) together with $X^{d+1} > 0$ implies $X^d + X^{d+1} > 0$, which implies $z = \ell^2/(X^d + X^{d+1})$ must be positive-definite on $\bar{D}$. Using this constraint on z , one immediately sees that the conditions $X^d \pm X^0 > 0$ and $X^{d+1} > 0$ are translated into the conditions $(x^0 \pm \ell)^2 - \mathbf{x}^2 > 0$ and $\ell^2 - (x^0)^2 + \mathbf{x}^2 > 0$, which can be put into a single expression $|x^0| + |\mathbf{x}| < \ell$. Hence in the Poincaré coordinate system, the wedge region $\bar{D}$ is described as follows:

$$\bar{D} = \{(z, x^{\mu}) : z > 0 \text{ \& } |x^0| + |\mathbf{x}| < \ell\}. \quad (2.64)$$

The intersection of the wedge $\bar{D}$ and the Poincaré section then gives the well-known diamond (also known as the double-cone) of d -dimensional Minkowski space-time:

$$\begin{aligned} D &= \{X^a \in \bar{D} : X^d + X^{d+1} = \ell\} \\ &= \{x^{\mu} : |x^0| + |\mathbf{x}| < \ell\}. \end{aligned} \quad (2.65)$$

Let us next consider the isometries of the wedge $\bar{D}$. Just as in the previous sections, the wedge $\bar{D}$ is mapped to itself under the following hyperbolic rotation on the (X^0, X^d) -plane:

$$g(\theta) = \begin{pmatrix} \cosh \theta & & \sinh \theta \\ & \mathbf{1}_{d-1} & \\ \sinh \theta & & \cosh \theta \\ & & & 1 \end{pmatrix} \in SO(1, 1). \quad (2.66)$$

One can easily show that, if the initial point X^a satisfies the conditions $X^d \pm X^0 > 0$ and $X^{d+1} > 0$, then the transformed point $X^a(\theta) = g^a_b X^b$ also satisfies the conditions $X^d(\theta) \pm X^0(\theta) > 0$ and $X^{d+1}(\theta) > 0$ and hence (2.66) gives a globally well-defined transformation on the wedge $\bar{D}$. Substituting $X^a(\theta)$ into $z(\theta) = \ell^2/(X^d(\theta) + X^{d+1}(\theta))$ and $x^\mu(\theta) = \ell X^\mu(\theta)/(X^d(\theta) + X^{d+1}(\theta))$, one can also easily show that the one-parameter subgroup (2.66) induces the following flow on the wedge $\bar{D}$:

$$z(\theta) = e^{-\varphi} \frac{z}{1 - 2(b \cdot x) + (b \cdot b)(x \cdot x)}, \quad (2.67a)$$

$$x^\mu(\theta) = e^{-\varphi} \frac{x^\mu - b^\mu(x \cdot x)}{1 - 2(b \cdot x) + (b \cdot b)(x \cdot x)} + a^\mu, \quad (2.67b)$$

where

$$\begin{aligned} a^\mu &= \left(\ell \tanh\left(\frac{\theta}{2}\right), 0, \dots, 0 \right), \\ b^\mu &= \left(\frac{1}{\ell} \tanh\left(\frac{\theta}{2}\right), 0, \dots, 0 \right), \\ \varphi &= 2 \log \cosh\left(\frac{\theta}{2}\right). \end{aligned} \quad (2.68)$$

As is evident from these expressions, this flow — the diamond flow — can also be obtained by the following three successive transformations: (i) special conformal transformation (2.13); (ii) dilatation (2.9) and (iii) translation (2.11). In fact, the $SO(1, 1)$ matrix (2.66) admits the following decomposition:^o

$$\begin{pmatrix} 1_d & \frac{a^\mu}{\ell} & \frac{a^\mu}{\ell} \\ -\frac{a_\nu}{\ell} & 1 - \frac{1}{2} \frac{a \cdot a}{\ell^2} & -\frac{1}{2} \frac{a \cdot a}{\ell^2} \\ \frac{a_\nu}{\ell} & \frac{1}{2} \frac{a \cdot a}{\ell^2} & 1 + \frac{1}{2} \frac{a \cdot a}{\ell^2} \end{pmatrix} \begin{pmatrix} 1_d & & \\ & \cosh \varphi & \sinh \varphi \\ & \sinh \varphi & \cosh \varphi \end{pmatrix} \begin{pmatrix} 1_d & \ell b^\mu & -\ell b^\mu \\ -\ell b_\nu & 1 - \frac{\ell^2}{2} b \cdot b & \frac{\ell^2}{2} b \cdot b \\ -\ell b_\nu & -\frac{\ell^2}{2} b \cdot b & 1 + \frac{\ell^2}{2} b \cdot b \end{pmatrix}. \quad (2.69)$$

Multiplying these matrices on a point $X^a \in \bar{D}$ induces the composition of conformal transformations (2.13), (2.9) and (2.11) and results in the diamond flow (2.67a) and (2.67b). It is not difficult to show that (2.67b) can also be obtained as a solution

^oThe decomposition (2.69) can also be written as follows:

$$\exp\left[i\frac{\theta}{2}\left(\ell P^0 - \frac{1}{\ell} K^0\right)\right] = \exp\left[-i\ell \tanh\left(\frac{\theta}{2}\right) P^0\right] \exp\left[i2 \log \cosh\left(\frac{\theta}{2}\right) D\right] \exp\left[i\frac{1}{\ell} \tanh\left(\frac{\theta}{2}\right) K^0\right],$$

where D , P^0 , and K^0 are the $(d+2) \times (d+2)$ -matrix representations given in (3.26). Note that the set of generators $\{D, P^0, K^0\}$ forms the one-dimensional conformal algebra $\mathfrak{so}(2, 1) \subset \mathfrak{so}(2, d)$.

to the following flow equation:

$$\frac{\partial x^\mu}{\partial \epsilon_{d0}} = k^{\mu d0}(x), \quad (2.70)$$

where $k^{\mu d0}(x) = -k^{\mu 0d}(x)$ is the conformal Killing vector given in (2.20b) and $\epsilon^d_0 = \theta$. Namely, the diamond flow is nothing but the conformal Killing vector flow associated with $k^{\mu d0}$. Note that the denominator $1 - 2(b \cdot x) + (b \cdot b)(x \cdot x)$ does not vanish provided $x^\mu = (x^0, \mathbf{x})$ satisfies the condition $|x^0| + |\mathbf{x}| < \ell$. Note also that $(z(\theta), x^\mu(\theta)) \rightarrow (0, \pm\ell, \mathbf{0})$ as $\theta \rightarrow \pm\infty$.

Now we wish to give a physical interpretation of the diamond flow. As discussed in Ref. 23, the diamond flow coincides with the worldline for a uniformly accelerating observer with finite lifetime. To see this, let us first choose the initial point as $x^\mu(0) = (x^0, \mathbf{x}) = \left(0, \ell \frac{a\ell\mathbf{\Omega}}{1 + \sqrt{1 + a^2\ell^2}}\right) \in D$, where $\mathbf{\Omega}$ is a constant unit vector that satisfies $\mathbf{\Omega} \cdot \mathbf{\Omega} = 1$ and $a \in (-\infty, \infty)$ is another dimensionful parameter that plays the role of constant proper acceleration. Substituting these into (2.67b), we see that the diamond flow takes the following simple forms:

$$x^0(\theta) = \ell \frac{\sinh \theta}{\cosh \theta + \sqrt{1 + a^2\ell^2}}, \quad \mathbf{x}(\theta) = \ell \frac{a\ell\mathbf{\Omega}}{\cosh \theta + \sqrt{1 + a^2\ell^2}}. \quad (2.71)$$

An important observation here is that (2.71) satisfies the following equation:

$$-(x^0(\theta))^2 + \left(\mathbf{x}(\theta) - \frac{1}{a} \sqrt{1 + a^2\ell^2} \mathbf{\Omega}\right)^2 = \frac{1}{a^2}, \quad (2.72)$$

which implies that $x^0(\theta)$ and $\mathbf{x}(\theta)$ can be put into the following alternative equivalent forms:

$$x^0(\tau) = \frac{1}{a} \sinh(a\tau), \quad \mathbf{x}(\tau) = \frac{1}{a} (\sqrt{1 + a^2\ell^2} - \cosh(a\tau)) \mathbf{\Omega}. \quad (2.73)$$

This is nothing but the worldline for a uniformly accelerating observer moving on the $(x^0, \mathbf{\Omega})$ -plane, where the proper time τ and group parameter θ are related as follows:

$$\tau = \frac{1}{a} \log \left[\frac{\cosh\left(\frac{\theta + a\tau_0}{2}\right)}{\cosh\left(\frac{\theta - a\tau_0}{2}\right)} \right] \in (-\tau_0, \tau_0), \quad (2.74)$$

with $\tau_0 = (1/a) \operatorname{arcsinh}(a\ell) > 0$ being half the proper lifetime of uniformly accelerating observer. Hence, up to this parametrization difference, the diamond flow coincides with the worldline for a uniformly accelerating observer with finite lifetime $2\tau_0$. The conformal Killing vector flow associated with the one-parameter subgroup (2.66) is depicted in Fig. 1(c).

Let us next move on to construct a coordinate patch for the diamond D where the temporal coordinate is identified with the $SO(1,1)$ group parameter.^P Such

^PAgain, if we work in a coordinate system in which the temporal coordinate is identified with the proper time, we have to deal with time-dependent temperature.²³

coordinate system is turned out to be of the form:

$$X^a = \lambda(Y)Y^a \in D, \quad (2.75)$$

where $Y^a = (Y^0, \dots, Y^{d+1})$ is a parametrization for the foliation $\mathbb{H}^1 \times \mathbb{H}^{d-1} \subset \bar{D}$ given by

$$Y^a = (\ell \sinh(t/\ell), \ell H^i, \ell \cosh(t/\ell), \ell H^{d+1}) \in \bar{D} \quad (2.76)$$

and $\lambda(Y)$ is a coordinate-dependent scaling factor that pulls back a point Y^a on the wedge $\bar{D}$ to a point X^a on the diamond D given as follows:

$$\lambda(Y) = \frac{\ell}{Y^d + Y^{d+1}} = \frac{1}{\cosh(t/\ell) + H^{d+1}} > 0. \quad (2.77)$$

Here $H^a = (H^1, \dots, H^{d-1}, H^{d+1})$ parametrizes the $(d-1)$ -dimensional hyperbolic space $\mathbb{H}^{d-1}$ and is subject to the following conditions:

$$H \cdot H \equiv (H^1)^2 + \dots + (H^{d-1})^2 - (H^{d+1})^2 = -1 \quad \text{and} \quad H^{d+1} \geq 1. \quad (2.78)$$

The induced metric on the diamond takes the following form:

$$ds_D^2 = \frac{-dt^2 + \ell^2 dH \cdot dH}{(\cosh(t/\ell) + H^{d+1})^2}, \quad (2.79)$$

which is again conformal to the hyperbolic space-time $\mathbb{H}^1 \times \mathbb{H}^{d-1}$. The two-point function projected out onto the diamond then takes the following form:

$$G_\Delta(X, X')|_{X, X' \in D} = \lambda^{-\Delta}(Y) \lambda^{-\Delta}(Y') \left[\frac{1/(2\ell^2)}{-\cosh(\frac{t-t'}{\ell}) - H \cdot H'} \right]^\Delta. \quad (2.80)$$

The overall factors can be removed by a conformal transformation, after which (2.80) describes the thermal two-point function on the hyperbolic space-time $\mathbb{H}^1 \times \mathbb{H}^{d-1}$. The remaining discussions are exactly the same as those presented in Subsec. 2.2 such that we will stop here.

2.5. A few remarks on the one-dimensional case

Let us close our construction of thermal CFTs in d dimensions by making a few remarks on the case $d = 1$.

