Addenda to the General spin precession and betatron oscillation in storage ring

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We give the generalized expression of spin precession of extended bunch particles having both anomalous magnetic and electric dipole moments (EDMs) in storage ring in higher order than the previous work and in the presence of $\mathbf{E}$ field as well as $\mathbf{B}$ field. These addenda are essential since some experiments consider the focusing field in the second-order of the beam extent and in the presence of both $\mathbf{B}$ and $\mathbf{E}$ fields.

Keywords: Electric dipole moment; storage ring; anomalous g-2.

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In the recent paper, we discussed general spin precession and betatron oscillation in storage ring.¹ “General” has the duplicate meanings. One is that the beam has a spread profile not only in vertical directions but also radial directions. Another is that it incorporates permanent electric dipole moment (EDM) as well as anomalous magnetic dipole moment (MDM). The measurement of both dipole moments is the smoking gun of the new physics beyond the Standard Model (SM).² Recently, the frequency shifts induced by magnetic field gradients and $\mathbf{v} \times \mathbf{E}$ effects on the neutron EDM³ and muon g-2 MDM⁴ have been discussed. We emphasized the importance of four-dimensional spin vector to discuss the spin of moving particles around the storage ring in the inertial frame. Imaging the J-PARC muon g-2/EDM experiment,⁵ we considered $\mathbf{E} = 0$ in the latter part of the previous paper. In this paper, we consider the case $\mathbf{E} \neq 0$ but with the constant magnitude of the velocity like the case of BNL-E821⁶ and its successor at FNAL. Also taking the strength of focusing $\mathbf{B}$ and $\mathbf{E}$ fields into consideration, we make an approximation at higher order than the previous paper with respect to the small extents.

The kinetic energy (including the rest energy) satisfies

$$\frac{dE_{\text{kin}}}{dt} = e\mathbf{E} \cdot \mathbf{v} \quad (1)$$

with

$$E_{\text{kin}} \equiv \frac{m}{\sqrt{1-v^2}} \equiv \gamma m \quad (2)$$

and constant magnitude of the velocity requires

$$\mathbf{E} \cdot \mathbf{v} = 0. \quad (3)$$

We do not assume this until it will be specified later. Here and hereafter, we use the $\hbar = c = 1$ units. In the presence of both $\mathbf{B}$ and $\mathbf{E}$, the Lorentz equation is given by

$$\gamma m \frac{d\mathbf{v}}{dt} \equiv \gamma m \dot{\mathbf{v}} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B} - \mathbf{v}(\mathbf{v} \cdot \mathbf{E})). \quad (4)$$

The Lorentz equation is also expressed as

$$\frac{d(\gamma m \mathbf{v})}{dt} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \quad (5)$$

The angular velocity of the spin rotation of the generalized Thomas–Bargman–Michel–Telegdi equation in the laboratory system is given by⁷

$$\begin{aligned} \boldsymbol{\Omega}_s = & -\frac{e}{m} \left[\left(a + \frac{1}{\gamma} \right) \mathbf{B} - \frac{\gamma a}{\gamma + 1} (\mathbf{v} \cdot \mathbf{B}) \mathbf{v} - \left(a + \frac{1}{\gamma + 1} \right) \mathbf{v} \times \mathbf{E} \right. \\ & \left. + \frac{\eta}{2} \left(\mathbf{E} - \frac{\gamma}{\gamma + 1} (\mathbf{v} \cdot \mathbf{E}) \mathbf{v} + \mathbf{v} \times \mathbf{B} \right) \right]. \end{aligned} \quad (6)$$

One usually considers the spin motion relative to the beam direction. It follows from (1) and (4) that the unit vector in direction of the velocity (momentum), $\mathbf{N} = \mathbf{v}/v = \mathbf{p}/p$ obeys

$$\frac{d\mathbf{N}}{dt} = \frac{\dot{\mathbf{v}}}{v} - \frac{\mathbf{v}}{v^3} (\mathbf{v} \cdot \dot{\mathbf{v}}) = \boldsymbol{\Omega}_p \times \mathbf{N}, \quad \boldsymbol{\Omega}_p = \frac{e}{m\gamma} \left(\frac{\mathbf{N} \times \mathbf{E}}{v} - \mathbf{B} \right). \quad (7)$$