In $(0+1)$ -dimensional space-time, there is no immediate analogue of Rindler wedge because of the absence of spatial directions. However, we can still introduce the light-cones and diamond as half-lines and interval of the time axis, and CFT restricted on these regions become thermal as well under the identification of the temporal coordinates with the $SO(1,1) \subset SO(2,1)$ group parameters; see Fig. 3. The embedding space formalism for one-dimensional CFT can be developed equally well and the resultant thermal CFT turns out to reside on the $(0+1)$ -dimensional hyperbolic space $\mathbb{H}^1$. It is a straightforward exercise to show that the thermal two-point function for a scalar primary operator on $\mathbb{H}^1$ takes the following form:

$$\left[\frac{\pi^2 T^2}{-\sinh^2(\pi T(t-t'))} \right]^\Delta, \quad (2.81)$$

Fig. 3. Light-cones and diamond in $(0+1)$ -dimensional space-time. x^0 and the $SO(1,1)$ group parameter $t \in (-\infty, \infty)$ are related as (a) $x^0 = \pm \ell e^{-t/\ell}$ and (b) $x^0 = \ell \tanh(t/(2\ell))$.

which of course satisfies the KMS condition (under the appropriate $i\epsilon$ -prescription). Note that (2.81) exactly coincides with the result of conformal quantum mechanics.³⁵

3. Intertwining Operator in CFT_d

So far, we have constructed thermal CFTs in d dimensions by using the embedding space formalism. Let us now turn to the representation-theoretic aspect of conformal two-point functions: the intertwining operators. Though it had been studied in the 1970s^{1–4} that two-point functions of CFT are nothing but the integral kernels of intertwining operators for two equivalent representations of conformal algebra $\mathfrak{so}(2, d)$, this fact is hardly ever acknowledged except to experts. We will see in the rest of the paper that the intertwining operator and the resulting intertwining relations provide powerful constraints on the structure of momentum-space thermal two-point functions. Before embarking on this, however, in this section, we first review the basics of intertwining operators in d -dimensional CFT and prepare necessary ingredients for our analysis. Then we show as an application that, for the case of zero-temperature CFT, the intertwining relations reduce to the well-known conformal Ward–Takahashi identities for momentum-space two-point functions.

We emphasize that the presentation of this section is meant to be rather sketchy, yet enough for the purpose of this paper, leaving mathematical rigors to the existing literature. For more details of intertwining operators in CFT, we refer to Refs. 1–4 (see also Ref. 36 for more recent expositions). We also note that in this section, we will use the standard d -dimensional approach to d -dimensional CFT: we will not return to the embedding space formalism in the remainder of the paper, though it may be possible to reformulate everything in the subsequent sections in the $(d+2)$ -dimensional language.

Throughout this section, we just focus on scalar two-point functions for simplicity.

3.1. Intertwining kernel = two-point function

To begin with, let $x \mapsto x_g$ be a generic conformal transformation associated with a group element $g \in SO(2, d)$. Let $\mathcal{O}_\Delta(x)$ be a scalar primary operator of scaling dimension Δ that satisfies the following transformation law:

$$U(g)\mathcal{O}_\Delta(x)U^{-1}(g) = \left| \frac{\partial x_g}{\partial x} \right|^{\Delta/d} \mathcal{O}_\Delta(x_g), \quad (3.1)$$

where $U(g)$ is a unitary representation of $g \in SO(2, d)$ and $|\partial x_g/\partial x|$ stands for the Jacobian of conformal transformation $x \mapsto x_g = x_g(x)$. Let $|\Omega\rangle$ be a conformally invariant Minkowski vacuum satisfying $U(g)|\Omega\rangle = |\Omega\rangle$ for any $g \in SO(2, d)$. Then the positive-frequency two-point Wightman function $G_\Delta(x, y)$, for example, is given by

$$G_\Delta(x, y) = \langle \Omega | \mathcal{O}_\Delta(x) \mathcal{O}_\Delta(y) | \Omega \rangle, \quad (3.2)$$

which satisfies the following identity (conformal Ward–Takahashi identity):

$$G_\Delta(x, y) = \left| \frac{\partial x_g}{\partial x} \right|^{\Delta/d} \left| \frac{\partial y_g}{\partial y} \right|^{\Delta/d} G_\Delta(x_g, y_g). \quad (3.3)$$

We note that, though (3.2) is the simplest example, for the following discussion $G_\Delta(x, y)$ is not necessarily to be the positive-frequency two-point Wightman function. The following discussion can be applied to any other two-point functions that satisfy the conformal Ward–Takahashi identity.

Now, let us next introduce an intertwining operator G_Δ , which is defined through the following integral transform whose kernel is the two-point function:

$$G_\Delta : \mathcal{O}_{d-\Delta}(x) \mapsto (G_\Delta \mathcal{O}_{d-\Delta})(x) := \int d^d y G_\Delta(x, y) \mathcal{O}_{d-\Delta}(y). \quad (3.4)$$

This operator G_Δ maps a primary operator of scaling dimension $d - \Delta$, which is called the “shadow operator,”⁵ to another primary operator of scaling dimension Δ . Indeed, as easily checked, the operator $(G_\Delta \mathcal{O}_{d-\Delta})(x)$ satisfies the following transformation law:

$$U(g)(G_\Delta \mathcal{O}_{d-\Delta})(x)U^{-1}(g) = \left| \frac{\partial x_g}{\partial x} \right|^{\Delta/d} (G_\Delta \mathcal{O}_{d-\Delta})(x_g). \quad (3.5)$$

Hence $(G_\Delta \mathcal{O}_{d-\Delta})(x)$ is indeed a primary operator of scaling dimension Δ . The proof of (3.5) is straightforward and goes as follows:

$$\begin{aligned} U(g)(G_\Delta \mathcal{O}_{d-\Delta})(x)U^{-1}(g) &= U(g) \left(\int d^d y G_\Delta(x, y) \mathcal{O}_{d-\Delta}(y) \right) U^{-1}(g) \\ &= \int d^d y G_\Delta(x, y) U(g) \mathcal{O}_{d-\Delta}(y) U^{-1}(g) \end{aligned}$$

$$\begin{aligned}
&= \int d^d y \left| \frac{\partial x_g}{\partial x} \right|^{\Delta/d} \left| \frac{\partial y_g}{\partial y} \right|^{\Delta/d} G_{\Delta}(x_g, y_g) \left| \frac{\partial y_g}{\partial y} \right|^{(d-\Delta)/d} \mathcal{O}_{\Delta}(y_g) \\
&= \left| \frac{\partial x_g}{\partial x} \right|^{\Delta/d} \int d^d y \left| \frac{\partial y_g}{\partial y} \right| G_{\Delta}(x_g, y_g) \mathcal{O}_{\Delta}(y_g) \\
&= \left| \frac{\partial x_g}{\partial x} \right|^{\Delta/d} \int d^d y_g G_{\Delta}(x_g, y_g) \mathcal{O}_{\Delta}(y_g) \\
&= \left| \frac{\partial x_g}{\partial x} \right|^{\Delta/d} (G_{\Delta} \mathcal{O}_{d-\Delta})(x_g), \tag{3.6}
\end{aligned}$$

where the third equality follows from (3.1) and (3.3). The fifth equality follows from the change of integration variable, $\int d^d y |\partial y_g / \partial y| = \int d^d y_g$. As is evident from this proof, $U(g)(G_{\Delta} \mathcal{O}_{d-\Delta})(x)U^{-1}(g)$ and $(G_{\Delta} U(g) \mathcal{O}_{d-\Delta} U^{-1}(g))(x)$ give the same result. Hence the intertwining operator G_{Δ} should satisfy the following operator identity (intertwining relation):

$$U(g)G_{\Delta} = G_{\Delta}U(g), \quad \forall g \in SO(2, d). \tag{3.7}$$

For practical calculations, however, it is more convenient to introduce the infinitesimal form of (3.7). To this end, let us next consider an infinitesimal conformal transformation associated with a group element $g^a_b = \delta^a_b + \epsilon^a_b$, where $\epsilon_{ab} = -\epsilon_{ba}$ are infinitesimal parameters. The conformal transformation $x \mapsto x_{1+\epsilon} = x_{1+\epsilon}(x)$ associated with such a group element $g = 1 + \epsilon$ can be Taylor-expanded as follows:

$$x_{1+\epsilon}^{\mu} = x^{\mu} + \frac{1}{2} \epsilon_{ab} k^{\mu ab}(x) \quad \text{with} \quad k^{\mu ab}(x) := \left. \frac{\partial x_{1+\epsilon}^{\mu}}{\partial \epsilon_{ab}} \right|_{\epsilon_{ab}=0}, \tag{3.8}$$

where $k^{\mu ab}(x) = -k^{\mu ba}(x)$ are conformal Killing vectors which of course coincide with those given in (2.20a)–(2.20d). Similarly, the unitary operator $U(g)$ associated with $g = 1 + \epsilon$ can also be Taylor-expanded:

$$U(1 + \epsilon) = 1 + \frac{i}{2} \epsilon_{ab} J^{ab}, \tag{3.9}$$

where $J^{ab} = -J^{ba}$ are Hermitian generators of the Lie algebra $\mathfrak{so}(2, d)$ satisfying the following commutation relations:

$$[J^{ab}, J^{cd}] = i(\eta^{ac} J^{bd} - \eta^{ad} J^{bc} - \eta^{bc} J^{ad} + \eta^{bd} J^{ac}). \tag{3.10}$$

Substituting (3.8) and (3.9) into (3.1), we get

$$\left(1 + \frac{i}{2} \epsilon_{ab} J^{ab}\right) \mathcal{O}_{\Delta}(x) \left(1 - \frac{i}{2} \epsilon_{ab} J^{ab}\right) = \left(1 + \frac{\Delta}{2d} \epsilon_{ab} \partial_{\mu} k^{\mu ab}\right) \mathcal{O}_{\Delta}\left(x + \frac{1}{2} \epsilon k\right), \tag{3.11}$$

which, at the linear order of ϵ_{ab} , reduces to the following form:

$$[J^{ab}, \mathcal{O}_{\Delta}(x)] = -J_{\Delta}^{ab} \mathcal{O}_{\Delta}(x), \tag{3.12}$$

where $J_{\Delta}^{ab} = -J_{\Delta}^{ba}$ are coordinate realizations of the $SO(2, d)$ generators given by

$$J_{\Delta}^{ab} = i \left(k^{\mu ab} \partial_{\mu} + \frac{\Delta}{d} \partial_{\mu} k^{\mu ab} \right). \quad (3.13)$$

Similarly, for the infinitesimal conformal transformations (3.8) and (3.9), the conformal Ward–Takahashi identity (3.3) becomes

$$\begin{aligned} G_{\Delta}(x, y) &= \left(1 + \frac{\Delta}{2d} \epsilon_{ab} \partial_{\mu} k^{\mu ab}(x) \right) \left(1 + \frac{\Delta}{2d} \epsilon_{ab} \partial_{\mu} k^{\mu ab}(y) \right) \\ &\times G_{\Delta} \left(x + \frac{1}{2} \epsilon k(x), y + \frac{1}{2} \epsilon k(y) \right), \end{aligned} \quad (3.14)$$

which, at the linear order of ϵ_{ab} , reduces to the following linear differential equations (infinitesimal conformal Ward–Takahashi identities):

$$(J_{\Delta}^{ab}(x, \partial_x) + J_{\Delta}^{ab}(y, \partial_y)) G_{\Delta}(x, y) = 0, \quad (3.15)$$

where $J_{\Delta}^{ab}(x, \partial_x)$ and $J_{\Delta}^{ab}(y, \partial_y)$ are the notations to emphasize that these are the differential operators acting on x and y , respectively.

Now let us move on to the infinitesimal form of intertwining relation (3.7). At the linear order of ϵ_{ab} , the intertwining relation (3.7) becomes $J^{ab} G_{\Delta} = G_{\Delta} J^{ab}$, or, equivalently,

$$(J_{\Delta}^{ab} G_{\Delta} \mathcal{O}_{d-\Delta})(x) = (G_{\Delta} J_{d-\Delta}^{ab} \mathcal{O}_{d-\Delta})(x). \quad (3.16)$$

Indeed, a straightforward calculation gives

$$\begin{aligned} (J_{\Delta}^{ab} G_{\Delta} \mathcal{O}_{d-\Delta})(x) &= \int d^d y (J_{\Delta}^{ab}(x, \partial_x) G_{\Delta}(x, y)) \mathcal{O}_{d-\Delta}(y) \\ &= \int d^d y (-J_{\Delta}^{ab}(y, \partial_y) G_{\Delta}(x, y)) \mathcal{O}_{d-\Delta}(y) \\ &= \int d^d y \left[-i \left(k^{\mu ab}(y) \frac{\partial}{\partial y^{\mu}} + \frac{\Delta}{d} \frac{\partial k^{\mu ab}(y)}{\partial y^{\mu}} \right) G_{\Delta}(x, y) \right] \mathcal{O}_{d-\Delta}(y) \\ &= \int d^d y G_{\Delta}(x, y) \left[i \left(k^{\mu ab}(y) \frac{\partial}{\partial y^{\mu}} + \frac{d-\Delta}{d} \frac{\partial k^{\mu ab}(y)}{\partial y^{\mu}} \right) \mathcal{O}_{d-\Delta}(y) \right] \\ &= \int d^d y G_{\Delta}(x, y) (J_{d-\Delta}^{ab}(y, \partial_y) \mathcal{O}_{d-\Delta}(y)) \\ &= (G_{\Delta} J_{d-\Delta}^{ab} \mathcal{O}_{d-\Delta})(x), \end{aligned} \quad (3.17)$$

where in the second line, we have used the infinitesimal conformal Ward–Takahashi identity (3.15) and in the third line, we have used the explicit expression (3.13) for the generator. The fourth equality follows from the integration by parts where we have ignored the surface term which is of no significance for the following discussions. It is obvious that one can also prove the identity $(J_{d-\Delta}^{ab} G_{d-\Delta} \mathcal{O}_{\Delta})(x) = (G_{d-\Delta} J_{\Delta}^{ab} \mathcal{O}_{\Delta})(x)$ in exactly the same way. It is also obvious that, since $\mathcal{O}_{d-\Delta}(x)$

Fig. 4. Commutative diagram of intertwining operators.

and $\mathcal{O}_{\Delta}(x)$ are arbitrary test primary operators in the above discussions, the following operator identities must hold:

$$J_{\alpha}^{ab} G_{\alpha} = G_{\alpha} J_{d-\alpha}^{ab}, \quad \alpha \in \{\Delta, d-\Delta\}. \quad (3.18)$$

It is these intertwining relations that we will use in the rest of the paper.

Several comments are in order.

- *Primary states.* The intertwining operator G_{α} also maps a primary state of scaling dimension $d-\alpha$ to another primary state of scaling dimension α , where α is either Δ or $d-\Delta$. In fact, by defining the coordinate-dependent primary state as

$$|\mathcal{O}_{\alpha}(x)\rangle := \mathcal{O}_{\alpha}(x)|\Omega\rangle, \quad \alpha \in \{\Delta, d-\Delta\}, \quad (3.19)$$

which satisfies the transformation law $U(g)|\mathcal{O}_{\alpha}(x)\rangle = |\partial x_g/\partial x|^{\alpha/d}|\mathcal{O}_{\alpha}(x)\rangle$, one can easily show that the state $G_{\alpha}|\mathcal{O}_{d-\alpha}(x)\rangle := (G_{\alpha}\mathcal{O}_{d-\alpha})(x)|\Omega\rangle$ also satisfies the transformation law $U(g)G_{\alpha}|\mathcal{O}_{d-\alpha}(x)\rangle = |\partial x_g/\partial x|^{\alpha/d}G_{\alpha}|\mathcal{O}_{d-\alpha}(x)\rangle$ and hence is the coordinate-dependent primary state of scaling dimension α . $|\mathcal{O}_{\alpha}(x)\rangle$ is an element of the representation space $\mathcal{V}_{\alpha}$ of conformal algebra $\mathfrak{so}(2, d)$ in which the quadratic Casimir operator $C_2[\mathfrak{so}(2, d)]$ takes the value $\Delta(d-\Delta)$. The intertwining relations between $\mathcal{V}_{d-\Delta}$ and $\mathcal{V}_{\Delta}$ are schematically summarized in the commutative diagram in Fig. 4.

- *Normalization.* The intertwining operators G_{Δ} and $G_{d-\Delta}$ are basically inverse to each other. Namely, under appropriate normalizations, they can be chosen to satisfy the following identity:⁹

$$G_{\alpha}G_{d-\alpha} = 1_{\mathcal{V}_{\alpha}}, \quad \alpha \in \{\Delta, d-\Delta\} \quad (3.20)$$

or, equivalently,

$$\int d^d z G_{\alpha}(x, z) G_{d-\alpha}(z, y) = \delta^{(d)}(x - y), \quad (3.21)$$

⁹There is, however, a subtle problem in the identity (3.20). A careful analysis shows that the identity (3.21) (or $\tilde{G}_{\Delta}(p)\tilde{G}_{d-\Delta}(p) = 1$) does not hold when $\Delta = d/2 + n$ ($n \in \mathbb{Z}_{\geq 0}$), which may well be related to the conformal anomaly; see, e.g. Ref. 37. We will not address this issue in the present paper.

where $1_{\mathcal{V}_\alpha}$ stands for the identity operator in the representation space $\mathcal{V}_\alpha$. Note that Eq. (3.21) is most simply expressed in the momentum space as $\tilde{G}_\Delta(p)\tilde{G}_{d-\Delta}(p) = 1$, where $\tilde{G}_\alpha(p)$ stands for the Fourier transform of two-point function $G_\alpha(x, y)$.

- *Unitarity bound.* It is well known that in d -dimensional space-time, the unitarity bound for scaling dimension Δ of scalar primary operator is $\Delta > (d-2)/2$, where the lower bound is the scaling dimension of free scalar field. In general, $d - \Delta$ violates this bound if Δ is big enough. However, there exists a range of Δ in which both the scaling dimensions of $\mathcal{O}_\Delta(x)$ and its shadow $\mathcal{O}_{d-\Delta}(x)$ fulfill the unitarity bound:

$$\frac{d-2}{2} < \Delta < \frac{d+2}{2}. \quad (3.22)$$

In the AdS/CFT correspondence this range of Δ corresponds to the well-known range of scalar field mass-squared $-d^2/4 < (mR)^2 < -d^2/4 + 1$ (R : AdS radius) in which there exist two different quantizations.³⁸

- *Conformal algebra.* It is worth to write down here the d -dimensional conformal algebra because the relation between the conformal generators $\{D, P^\mu, K^\mu, M^{\mu\nu}\}$ and $SO(2, d)$ generators J^{ab} often differs in the literature. In our convention, the dilatation generator D , momentum generators P^μ , special conformal transformation generators K^μ , and Lorentz transformation generators $M^{\mu\nu}$ that are consistent with (2.8), (2.10), (2.12) and (2.14) are given by the following linear combinations:

$$\begin{aligned} D &= J^{d, d+1}, \quad P^\mu = \frac{1}{\ell}(J^{d+1, \mu} + J^{d\mu}), \\ K^\mu &= \ell(J^{d+1, \mu} - J^{d\mu}), \quad M^{\mu\nu} = J^{\mu\nu}, \end{aligned} \quad (3.23)$$

which satisfy the commutation relations of d -dimensional conformal algebra:

$$\begin{aligned} [D, P^\mu] &= iP^\mu, \quad [D, K^\mu] = -iK^\mu, \quad [P^\mu, K^\nu] = 2i(\eta^{\mu\nu}D - J^{\mu\nu}), \\ [M^{\mu\nu}, P^\rho] &= i(\eta^{\mu\rho}P^\nu - \eta^{\nu\rho}P^\mu), \\ [M^{\mu\nu}, K^\rho] &= i(\eta^{\mu\rho}K^\nu - \eta^{\nu\rho}K^\mu), \\ [M^{\mu\nu}, M^{\rho\sigma}] &= i(\eta^{\mu\rho}M^{\nu\sigma} - \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\nu\rho}M^{\mu\sigma} + \eta^{\nu\sigma}M^{\mu\rho}). \end{aligned} \quad (3.24)$$

With this definition, the infinitesimal conformal transformation $U(1 + \epsilon) = 1 + \frac{i}{2}\epsilon_{ab}J^{ab}$ is expressed as follows:

$$U(1 + \epsilon) = 1 + i\varphi D - ia_\mu P^\mu + ib_\mu K^\mu + \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}, \quad (3.25)$$

where the infinitesimal parameters ϵ^a_b and $(\varphi, a^\mu, b^\mu, \omega^\mu_\nu)$ are identified under the rule (2.16). Finite conformal transformations are obtained by exponentiating the last four terms in (3.25). For example, one can easily check that, by using

the $(d+2) \times (d+2)$ -matrix representation of conformal generators

$$\begin{aligned}
 (D)^a{}_b &= -i \begin{pmatrix} \mathbf{0}_d & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}, \\
 (M^{\mu\nu})^a{}_b &= -i \begin{pmatrix} \eta^{\mu a} \delta^\nu{}_b - \eta^{\nu a} \delta^\mu{}_b & & \\ & 0 & 0 \\ & 0 & 0 \end{pmatrix}, \\
 (P^\mu)^a{}_b &= \frac{i}{\ell} \begin{pmatrix} \mathbf{0}_d & \eta^{\mu a} & \eta^{\mu a} \\ -\delta^\mu{}_b & 0 & 0 \\ +\delta^\mu{}_b & 0 & 0 \end{pmatrix}, \\
 (K^\mu)^a{}_b &= -i\ell \begin{pmatrix} \mathbf{0}_d & \eta^{\mu a} & -\eta^{\mu a} \\ -\delta^\mu{}_b & 0 & 0 \\ -\delta^\mu{}_b & 0 & 0 \end{pmatrix},
 \end{aligned} \tag{3.26}$$

which follow from the matrix representation of $SO(2, d)$ generators $i(J^{ab})^c{}_d = \eta^{ac} \delta^b{}_d - \eta^{bc} \delta^a{}_d$, the $(d+2) \times (d+2)$ matrices $\exp(i\varphi D) \in SO(1, 1)$, $\exp(-ia_\mu P^\mu) \in E(1)^d$ and $\exp(ib_\mu K^\mu) \in E(1)^d$ exactly coincides with (2.8), (2.10) and (2.12). (These matrix representations are not unitary representations of course.) For our purpose, however, the coordinate realization of conformal generators is more important. Substituting the conformal Killing vectors into (3.13), one can easily check that the coordinate realization $\{D_\alpha, P^\mu_\alpha, K^\mu_\alpha, M^{\mu\nu}_\alpha\}$ that acts on the representation space $\mathcal{V}_\alpha$ of scaling dimension $\alpha \in \{\Delta, d - \Delta\}$ is given as follows:

$$D_\alpha = -i(x \cdot \partial + \alpha), \tag{3.27a}$$

$$P^\mu_\alpha = -i\partial^\mu, \tag{3.27b}$$

$$K^\mu_\alpha = -i[(x \cdot x)\partial^\mu - 2x^\mu(x \cdot \partial + \alpha)], \tag{3.27c}$$

$$M^{\mu\nu}_\alpha = -i(x^\mu \partial^\nu - x^\nu \partial^\mu). \tag{3.27d}$$

We shall use these differential operators in the next section.

3.2. Intertwining relations in the $E(1)$ basis

The intertwining relations are the operator identities in the representation space $\mathcal{V}_\alpha$ of fixed scaling dimension $\alpha \in \{\Delta, d - \Delta\}$. In the last section, we have seen that the operator identities (3.18) hold when they act on primary operators or primary states. However, the intertwining relations still hold when they are applied to the basis of representation space $\mathcal{V}_\alpha$. In this section, we study the intertwining relations in the basis where the $E(1)$ transformations (2.10) become diagonal. In other words,

we apply the identity (3.18) to the simultaneous eigenfunctions of $E(1)$ generators (3.27b) which provide a complete orthonormal basis of $\mathcal{V}_\alpha$. We will see that in this case, the intertwining relations just reduce to the well-known conformal Ward–Takahashi identities for momentum-space two-point function of zero-temperature CFT.

To begin with, let $\{f_p(x)\}$ be a complete orthonormal basis that satisfies the following orthonormality and completeness relations:

$$\int d^d x f_p^*(x) f_q(x) = (2\pi)^d \delta^{(d)}(p - q), \quad (3.28a)$$

$$\int \frac{d^d p}{(2\pi)^d} f_p(x) f_p^*(y) = \delta^{(d)}(x - y). \quad (3.28b)$$

In the flat Minkowski space-time, such complete orthonormal basis is of course given by the plane wave, $f_p(x) = e^{ip \cdot x}$, which is nothing but the simultaneous eigenfunction of $E(1)$ generators $P_\alpha^\mu = -i\partial^\mu$ with eigenvalue p^μ . The primary operator and two-point function can then be expanded in terms of this complete orthonormal basis:

$$\mathcal{O}_\alpha(x) = \int \frac{d^d p}{(2\pi)^d} \tilde{\mathcal{O}}_\alpha(p) f_p(x), \quad (3.29a)$$

$$G_\alpha(x, y) = \int \frac{d^d p}{(2\pi)^d} \tilde{G}_\alpha(p) f_p(x) f_p^*(y), \quad \alpha \in \{\Delta, d - \Delta\}, \quad (3.29b)$$

which are of course just the Fourier transforms. Plugging these into the left- and right-hand sides of the identity $(J_\Delta^{ab} G_\Delta \mathcal{O}_{d-\Delta})(x) = (G_\Delta J_{d-\Delta}^{ab} \mathcal{O}_{d-\Delta})(x)$, we get

$$(J_\Delta^{ab} G_\Delta \mathcal{O}_{d-\Delta})(x) = \int \frac{d^d p}{(2\pi)^d} \int \frac{d^d q}{(2\pi)^d} \langle p | J_\Delta^{ab} | q \rangle \tilde{G}_\Delta(q) \tilde{\mathcal{O}}_{d-\Delta}(q) f_p(x), \quad (3.30a)$$

$$(G_\Delta J_{d-\Delta}^{ab} \mathcal{O}_{d-\Delta})(x) = \int \frac{d^d p}{(2\pi)^d} \int \frac{d^d q}{(2\pi)^d} \tilde{G}_\Delta(p) \langle p | J_{d-\Delta}^{ab} | q \rangle \tilde{\mathcal{O}}_{d-\Delta}(q) f_p(x), \quad (3.30b)$$

where $\langle p | J_\alpha^{ab} | q \rangle$ are “matrix elements” given by

$$\langle p | J_\alpha^{ab} | q \rangle := \int d^d x f_p^*(x) J_\alpha^{ab} f_q(x), \quad \alpha \in \{\Delta, d - \Delta\}. \quad (3.31)$$

Comparing the expressions (3.30a) and (3.30b) and noting $\tilde{\mathcal{O}}_{d-\Delta}(q)$ is an arbitrary primary operator, we arrive at the following identities for the momentum-space two-point function:

$$\langle p | J_\Delta^{ab} | q \rangle \tilde{G}_\Delta(q) = \tilde{G}_\Delta(p) \langle p | J_{d-\Delta}^{ab} | q \rangle. \quad (3.32)$$

Though it is less obvious, the intertwining relation (3.32) is just another form of the conformal Ward–Takahashi identity. To see this, let us first compute the “matrix elements” $\langle p | J_\alpha^{ab} | q \rangle$ by using the coordinate realizations (3.13). Substituting (3.27a)–(3.27d) into (3.31), we get the following “matrix elements”:

$$\langle p | D_\alpha | q \rangle = -i(q \cdot \partial_q + \alpha) \delta^{(d)}(p - q), \quad (3.33a)$$

$$\langle p|P_\alpha^\mu|q\rangle = q^\mu \delta^{(d)}(p-q), \quad (3.33b)$$

$$\langle p|K_\alpha^\mu|q\rangle = [-q^\mu \partial_q \cdot \partial_q + 2(q \cdot \partial_q + \alpha) \partial_q^\mu] \delta^{(d)}(p-q), \quad (3.33c)$$

$$\langle p|M_\alpha^{\mu\nu}|q\rangle = i(q^\mu \partial_q^\nu - q^\nu \partial_q^\mu) \delta^{(d)}(p-q). \quad (3.33d)$$

Now it is a straightforward exercise to show that the intertwining relation (3.32) boils down to the following differential equations for the momentum-space two-point function:

$$(p^\mu \partial^\nu - p^\nu \partial^\mu) \tilde{G}_\Delta(p) = 0, \quad (3.34a)$$

$$[p^\mu \partial \cdot \partial - 2(p \cdot \partial - \Delta + d) \partial^\mu] \tilde{G}_\Delta(p) = 0, \quad (3.34b)$$

$$(p \cdot \partial - 2\Delta + d) \tilde{G}_\Delta(p) = 0, \quad (3.34c)$$

which are nothing but the well-known conformal Ward–Takahashi identities in the momentum space. These differential equations are solved as follows. First, Eq. (3.34a) implies $\tilde{G}_\Delta$ must be a function of the Lorentz invariant $p \cdot p$. And, for such a function, Eq. (3.34c) reduces to $[(p \cdot p) \frac{d}{d(p \cdot p)} - (\Delta - \frac{d}{2})] \tilde{G}_\Delta = 0$, which is easily solved with the result

$$\tilde{G}_\Delta(p) \propto (p \cdot p)^{\Delta-d/2}. \quad (3.35)$$

Note that the solution (3.35) automatically satisfies (3.34b).

Before closing this section, it is wise to reexamine here the intertwining relations (3.32) in a little bit more abstract way. To this end, we first define the action of intertwining operator G_α on the basis function $f_p(x)$ as follows:

$$(G_\alpha f_p)(x) := \int d^d y G_\alpha(x, y) f_p(y), \quad (3.36)$$

which, after substituting the expansion (3.29b), can be reduced to the following:

$$(G_\alpha f_p)(x) = \tilde{G}_\alpha(p) f_p(x). \quad (3.37)$$

This is best expressed in the following bra–ket notation:

$$G_\alpha|p\rangle = \tilde{G}_\alpha(p)|p\rangle. \quad (3.38)$$

That is to say, the momentum-space two-point function $\tilde{G}_\alpha(p)$ is the eigenvalue of intertwining operator G_α . Now it is obvious that the intertwining relations (3.32) are equivalent to $\langle p|J_\Delta^{ab} G_\Delta|q\rangle = \langle p|G_\Delta J_{d-\Delta}^{ab}|q\rangle$, which is nothing but the “matrix elements” of the operator identities (3.18).

To summarize, in zero-temperature CFT, the intertwining relations are just the conformal Ward–Takahashi identities in disguise. At finite temperature, however, we shall see that the intertwining relations lead to novel constraints on momentum-space two-point functions.

4. Thermal Correlator Recursions

In the last section, we have studied the intertwining relations in the $E(1)$ basis in which the $E(1)$ generators P_α^μ become diagonal. For the case of finite-temperature CFT, however, we need to work on the basis in which the $SO(1,1)$ generator becomes diagonal because, as we have seen in Sec. 2, the time-translation generator generates the one-parameter subgroup $SO(1,1)$ in geometrically thermalized CFTs. In this section, we shall first develop the representation of conformal algebra $\mathfrak{so}(2,d)$ in the $SO(1,1)$ basis and derive the basis function explicitly. We then show that the intertwining relations in the $SO(1,1)$ basis result in the recurrence relations for momentum-space thermal two-point functions.

For the sake of notational brevity, throughout this section, we will work in the units $\ell = 1$ (i.e. $2\pi T = 1$). The Unruh temperature T is easily restored by dimensional analysis.

4.1. Representation of the Lie algebra $\mathfrak{so}(2,d)$ in the $SO(1,1)$ basis

Let us first construct the $SO(1,1)$ basis for the representation space of the Lie algebra $\mathfrak{so}(2,d)$. As we have seen in Subsecs. 2.2–2.4, in thermal CFTs on the Rindler wedge, light-cone and diamond, the time translations are given by the $SO(1,1)$ hyperbolic rotations on the (X^0, X^1) -, (X^d, X^{d+1}) - and (X^0, X^d) -planes. This motivates us to introduce the following linear combinations of the $SO(2,d)$ generators:

$$\text{Rindler wedge: } H = J^{10}, \quad E^{\pm a} = J^{0a} \pm J^{1a}, \quad M^{ab} = J^{ab}, \quad (4.1a)$$

$$\text{Light-cone: } H = J^{d,d+1}, \quad E^{\pm a} = J^{d+1,a} \pm J^{da}, \quad M^{ab} = J^{ab}, \quad (4.1b)$$

$$\text{Diamond: } H = J^{d0}, \quad E^{\pm a} = J^{0a} \pm J^{da}, \quad M^{ab} = J^{ab}, \quad (4.1c)$$

where the indices a and b run through $\{2, \dots, d+1\}$, $\{0, \dots, d-1\}$, and $\{1, \dots, d-1, d+1\}$ for the Rindler wedge, light-cone and diamond, respectively. Irrespective of these linear combinations, it follows from (3.10) that the set of operators $\{H, E^{\pm a}, M^{ab}\}$ satisfy the following same commutation relations:

$$[H, E^{\pm a}] = \pm i E^{\pm a}, \quad (4.2a)$$

$$[E^{+a}, E^{-b}] = 2i(\eta^{ab}H - M^{ab}), \quad (4.2b)$$

$$[M^{ab}, E^{\pm c}] = i(\eta^{ac}E^{\pm b} - \eta^{bc}E^{\pm a}), \quad (4.2c)$$

$$[M^{ab}, M^{cd}] = i(\eta^{ac}M^{bd} - \eta^{ad}M^{bc} - \eta^{bc}M^{ad} + \eta^{bd}M^{ac}), \quad (4.2d)$$

with other commutators vanishing. The quadratic Casimir operator of the Lie algebra $\mathfrak{so}(2,d)$ is then expressed as follows:

$$C_2[\mathfrak{so}(2,d)] = \frac{1}{2}J^{ab}J_{ab} = -H(H \pm id) - \eta_{ab}E^{\mp a}E^{\pm b} + C_2[\mathfrak{so}(1,d-1)], \quad (4.3)$$

where $C_2[\mathfrak{so}(1, d-1)]$ is the quadratic Casimir operator of the subalgebra $\mathfrak{so}(1, d-1)$ given by

$$C_2[\mathfrak{so}(1, d-1)] = \frac{1}{2} M^{ab} M_{ab}. \quad (4.4)$$

Note that $C_2[\mathfrak{so}(1, d-1)]$ commutes with H and M^{ab} , whereas it does not commute with $E^{\pm a}$. The quadratic Casimir operator of the whole algebra $C_2[\mathfrak{so}(2, d)]$, on the other hand, commutes with all the generators of course.

Now, we wish to find a complete orthonormal basis for the representation space $\mathcal{V}_\alpha$ of scaling dimension $\alpha \in \{\Delta, d - \Delta\}$ where the time-translation generator H becomes diagonal. This can be done as follows. Since H commutes with the subalgebra $\mathfrak{so}(1, d-1)$, there exists a simultaneous eigenstate of $C_2[\mathfrak{so}(2, d)]$, H , and $C_2[\mathfrak{so}(1, d-1)]$ that satisfies the following eigenvalue equations:

$$C_2[\mathfrak{so}(2, d)]|\alpha, \omega, j; \sigma\rangle = \Delta(\Delta - d)|\alpha, \omega, j; \sigma\rangle, \quad (4.5a)$$

$$H|\alpha, \omega, j; \sigma\rangle = \omega|\alpha, \omega, j; \sigma\rangle, \quad (4.5b)$$

$$C_2[\mathfrak{so}(1, d-1)]|\alpha, \omega, j; \sigma\rangle = j(j - d + 2)|\alpha, \omega, j; \sigma\rangle, \quad (4.5c)$$

where the eigenvalues $\Delta(\Delta - d)$, ω and $j(j - d + 2)$ are all real. The label σ denotes the set of eigenvalues for all other operators that commute with H and $C_2[\mathfrak{so}(1, d-1)]$, which, as we will see shortly for the case of Rindler wedge, can be chosen to the eigenvalues of momentum operators perpendicular to the (x^0, x^1) -plane. Before going to construct the eigenstate $|\alpha, \omega, j; \sigma\rangle$, it is worthwhile to point out here the meaning of the operators $E^{\pm a}$. It just follows from the commutation relations $[H, E^{\pm a}] = \pm i E^{\pm a}$ that the states $E^{\pm a}|\alpha, \omega, j; \sigma\rangle$ satisfy the following eigenvalue equations:

$$H E^{\pm a}|\alpha, \omega, j; \sigma\rangle = (\omega \pm i) E^{\pm a}|\alpha, \omega, j; \sigma\rangle, \quad (4.6)$$

which implies the operators $E^{\pm a}$ raise and lower the eigenvalue ω by $\pm i$. Thus, one may write

$$E^{\pm a}|\alpha, \omega, j; \sigma\rangle = \sum_{j' \sigma'} c_{j \sigma; j' \sigma'}^{\pm a}(\alpha, \omega) |\alpha, \omega \pm i, j'; \sigma'\rangle. \quad (4.7)$$

We shall see that it is these coefficients $c_{j \sigma; j' \sigma'}^{\pm a}(\alpha, \omega)$ that determines the momentum-space two-point function of thermal CFT on the hyperbolic space-time $\mathbb{H}^1 \times \mathbb{H}^{d-1}$. In order to find these coefficients, we first need to solve the eigenvalue equations (4.5a)–(4.5c) and construct the eigenstate $|\alpha, \omega, j; \sigma\rangle$. To this end, from now on, we shall focus on the linear combinations (4.1a) and utilize the coordinate realizations of the generators in the Rindler coordinate system $(t, x, \mathbf{x}_\perp)$. A straightforward (yet a little bit tedious) calculation shows that, in the Rindler coordinate system given in (2.48), the coordinate realizations of the operators $C_2[\mathfrak{so}(2, d)]_\alpha = \frac{1}{2} \sum_{a,b=0}^{d+1} J_\alpha^{ab} J_{\alpha,ab}$, $H_\alpha = J_\alpha^{10}$ and $C_2[\mathfrak{so}(1, d-1)]_\alpha = \frac{1}{2} \sum_{a,b=2}^{d+1} J_\alpha^{ab} J_{\alpha,ab}$ that act on

the representation space $\mathcal{V}_\alpha$ of scaling dimension $\alpha \in \{\Delta, d - \Delta\}$ take the following forms:^r

$$C_2[\mathfrak{so}(2, d)]_\alpha = \Delta(\Delta - d), \quad (4.8a)$$

$$H_\alpha = i\partial_t, \quad (4.8b)$$

$$C_2[\mathfrak{so}(1, d - 1)]_\alpha = x^2 \left(\Delta_{\mathbb{R}^{d-2}} + \partial_x^2 + \frac{2\alpha - d + 3}{x} \partial_x + \frac{\alpha(\alpha - d + 2)}{x^2} \right), \quad (4.8c)$$

where $\Delta_{\mathbb{R}^{d-2}} = \sum_{i=2}^{d-1} \frac{\partial}{\partial x_\perp^i} \frac{\partial}{\partial x_\perp^i}$ is the Laplacian of $(d - 2)$ -dimensional space $\mathbb{R}^{d-2}$ that is perpendicular to the (x^0, x^1) -plane. Note that the quadratic Casimir operator (4.8a) becomes constant and hence the eigenvalue equation (4.5a) does not give us any information. What we need to solve is therefore the following differential equations:

$$i\partial_t f = \omega f, \quad (4.9a)$$

$$x^2 \left(\Delta_{\mathbb{R}^{d-2}} + \partial_x^2 + \frac{2\alpha - d + 3}{x} \partial_x + \frac{\alpha(\alpha - d + 2)}{x^2} \right) f = j(j - d + 2)f. \quad (4.9b)$$

The first equation implies that the t -dependence of f is just the plane wave $e^{-i\omega t}$. In order to solve the second equation, we first note that Eq. (4.9b) can be cast into the following Schrödinger-like equation:

$$\left[-\partial_x^2 + \frac{\left(j - \frac{d-2}{2}\right)^2 - \frac{1}{4}}{x^2} \right] \bar{f} = -\mathbf{p}_\perp^2 \bar{f}, \quad (4.10)$$

where $\bar{f} := x^{\alpha-(d-3)/2} f$ and $\mathbf{p}_\perp^2$ is the eigenvalue of $-\Delta_{\mathbb{R}^{d-2}}$ whose eigenfunction is the plane wave $e^{i\mathbf{p}_\perp \cdot \mathbf{x}_\perp}$. Note that Eq. (4.10) is the bound-state problem of inverse-square potential, which has no solution unless $(j - (d - 2)/2)^2$ becomes negative. Hence j must be of the form

$$j = \frac{d-2}{2} + ik, \quad k \in (0, \infty). \quad (4.11)$$

In other words, the representation of the subalgebra $\mathfrak{so}(1, d - 1)$ must be the principal series representation, which is one of the continuous series representations of the indefinite orthogonal group. It should be emphasized that, though j is complex, the combination $j(j - d + 2) = -k^2 - (d - 2)^2/4$ is real. The solution to the bound-state problem (4.10) that converges as $x \rightarrow \infty$ is given by $x^{1/2} K_{ik}(|\mathbf{p}_\perp| x)$, where $K_\nu(z)$ is the modified Bessel function of the second kind. Collecting the above pieces, we

^rTo derive these, substitute the following into (3.13) with (2.20a)–(2.20d):

$$\begin{aligned} x^0 &= \pm x \sinh t, & x^1 &= \pm x \cosh t, \\ \partial_0 &= \pm \frac{\cosh t}{x} \partial_t \mp \sinh t \partial_x, & \partial_1 &= \mp \frac{\sinh t}{x} \partial_t \pm \cosh t \partial_x. \end{aligned}$$

arrive at the following eigenfunction:

$$f_{\alpha,\omega,k;\mathbf{p}_\perp}(t,x,\mathbf{x}_\perp) = \sqrt{\frac{4k \sinh(\pi k)}{\pi}} x^{-\alpha+(d-2)/2} K_{ik}(|\mathbf{p}_\perp|x) e^{-i\omega t + i\mathbf{p}_\perp \cdot \mathbf{x}_\perp}, \quad (4.12)$$

where the normalization factor is chosen for later convenience. Note that $\mathbf{p}_\perp$ corresponds to σ in Eqs. (4.5a)–(4.5c). It can be shown that the eigenfunction (4.12) provides a complete orthonormal basis on the Rindler wedge; that is, it satisfies the orthonormality^s

$$\begin{aligned} \int_{t,x,\mathbf{x}_\perp} f_{\alpha,\omega,k,\mathbf{p}_\perp}^*(t,x,\mathbf{x}_\perp) f_{d-\alpha,\omega',k',\mathbf{p}'_\perp}(t,x,\mathbf{x}_\perp) \\ = (2\pi)^d \delta(\omega - \omega') \delta(k - k') \delta^{(d-2)}(\mathbf{p}_\perp - \mathbf{p}'_\perp) \end{aligned} \quad (4.13a)$$

and the completeness

$$\begin{aligned} \int_{\omega,k,\mathbf{p}_\perp} f_{d-\alpha,\omega,k,\mathbf{p}_\perp}(t,x,\mathbf{x}_\perp) f_{\alpha,\omega,k,\mathbf{p}_\perp}^*(t',x',\mathbf{x}'_\perp) \\ = \frac{1}{x} \delta(t - t') \delta(x - x') \delta^{(d-2)}(\mathbf{x}_\perp - \mathbf{x}'_\perp), \end{aligned} \quad (4.13b)$$

where

$$\int_{t,x,\mathbf{x}_\perp} := \int_{-\infty}^{\infty} dt \int_0^{\infty} x dx \int d^{d-2} \mathbf{x}_\perp, \quad (4.14a)$$

$$\int_{\omega,k,\mathbf{p}_\perp} := \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_0^{\infty} \frac{dk}{2\pi} \int \frac{d^{d-2} \mathbf{p}_\perp}{(2\pi)^{d-2}}. \quad (4.14b)$$

The two-point function on the Rindler wedge can then be expanded as follows:

$$G_\alpha(t,x,\mathbf{x}_\perp; t',x',\mathbf{x}'_\perp) = \int_{\omega,k,\mathbf{p}_\perp} \tilde{G}_\alpha(\omega,k) f_{\alpha,\omega,k,\mathbf{p}_\perp}(t,x,\mathbf{x}_\perp) f_{\alpha,\omega,k,\mathbf{p}_\perp}^*(t',x',\mathbf{x}'_\perp). \quad (4.15)$$

It should be noted that, because G_α is an $SO(1,1) \times SO(1,d-1)$ scalar, its Fourier transform $\tilde{G}_\Delta$ only depends on the $SO(1,1)$ eigenvalue ω and the $SO(1,d-1)$ invariant k : it does not depend on the eigenvalues $\mathbf{p}_\perp$ of the generators of the subalgebra $\mathfrak{so}(1,d-1)$.^t Note also that, just as G_Δ maps a primary operator of scaling

^sEquations (4.13a) and (4.13b) follow from the following identities for the modified Bessel function of the second kind (also known as the Macdonald function) with imaginary index (see, e.g. Refs. 39–41):

$$\begin{aligned} \frac{4k \sinh(\pi k)}{\pi} \int_0^{\infty} \frac{dx}{x} K_{ik}(x) K_{ik'}(x) &= 2\pi \delta(k - k') \quad \text{for } k, k' > 0; \\ \int_0^{\infty} \frac{dk}{2\pi} \frac{4k \sinh(\pi k)}{\pi} K_{ik}(x) K_{ik}(x') &= x \delta(x - x') \quad \text{for } x, x' > 0. \end{aligned}$$

We note that these identities are mutually related through the Kontorovich–Lebedev transform.

^tThis would not be the case for two-point functions of generic primary tensors.

dimension $d - \Delta$ to another primary operator of scaling dimension Δ , the intertwining operator also maps a basis function $f_{d-\Delta, \omega, k; \mathbf{p}_\perp}(t, x, \mathbf{x}_\perp)$ of the representation space $\mathcal{V}_{d-\Delta}$ to another basis function $f_{\Delta, \omega, k; \mathbf{p}_\perp}(t, x, \mathbf{x}_\perp)$ of the representation space $\mathcal{V}_\Delta$ and satisfies the following equation:

$$\begin{aligned} (G_\Delta f_{d-\Delta, \omega, k; \mathbf{p}_\perp})(t, x, \mathbf{x}_\perp) &:= \int_{t', x', \mathbf{x}'_\perp} G_\Delta(t, x, \mathbf{x}_\perp; t', x', \mathbf{x}'_\perp) f_{d-\Delta, \omega, k; \mathbf{p}_\perp}(t', x', \mathbf{x}'_\perp) \\ &= \tilde{G}_\Delta(\omega, k) f_{\Delta, \omega, k; \mathbf{p}_\perp}(t, x, \mathbf{x}_\perp), \end{aligned} \quad (4.16)$$

where the last equality follows from Eqs. (4.15) and (4.13a). Equation (4.16) is most simply expressed by the bra-ket notation:

$$G_\Delta |d - \Delta, \omega, k; \mathbf{p}_\perp\rangle = \tilde{G}_\Delta(\omega, k) |\Delta, \omega, k; \mathbf{p}_\perp\rangle. \quad (4.17)$$

This equation, together with the ladder equations (4.7), enables us to extract powerful constraints on $\tilde{G}_\Delta(\omega, k)$ from the intertwining relations. To see this, we need to explicitly compute the coefficients $c_{j\sigma; j'\sigma'}^{\pm a}(\alpha, \omega)$ in (4.7). We shall do this in the next section and show that the intertwining relations result in certain linear recurrence relations for $\tilde{G}_\Delta(\omega, k)$ in the complex momentum space.

4.2. Intertwining relations in the $SO(1, 1)$ basis

In general, it is quite complicated to compute all the coefficients of the ladder equations (4.7). For our purpose, however, there is no need to compute all those coefficients. It turns out that it is sufficient to consider the ladder equations for the linear combinations $E_\alpha^{\pm d} + E_\alpha^{\pm(d+1)}$, which take the following simple coordinate realizations in the Rindler coordinate system:^u

$$E_\alpha^{\pm d} + E_\alpha^{\pm(d+1)} = i(-\partial_0 \pm \partial_1) = e^{\pm t} \left(-\frac{i\partial_t}{x} \pm i\partial_x \right). \quad (4.18)$$

By applying this to the eigenfunction (4.12) and using the recurrence relations of the modified Bessel function

$$\frac{1}{z} K_\nu(z) = -\frac{K_{\nu-1}(z) - K_{\nu+1}(z)}{2\nu} \quad \text{and} \quad \frac{d}{dz} K_\nu(z) = -\frac{K_{\nu-1}(z) + K_{\nu+1}(z)}{2}, \quad (4.19)$$

one immediately arrives, in the bra-ket notation, at the following ladder equations:

$$\begin{aligned} &(E_\alpha^{\pm d} + E_\alpha^{\pm(d+1)}) |\alpha, \omega, k; \mathbf{p}_\perp\rangle \\ &= A^\pm \left[\alpha - \frac{d-2}{2} \mp i(\omega \pm k) \right] |\alpha, \omega \pm i, k + i; \mathbf{p}_\perp\rangle \\ &\quad + B^\pm \left[\alpha - \frac{d-2}{2} \mp i(\omega \mp k) \right] |\alpha, \omega \pm i, k - i; \mathbf{p}_\perp\rangle, \end{aligned} \quad (4.20)$$

where $A^\pm$ and $B^\pm$ are α -independent irrelevant factors.

^uThese are the coordinate realizations in W_R . In W_L , just replace x and ∂_x to $-x$ and $-\partial_x$.

Now we use the intertwining relations:

$$\left(E_{\Delta}^{\pm d} + E_{\Delta}^{\pm(d+1)}\right)G_{\Delta} = G_{\Delta}\left(E_{d-\Delta}^{\pm d} + E_{d-\Delta}^{\pm(d+1)}\right). \quad (4.21)$$

Applying this to the basis $|d - \Delta, \omega, k; \mathbf{p}_{\perp}\rangle$, one immediately sees that the intertwining relations (4.21) result in the following identities:

$$\begin{aligned} \left[\Delta - \frac{d-2}{2} \mp i(\omega \mp k)\right] \tilde{G}_{\Delta}(\omega, k) \\ = \left[\tilde{\Delta} - \frac{d-2}{2} \mp i(\omega \mp k)\right] \tilde{G}_{\Delta}(\omega \pm i, k - i), \end{aligned} \quad (4.22a)$$

$$\begin{aligned} \left[\Delta - \frac{d-2}{2} \mp i(\omega \pm k)\right] \tilde{G}_{\Delta}(\omega, k) \\ = \left[\tilde{\Delta} - \frac{d-2}{2} \mp i(\omega \pm k)\right] \tilde{G}_{\Delta}(\omega \pm i, k + i) \end{aligned} \quad (4.22b)$$

or, equivalently, to the following recurrence relations in the complex momentum space:

$$\tilde{G}_{\Delta}(\omega \pm i, k \pm i) = \frac{\Delta - (d-2)/2 \mp i(\omega + k)}{\tilde{\Delta} - (d-2)/2 \mp i(\omega + k)} \tilde{G}_{\Delta}(\omega, k), \quad (4.23a)$$

$$\tilde{G}_{\Delta}(\omega \pm i, k \mp i) = \frac{\Delta - (d-2)/2 \mp i(\omega - k)}{\tilde{\Delta} - (d-2)/2 \mp i(\omega - k)} \tilde{G}_{\Delta}(\omega, k), \quad (4.23b)$$

where $\tilde{\Delta} = d - \Delta$ is the scaling dimension of the “shadow operator.”

Now, the problem is to reduce to a problem to solve these recurrence relations exactly. In fact, the solution is not unique: there are a number of nontrivial solutions that satisfy (4.23a) and (4.23b). Among them are the following “minimal” solutions:

$$\tilde{G}_{\Delta}^{A/R}(\omega, k) = \frac{\Gamma\left(\frac{\Delta - (d-2)/2 \pm i(\omega + k)}{2}\right) \Gamma\left(\frac{\Delta - (d-2)/2 \pm i(\omega - k)}{2}\right)}{\Gamma\left(\frac{\tilde{\Delta} - (d-2)/2 \pm i(\omega + k)}{2}\right) \Gamma\left(\frac{\tilde{\Delta} - (d-2)/2 \pm i(\omega - k)}{2}\right)}, \quad (4.24a)$$

$$\begin{aligned} \tilde{G}_{\Delta}^{\pm}(\omega, k) &= e^{\pm \pi \omega} \left| \Gamma\left(\frac{\Delta - (d-2)/2 + i(\omega + k)}{2}\right) \right|^2 \\ &\times \left| \Gamma\left(\frac{\Delta - (d-2)/2 + i(\omega - k)}{2}\right) \right|^2, \end{aligned} \quad (4.24b)$$

where, for simplicity, we have set the overall normalization factors to be unity. Note that $|\Gamma(x + iy)|^2 = \Gamma(x + iy)\Gamma(x - iy)$ for $x, y \in \mathbb{R}$. These solutions are the “minimal” solutions in the sense that they do not contain a factor that is invariant under the shifts $(\omega, k) \rightarrow (\omega \pm i, k \pm i)$ and $(\omega, k) \rightarrow (\omega \pm i, k \mp i)$. A typical example of such factor is $g = g(e^{2\pi\omega}, e^{2\pi k})$, where g is an arbitrary function. One can easily check that (4.24a) and (4.24b) indeed satisfy (4.23a) and (4.23b) and hence give the solutions to the recurrence relations.

Finally, let us give physical interpretations of the solutions (4.24a) and (4.24b). $\tilde{G}_{\Delta}^{A/R}(\omega, k)$ are interpreted as the advanced and retarded two-point functions. Indeed, they have desired analytic structures in the complex ω -plane: $\tilde{G}_{\Delta}^R(\omega, k)$ has simple poles at $\omega = \pm k - i(\Delta - (d-2)/2 + 2n)$ ($n \in \mathbb{Z}_{\geq 0}$) in the lower half complex ω -plane, whereas $\tilde{G}_{\Delta}^A(\omega, k)$ has simple poles at $\omega = \pm k + i(\Delta - (d-2)/2 + 2n)$ ($n \in \mathbb{Z}_{\geq 0}$) in the upper half complex ω -plane, both of which are consistent with the causal structures of retarded and advanced two-point functions.^v (Note that we have assumed that the scaling dimension satisfies the unitarity bound $\Delta > (d-2)/2$.) $\tilde{G}_{\Delta}^{\pm}(\omega, k)$, on the other hand, are interpreted as the positive- and negative-frequency two-point Wightman functions. Indeed, they satisfy the KMS condition in the momentum space (see Eq. (2.44)):

$$\tilde{G}_{\Delta}^{+}(\omega, k) = e^{2\pi\omega} \tilde{G}_{\Delta}^{-}(\omega, k). \quad (4.25)$$

In App. A.2, we check that the Fourier transforms of positive- and negative-frequency two-point Wightman functions indeed coincide with the solutions (4.24b). It should be noted that, since the recurrence relations are linear, any linear combination of the solutions also satisfies (4.24a) and (4.24b) and hence gives another solution to the recurrence relations.

5. Conclusions

It is well known that d -dimensional CFT can be easily thermalized by just putting it on the Rindler wedge, light-cone or diamond under the identifications of temporal coordinates with $SO(1, 1) \subset SO(2, d)$ group parameters. Though this is essentially the Unruh effect that has been vastly studied over the last four decades, correlation functions of thus obtained thermal CFT have not been well-explored before. In this paper, we studied thermal two-point functions for scalar primary operators by using the intertwining operator. Though it has long been known that two-point functions of CFT are nothing but the integral kernels of intertwining operators, its implications to thermal CFT have not been well-explored either. Much inspired by the Kerimov's intertwining operator approach to exact S-matrix,⁶ in this work, we showed that by using the unconventional $SO(1, 1)$ continuous basis for the representation space of conformal algebra $\mathfrak{so}(2, d)$, the intertwining relations result in the recurrence relations for thermal two-point functions in the complex momentum space. By solving these recurrence relations, we obtained the advanced/retarded two-point functions as well as the positive/negative-frequency two-point Wightman functions in the momentum space.

^vHere the causal structures mean that the position-space representations $G_{\Delta}^{R/A}(t, x, \mathbf{x}_{\perp}; t', x', \mathbf{x}'_{\perp})$ given through (4.15) vanish for $t - t' < 0$ ($t - t' > 0$). (Note that the integral (4.15) contains the factor $e^{-i\omega(t-t')}$. And thanks to this factor, the integration contour can be deformed to the infinite semicircle in the upper (lower) half ω -plane for $t - t' < 0$ ($t - t' > 0$), which results in $G_{\Delta}^{R/A}(t, x, \mathbf{x}_{\perp}; t', x', \mathbf{x}'_{\perp}) = 0$ for $t - t' < 0$ ($t - t' > 0$) if there is no simple poles in the upper (lower) half ω -plane.)

There are several remaining issues that deserve to be addressed in the future. In this paper, we just focused on scalar correlation functions for simplicity. For practical applications, however, it is important to understand the structure of momentum-space correlation functions for generic primary tensors. It is also interesting to generalize our approach to the AdS/CFT correspondence. Since thermal CFT on $\mathbb{H}^1 \times \mathbb{H}^{d-1}$ can also be obtained as a boundary theory of certain AdS wedge regions, it must be possible to develop a similar algebraic approach to momentum-space two-point functions by utilizing the intertwining operators developed in Refs. 42 and 43. We would like to address this issue elsewhere.

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Appendix A. Fourier Transform

Thermal correlation functions are notorious for being complicated to Fourier-transform even in the well-studied two-dimensional CFT. In this appendix, we present computational details for the Fourier transforms of position-space two-point Wightman functions of thermal CFT living on the hyperbolic space-time $\mathbb{H}^1 \times \mathbb{H}^{d-1}$. To do this, we first need to find a complete orthonormal basis on $\mathbb{H}^1 \times \mathbb{H}^{d-1}$ in an appropriate coordinate system. This is done in App. A.1. In App. A.2, we compute the Fourier transforms and show that the resulting momentum-space two-point functions exactly coincide with (4.24b).

Throughout this appendix, we will work in the units $\ell = 1$ (i.e. $2\pi T = 1$).

A.1. Harmonic analysis on the hyperbolic space-time

The d -dimensional hyperbolic space-time $\mathbb{H}^1 \times \mathbb{H}^{d-1}$ is a foliation of the $(d+1)$ -dimensional cone (2.1) and can be embedded into the $(d+2)$ -dimensional space $\mathbb{R}^{2,d} \ni Y^a = (Y^0, \dots, Y^{d+1})$ as follows:

$$\begin{aligned} (Y^0)^2 - (Y^1)^2 &= -1 = (Y^2)^2 + \dots + (Y^d)^2 - (Y^{d+1})^2, \\ Y^1 &\geq 1, \quad Y^{d+1} \geq 1. \end{aligned} \tag{A.1}$$

In order to find a complete orthonormal basis on this space-time, we first need to introduce an appropriate coordinate patch. The most convenient coordinate system is turned out to be of the form:

$$\begin{aligned} Y^0 &= \sinh t, \quad Y^1 = \cosh t, \quad Y^i = \frac{x_\perp^i}{x}, \\ Y^d &= \frac{1 - x^2 - \mathbf{x}_\perp^2}{2x}, \quad Y^{d+1} = \frac{1 + x^2 + \mathbf{x}_\perp^2}{2x}, \end{aligned} \tag{A.2}$$

where $i \in \{2, \dots, d-1\}$, $t \in (-\infty, \infty)$, $x \in (0, \infty)$ and $\mathbf{x}_\perp \in \mathbb{R}^{d-2}$. The induced metric on the hyperbolic space-time $\mathbb{H}^1 \times \mathbb{H}^{d-1}$ then takes the following form:

$$\begin{aligned} ds_{\mathbb{H}^1 \times \mathbb{H}^{d-1}}^2 &= -(dY^0)^2 + (dY^1)^2 + \dots + (dY^d)^2 - (dY^{d+1})^2 \Big|_{Y \in \mathbb{H}^1 \times \mathbb{H}^{d-1}} \\ &= -dt^2 + \frac{dx^2 + d\mathbf{x}_\perp^2}{x^2}. \end{aligned} \quad (\text{A.3})$$

The Laplace–Beltrami operator on $\mathbb{H}^1 \times \mathbb{H}^{d-1}$ is therefore given by

$$\Delta_{\mathbb{H}^1 \times \mathbb{H}^{d-1}} = \frac{1}{\sqrt{|g|}} \partial_\mu \sqrt{|g|} g^{\mu\nu} \partial_\nu = -\partial_t^2 + \Delta_{\mathbb{H}^{d-1}}, \quad (\text{A.4})$$

where

$$\Delta_{\mathbb{H}^{d-1}} = x^2 \left(x^{d-3} \partial_x \frac{1}{x^{d-3}} \partial_x + \Delta_{\mathbb{R}^{d-2}} \right). \quad (\text{A.5})$$

Here $\Delta_{\mathbb{R}^{d-2}}$ is the Laplacian on $\mathbb{R}^{d-2}$. Note that the metric (A.3) is time-independent, which means that in this coordinate system, the time translation is an isometry of $\mathbb{H}^1 \times \mathbb{H}^{d-1}$. In order to find the complete orthonormal basis on the hyperbolic space-time, we then need to solve the following eigenvalue equations:

$$i\partial_t f = \omega f, \quad (\text{A.6a})$$

$$\Delta_{\mathbb{H}^{d-1}} f = j(j-d+2)f, \quad (\text{A.6b})$$

$$-\Delta_{\mathbb{R}^{d-2}} f = \mathbf{p}_\perp^2 f. \quad (\text{A.6c})$$

The solutions to the first and third equations are just the plane waves $e^{-i\omega t}$ and $e^{i\mathbf{p}_\perp \cdot \mathbf{x}_\perp}$, so in what follows, we shall focus on the second equation (A.6b). Substituting (A.5) into (A.6b), we see that the eigenvalue equation (A.6b) reduces to the following Schrödinger-like equation:

$$\left[-\partial_x^2 + \frac{\left(j - \frac{d-2}{2}\right)^2 - \frac{1}{4}}{x^2} \right] \bar{f} = -\mathbf{p}_\perp^2 \bar{f}, \quad (\text{A.7})$$

where

$$\bar{f} = x^{-(d-3)/2} f. \quad (\text{A.8})$$

The Schrödinger-like equation (A.7) does not admit any normalizable solutions unless $(j - (d-2)/2)^2$ becomes negative. Hence j must be of the form

$$j = \frac{d-2}{2} + ik, \quad k \in (0, \infty). \quad (\text{A.9})$$

Note that though j is complex, the eigenvalue of the Laplace–Beltrami operator $\Delta_{\mathbb{H}^{d-1}}$ itself is real, $j(j-d+2) = -k^2 - (d-2)^2/4$. The solution to Eq. (A.7) that converges as $x \rightarrow \infty$ is $x^{1/2} K_{ik}(|\mathbf{p}_\perp| x)$, where $K_\nu(z)$ is the modified Bessel function of the second kind. Collecting the above pieces, we find the following

complete orthonormal basis on the hyperbolic space-time $\mathbb{H}^1 \times \mathbb{H}^{d-1}$:

$$f_{\omega,k,\mathbf{p}_\perp}(t, x, \mathbf{x}_\perp) = \sqrt{\frac{4k \sinh(\pi k)}{\pi}} x^{(d-2)/2} K_{ik}(|\mathbf{p}_\perp| x) e^{-i\omega t + i\mathbf{p}_\perp \cdot \mathbf{x}_\perp}, \quad (\text{A.10})$$

which satisfies the orthonormality

$$\begin{aligned} \int_{-\infty}^{\infty} dt \int_0^{\infty} \frac{dx}{x^{d-1}} \int d^{d-2} \mathbf{x}_\perp f_{\omega,k,\mathbf{p}_\perp}^*(t, x, \mathbf{x}_\perp) f_{\omega',k',\mathbf{p}'_\perp}(t, x, \mathbf{x}_\perp) \\ = (2\pi)^d \delta(\omega - \omega') \delta(k - k') \delta^{(d-2)}(\mathbf{p}_\perp - \mathbf{p}'_\perp) \end{aligned} \quad (\text{A.11a})$$

and the completeness

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_0^{\infty} \frac{dk}{2\pi} \int \frac{d^{d-2} \mathbf{p}_\perp}{(2\pi)^{d-2}} f_{\omega,k,\mathbf{p}_\perp}(t, x, \mathbf{x}_\perp) f_{\omega,k,\mathbf{p}_\perp}^*(t', x', \mathbf{x}'_\perp) \\ = x^{d-1} \delta(t - t') \delta(x - x') \delta^{(d-2)}(\mathbf{x}_\perp - \mathbf{x}'_\perp). \end{aligned} \quad (\text{A.11b})$$

A.2. Two-point Wightman function in the momentum space

Let us next turn to the problem of calculating the Fourier transform of two-point functions. As noted in Subsec. 2.2, in the coordinate system (A.2), the positive- and negative-frequency two-point Wightman functions are given by

$$G_{\Delta}^{\pm}(Y, Y') = \frac{1}{(-2Y \cdot Y')^{\Delta}} = \left[\frac{1/2}{-\cosh(t - t' \mp i\epsilon) + \frac{x^2 + x'^2 + |\mathbf{x}_\perp - \mathbf{x}'_\perp|^2}{2xx'}} \right]^{\Delta}, \quad (\text{A.12})$$

where ϵ is a positive infinitesimal. In order to compute the momentum-space representations of (A.12), we first note that the two-point functions (A.12) can be expanded as follows:

$$G_{\Delta}^{\pm}(Y, Y') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_0^{\infty} \frac{dk}{2\pi} \int \frac{d^{d-2} \mathbf{p}_\perp}{(2\pi)^{d-2}} \tilde{G}_{\Delta}^{\pm}(\omega, k) f_{\omega,k,\mathbf{p}_\perp}(Y) f_{\omega,k,\mathbf{p}_\perp}^*(Y'), \quad (\text{A.13})$$

where $f_{\omega,k,\mathbf{p}_\perp}(Y) \equiv f_{\omega,k,\mathbf{p}_\perp}(t, x, \mathbf{x}_\perp)$. It then follows from the orthonormality (A.11a) that the following equality holds:

$$\begin{aligned} \left(G_{\Delta}^{\pm} f_{\omega,k,\mathbf{p}_\perp} \right)(Y) &:= \int_{\mathbb{H}^1 \times \mathbb{H}^{d-1}} dY' G_{\Delta}^{\pm}(Y, Y') f_{\omega,k,\mathbf{p}_\perp}(Y') \\ &= \tilde{G}_{\Delta}^{\pm}(\omega, k) f_{\omega,k,\mathbf{p}_\perp}(Y), \end{aligned} \quad (\text{A.14})$$

where $\int_{\mathbb{H}^1 \times \mathbb{H}^{d-1}} dY'$ is a shorthand for $\int_{-\infty}^{\infty} dt' \int_0^{\infty} dx' x'^{-d+1} \int d^{d-2} \mathbf{x}'_\perp$. The momentum-space two-point Wightman functions $\tilde{G}_{\Delta}^{\pm}(\omega, k)$ can be obtained from this identity. In other words, $\tilde{G}_{\Delta}^{\pm}(\omega, k)$ are obtained by calculating the following

d -dimensional integral:

$$\begin{aligned} \left(G_{\Delta}^{\pm} f_{\omega, k, \mathbf{p}_{\perp}}\right)(Y) &= \sqrt{\frac{4k \sinh(\pi k)}{\pi}} \int_{-\infty}^{\infty} dt' \int_0^{\infty} \frac{dx'}{x'^{d-1}} \int d^{d-2} \mathbf{x}'_{\perp} \\ &\times \left[\frac{1/2}{-\cosh(t-t' \mp i\epsilon) + \frac{x^2+x'^2+|\mathbf{x}_{\perp}-\mathbf{x}'_{\perp}|^2}{2xx'}} \right]^{\Delta} \\ &\times x'^{\frac{d-2}{2}} K_{ik}(|\mathbf{p}_{\perp}|x') e^{-i\omega t' + i\mathbf{p}_{\perp} \cdot \mathbf{x}'_{\perp}}. \end{aligned} \quad (\text{A.15})$$

In order to compute this integral, we first change the integration variables as $t' \rightarrow t' + t$ and $\mathbf{x}'_{\perp} \rightarrow \mathbf{x}'_{\perp} + \mathbf{x}_{\perp}$, and then shift the t' -integration contour slightly upward/downward by $\pm i\pi$ for $G_{\Delta}^{\pm}$. The resultant integral takes the following form:

$$\begin{aligned} \left(G_{\Delta}^{\pm} f_{\omega, k, \mathbf{p}_{\perp}}\right)(Y) &= \sqrt{\frac{4k \sinh(\pi k)}{\pi}} e^{-i\omega t + i\mathbf{p}_{\perp} \cdot \mathbf{x}_{\perp}} \\ &\times \int_{-\infty \pm i\pi}^{\infty \pm i\pi} dt' \int_0^{\infty} dx' x'^{-d/2} \int d^{d-2} \mathbf{x}'_{\perp} \\ &\times \left[\frac{1/2}{-\cosh t' + \frac{x^2+x'^2+\mathbf{x}_{\perp}^{\prime 2}}{2xx'}} \right]^{\Delta} K_{ik}(|\mathbf{p}_{\perp}|x') e^{-i\omega t' + i\mathbf{p}_{\perp} \cdot \mathbf{x}'_{\perp}} \\ &= e^{\pm \pi \omega} \sqrt{\frac{4k \sinh(\pi k)}{\pi}} e^{-i\omega t + i\mathbf{p}_{\perp} \cdot \mathbf{x}_{\perp}} \\ &\times \int_{-\infty}^{\infty} dt' \int_0^{\infty} dx' x'^{-d/2} \int d^{d-2} \mathbf{x}'_{\perp} \\ &\times \left[\frac{1/2}{\cosh t' + \frac{x^2+x'^2+\mathbf{x}_{\perp}^{\prime 2}}{2xx'}} \right]^{\Delta} K_{ik}(|\mathbf{p}_{\perp}|x') e^{-i\omega t' + i\mathbf{p}_{\perp} \cdot \mathbf{x}'_{\perp}}, \end{aligned} \quad (\text{A.16})$$

where, in the second equality, we have changed the integration variable as $t' \rightarrow t' \pm i\pi$. Next, we use the following identity:

$$\left[\frac{1/2}{\cosh t' + \frac{x^2+x'^2+\mathbf{x}_{\perp}^{\prime 2}}{2xx'}} \right]^{\Delta} = \frac{1}{2^{\Delta} \Gamma(\Delta)} \int_0^{\infty} dz z^{\Delta-1} e^{-z \left(\cosh t + \frac{x^2+x'^2+\mathbf{x}_{\perp}^{\prime 2}}{2xx'} \right)}, \quad (\text{A.17})$$

which just follows from the integral representation of the Gamma function. Substituting this into (A.16), we get

$$\begin{aligned} \left(G_{\Delta}^{\pm} f_{\omega, k, \mathbf{p}_{\perp}}\right)(Y) &= \frac{e^{\pm \pi \omega}}{2^{\Delta} \Gamma(\Delta)} \sqrt{\frac{4k \sinh(\pi k)}{\pi}} e^{-i\omega t + i\mathbf{p}_{\perp} \cdot \mathbf{x}_{\perp}} \\ &\times \int_0^{\infty} dz z^{\Delta-1} \int_{-\infty}^{\infty} dt' e^{-i\omega t' - z \cosh t'} \end{aligned}$$

$$\begin{aligned} & \times \int_0^\infty dx' x'^{-d/2} K_{ik}(|\mathbf{p}_\perp|x') e^{-\frac{z}{2xx'}(x^2+x'^2)} \\ & \times \int d^{d-2} \mathbf{x}'_\perp e^{i\mathbf{p}_\perp \cdot \mathbf{x}'_\perp - \frac{z}{2xx'} \mathbf{x}'_\perp{}^2}. \end{aligned} \quad (\text{A.18})$$

The t' - and $\mathbf{x}'_\perp$ -integrals are calculated as follows:

$$\int_{-\infty}^\infty dt' e^{-i\omega t' - z \cosh t'} = K_{i\omega}(z), \quad (\text{A.19a})$$

$$\int d^{d-2} \mathbf{x}'_\perp e^{i\mathbf{p}_\perp \cdot \mathbf{x}'_\perp - \frac{z}{2xx'} \mathbf{x}'_\perp{}^2} = \left(\frac{2\pi x x'}{z} \right)^{(d-2)/2} e^{-\frac{xx'}{2z} \mathbf{p}_\perp^2}, \quad (\text{A.19b})$$

from which we get

$$\begin{aligned} \left(G_\Delta^\pm f_{\omega,k,\mathbf{p}_\perp} \right)(Y) &= \frac{(2\pi)^{\frac{d-2}{2}} e^{\pm\pi\omega}}{2^\Delta \Gamma(\Delta)} \sqrt{\frac{4k \sinh(\pi k)}{\pi}} x^{(d-2)/2} e^{-i\omega t + i\mathbf{p}_\perp \cdot \mathbf{x}_\perp} \\ &\times \int_0^\infty dz z^{\Delta-d/2} K_{i\omega}(z) \\ &\times \int_0^\infty \frac{dx'}{x'} K_{ik}(|\mathbf{p}_\perp|x') e^{-\frac{z}{2xx'}(x^2+x'^2) - \frac{xx'}{2z} \mathbf{p}_\perp^2}. \end{aligned} \quad (\text{A.20})$$

Making use of the following identity for the modified Bessel function

$$\int_0^\infty \frac{dx'}{x'} K_{ik}(|\mathbf{p}_\perp|x') e^{-\frac{z}{2xx'}(x^2+x'^2) - \frac{xx'}{2z} \mathbf{p}_\perp^2} = 2K_{ik}(z) K_{ik}(|\mathbf{p}_\perp|x), \quad (\text{A.21})$$

we finally get

$$\left(G_\Delta^\pm f_{\omega,k,\mathbf{p}_\perp} \right)(Y) = \tilde{G}_\Delta^\pm(\omega, k) f_{\omega,k,\mathbf{p}_\perp}(Y), \quad (\text{A.22})$$

where

$$\tilde{G}_\Delta^\pm(\omega, k) = \frac{2(2\pi)^{\frac{d-2}{2}}}{2^\Delta \Gamma(\Delta)} e^{\pm\pi\omega} \int_0^\infty dz z^{\Delta-d/2} K_{i\omega}(z) K_{ik}(z). \quad (\text{A.23})$$

This integral is exactly calculable. To see this, we first use the following identities:

$$K_{i\omega}(z) = \int_{-\infty}^\infty dt e^{i\omega t - z \cosh z} \quad \text{and} \quad K_{ik}(z) = \int_{-\infty}^\infty dx e^{ikx - z \cosh z}. \quad (\text{A.24})$$

Substituting these into Eq. (A.23), we get

$$\begin{aligned} \tilde{G}_\Delta^\pm(\omega, k) &= \frac{2(2\pi)^{\frac{d-2}{2}}}{2^\Delta \Gamma(\Delta)} e^{\pm\pi\omega} \int_{-\infty}^\infty dt \int_{-\infty}^\infty dx e^{i\omega t + ikx} \int_0^\infty dz z^{\Delta-d/2} e^{-z(\cosh t + \cosh x)} \\ &= \frac{2\pi^{\frac{d-2}{2}} \Gamma(\Delta - \frac{d-2}{2})}{\Gamma(\Delta)} e^{\pm\pi\omega} \int_{-\infty}^\infty dt \\ &\times \int_{-\infty}^\infty dx \frac{e^{i\omega t + ikx}}{[4 \cosh(\frac{t+x}{2}) \cosh(\frac{t-x}{2})]^{\Delta-(d-2)/2}}, \end{aligned} \quad (\text{A.25})$$

where the second equality follows from the identity

$$\frac{1}{2^{\Delta-\frac{d-2}{2}}} \int_0^\infty dz z^{\Delta-d/2} e^{-z(\cosh t + \cosh x)} = \frac{\Gamma(\Delta - \frac{d-2}{2})}{[4 \cosh(\frac{t+x}{2}) \cosh(\frac{t-x}{2})]^{\Delta-(d-2)/2}}. \quad (\text{A.26})$$

Let us next introduce the light-cone coordinates $x^\pm = t \pm x$, in which the integration measure becomes $dt dx = (1/2) dx^+ dx^-$. The integral (A.25) is then evaluated as follows:

$$\begin{aligned} \tilde{G}_\Delta^\pm(\omega, k) &= \frac{\pi^{\frac{d-2}{2}} \Gamma(\Delta - \frac{d-2}{2})}{\Gamma(\Delta)} e^{\pm \pi \omega} \int_{-\infty}^\infty dx^+ \frac{e^{i \frac{\omega+k}{2} x^+}}{[2 \cosh(\frac{x^+}{2})]^{\Delta-\frac{d-2}{2}}} \\ &\quad \times \int_{-\infty}^\infty dx^- \frac{e^{i \frac{\omega-k}{2} x^-}}{[2 \cosh(\frac{x^-}{2})]^{\Delta-\frac{d-2}{2}}} \\ &= \frac{\pi^{\frac{d-2}{2}} \Gamma(\Delta - \frac{d-2}{2})}{\Gamma(\Delta)} e^{\pm \pi \omega} \int_0^\infty du \frac{u^{\frac{\Delta-(d-2)/2+i(\omega+k)}{2}-1}}{(1+u)^{\Delta-\frac{d-2}{2}}} \\ &\quad \times \int_0^\infty dv \frac{v^{\frac{\Delta-(d-2)/2+i(\omega-k)}{2}-1}}{(1+v)^{\Delta-\frac{d-2}{2}}} \\ &= \frac{\pi^{\frac{d-2}{2}} \Gamma(\Delta - \frac{d-2}{2})}{\Gamma(\Delta)} e^{\pm \pi \omega} \frac{\Gamma(\frac{\Delta-(d-2)/2+i(\omega+k)}{2}) \Gamma(\frac{\Delta-(d-2)/2-i(\omega+k)}{2})}{\Gamma(\Delta - \frac{d-2}{2})} \\ &\quad \times \frac{\Gamma(\frac{\Delta-(d-2)/2+i(\omega-k)}{2}) \Gamma(\frac{\Delta-(d-2)/2-i(\omega-k)}{2})}{\Gamma(\Delta - \frac{d-2}{2})}, \end{aligned} \quad (\text{A.27})$$

where in the second equality, we have changed the integration variables as $u = e^{x^+}$ and $v = e^{x^-}$ and in the last equality, we have used the integration formula for the beta function

$$B(p, q) = \int_0^\infty dx \frac{x^{p-1}}{(1+x)^{p+q}} = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \quad \text{for } \text{Re } p, \text{Re } q > 0. \quad (\text{A.28})$$

To summarize, we have found that the positive- and negative-frequency two-point Wightman functions in the momentum space take the following forms:

$$\begin{aligned} \tilde{G}_\Delta^\pm(\omega, k) &= \frac{\pi^{\frac{d-2}{2}}}{\Gamma(\Delta)\Gamma(\Delta - \frac{d-2}{2})} e^{\pm \pi \omega} \left| \Gamma\left(\frac{\Delta - (d-2)/2 + i(\omega+k)}{2}\right) \right|^2 \\ &\quad \times \left| \Gamma\left(\frac{\Delta - (d-2)/2 + i(\omega-k)}{2}\right) \right|^2, \end{aligned} \quad (\text{A.29})$$

which, up to the normalization factor, exactly coincide with the solutions (4.24b).

References

1. K. Koller, *Commun. Math. Phys.* **40**, 15 (1975).
2. V. K. Dobrev, G. Mack, V. B. Petkova, S. G. Petrova and I. T. Todorov, *Harmonic Analysis on the n -Dimensional Lorentz Group and Its Application to Conformal Quantum Field Theory*, Lecture Notes in Physics, Vol. 63 (Springer-Verlag, Berlin, 1977).
3. E. S. Fradkin and M. Ya. Palchik, *Phys. Rep.* **44**, 249 (1978).
4. I. T. Todorov, M. C. Mintchev and V. B. Petkova, *Conformal Invariance in Quantum Field Theory*, Publications of the Scuola Normale Superiore (Edizioni della Normale, Pisa, 1978).
5. S. Ferrara, A. F. Grillo, G. Parisi and R. Gatto, *Lett. Nuovo Cimento* **4**, 115 (1972).
6. G. A. Kerimov, *Phys. Rev. Lett.* **80**, 2976 (1998).
7. G. A. Kerimov and M. Sezgin, *J. Phys. A* **31**, 7901 (1998).
8. G. A. Kerimov, *Phys. Atom. Nucl.* **65**, 1036 (2002).
9. G. A. Kerimov and M. Sezgin, *J. Phys. A* **35**, 6659 (2002).
10. F. Strocchi, *An Introduction to Non-Perturbative Foundations of Quantum Field Theory*, International Series of Monographs on Physics, Vol. 158 (Oxford University Press, Oxford, 2013).
11. K. Sogo, M. Uchinami, A. Nakamura and M. Wadati, *Prog. Theor. Phys.* **66**, 1284 (1981).
12. J. M. Maldacena and A. Strominger, *Phys. Rev. D* **56**, 4975 (1997), arXiv:hep-th/9702015.
13. S. S. Gubser, *Phys. Rev. D* **56**, 7854 (1997), arXiv:hep-th/9706100.
14. I. Bredberg, T. Hartman, W. Song and A. Strominger, *J. High Energy Phys.* **04**, 019 (2010), arXiv:0907.3477 [hep-th].
15. M. Becker, Y. Cabrera and N. Su, *J. High Energy Phys.* **09**, 157 (2014), arXiv:1407.3415 [hep-th].
16. S. Ohya, *J. High Energy Phys.* **12**, 011 (2013), arXiv:1309.2939 [hep-th].
17. S. Ohya, *Acta Polytech.* **54**, 142 (2014), arXiv:1312.7348 [hep-th].
18. G. L. Sewell, *Ann. Phys.* **141**, 201 (1982).
19. J. J. Bisognano and E. H. Wichmann, *J. Math. Phys.* **16**, 985 (1975).
20. J. J. Bisognano and E. H. Wichmann, *J. Math. Phys.* **17**, 303 (1976).
21. R. Haag, N. M. Hugenholtz and M. Winnink, *Commun. Math. Phys.* **5**, 215 (1967).
22. S. Takagi, *Prog. Theor. Phys. Suppl.* **88**, 1 (1986).
23. P. Martinetti and C. Rovelli, *Class. Quantum Grav.* **20**, 4919 (2003), arXiv:gr-qc/0212074.
24. D. Buchholz, On the structure of local quantum fields with nontrivial interaction, in *Operator Algebras, Ideals and their Applications in Theoretical Physics* (Teubner, Leipzig, 1978), pp. 146–153.
25. P. D. Hislop and R. Longo, *Commun. Math. Phys.* **84**, 71 (1982).
26. H. Casini, M. Huerta and R. C. Myers, *J. High Energy Phys.* **05**, 036 (2011), arXiv:1102.0440 [hep-th].
27. P. A. M. Dirac, *Ann. Math.* **37**, 429 (1936).
28. S. Weinberg, *Phys. Rev. D* **82**, 045031 (2010), arXiv:1006.3480 [hep-th].
29. M. S. Costa, J. Penedones, D. Poland and S. Rychkov, *J. High Energy Phys.* **11**, 071 (2011), arXiv:1107.3554 [hep-th].
30. M. S. Costa, J. Penedones, D. Poland and S. Rychkov, *J. High Energy Phys.* **11**, 154 (2011), arXiv:1109.6321 [hep-th].
31. T. Faulkner, *J. High Energy Phys.* **05**, 033 (2015), arXiv:1412.5648 [hep-th].

32. C. Agón and T. Faulkner, *J. High Energy Phys.* **08**, 118 (2016), arXiv:1511.07462 [hep-th].
33. R. Haag, *Local Quantum Physics: Fields, Particles, Algebras*, Texts and Monographs in Physics, 2nd edn. (Springer-Verlag, Berlin, 1996).
34. T. Faulkner, R. G. Leigh and O. Parrikar, *J. High Energy Phys.* **04**, 088 (2016), arXiv:1511.05179 [hep-th].
35. R. Nakayama, *Prog. Theor. Phys.* **127**, 393 (2012), arXiv:1112.1267 [hep-th].
36. E. S. Fradkin and M. Y. Palchik, *Conformal Quantum Field Theory in D-dimensions*, Mathematics and Its Applications, Vol. 376 (Kluwer, Dordrecht, 1996).
37. A. Bzowski, P. McFadden and K. Skenderis, *J. High Energy Phys.* **03**, 066 (2016), arXiv:1510.08442 [hep-th].
38. I. R. Klebanov and E. Witten, *Nucl. Phys. B* **556**, 89 (1999), arXiv:hep-th/9905104.
39. S. B. Yakubovich, *Opuscula Math.* **26**, 161 (2006).
40. A. Passian, H. Simpson, S. Kouckekian and S. B. Yakubovich, *J. Math. Anal. Appl.* **360**, 380 (2009).
41. R. Szmtykowski and S. Bielski, *J. Math. Anal. Appl.* **365**, 195 (2010), arXiv:0909.4232 [math.CA].
42. V. K. Dobrev, *Nucl. Phys. B* **553**, 559 (1999), arXiv:hep-th/9812194.
43. N. Aizawa and V. K. Dobrev, *Rep. Math. Phys.* **75**, 179 (2015), arXiv:1406.2129 [hep-th].