Thus, the angular velocity of the spin rotation relative to the beam direction (particle rest frame) is given by

$$\begin{aligned} \boldsymbol{\Omega}' = \boldsymbol{\Omega}_s - \boldsymbol{\Omega}_p = & -\frac{e}{m} \left[a\mathbf{B} - \frac{\gamma a}{\gamma + 1} (\mathbf{v} \cdot \mathbf{B}) \mathbf{v} - \left(a - \frac{1}{\gamma^2 - 1} \right) \mathbf{v} \times \mathbf{E} \right. \\ & \left. + \frac{\eta}{2} \left(\mathbf{E} - \frac{\gamma}{\gamma + 1} (\mathbf{v} \cdot \mathbf{E}) \mathbf{v} + \mathbf{v} \times \mathbf{B} \right) \right]. \end{aligned} \quad (8)$$

However, in the presence of the pitch correction, the particle rest frame is less convenient than the moving cylindrical frame, where particles move in horizontal

Fig. 1. The configuration of betatron oscillation. $\mathbf{e}_1$ (x -axis), $\mathbf{e}_2$ (y -axis) coordinates are those projected on the averaged plane (horizontal plane) of the storage ring. $\mathbf{e}_3$ (z -axis) is vertical to the horizontal plane. $\mathbf{v}$ is the particle velocity (emphasized to see the small pitch correction) which has both y , z components with main x component.

plane on average (reference orbit) and the deviation from the reference orbit is approximated small (Fig. 1). It is given by

$$\begin{aligned} \boldsymbol{\Omega} = \boldsymbol{\Omega}_s - \boldsymbol{\Omega}_{pz} = & -\frac{e}{m} \left[a\mathbf{B} - \frac{a\gamma}{\gamma+1} \mathbf{v}(\mathbf{v} \cdot \mathbf{B}) + \left(\frac{1}{\gamma^2-1} - a \right) (\mathbf{v} \times \mathbf{E}) \right. \\ & + \frac{1}{\gamma} \left\{ \mathbf{B}_{\parallel} - \frac{1}{v^2} (\mathbf{v} \times \mathbf{E}_{\parallel}) \right\} \\ & \left. + \frac{\eta}{2} \left(\mathbf{E} - \frac{\gamma}{\gamma+1} \mathbf{v}(\mathbf{v} \cdot \mathbf{E}) + \mathbf{v} \times \mathbf{B} \right) \right]. \end{aligned} \quad (9)$$

Here, $\boldsymbol{\Omega}_{pz}$ is the z -component of $\boldsymbol{\Omega}_p$, and $\mathbf{B}_{\parallel}$ indicates the projected part of $\mathbf{B}$ onto the plane spanned by the reference orbit, that is,

$$\mathbf{B} = (B_x, B_y, B_z) = (\mathbf{B}_{\parallel}, B_z), \quad \mathbf{E} = (\mathbf{E}_{\parallel}, E_z). \quad (10)$$

In this frame,

$$\frac{d\mathbf{e}_1}{dx} = -\frac{\mathbf{e}_2}{\rho}, \quad \frac{d\mathbf{e}_2}{dx} = \frac{\mathbf{e}_1}{\rho}, \quad \frac{d\mathbf{e}_3}{dx} = 0. \quad (11)$$

Here, $\mathbf{e}_1$ is the tangential unit vector (x -component) to the beam's averaged circular motion in the horizontal plane (spanned by x, y coordinates), $\mathbf{e}_2$ is the radial unit vector (y -component) in the horizontal plane and $\mathbf{e}_3$ is the vertical unit vector (z -component) to the horizontal plane. ρ is the radius of the averaged circle and

$$\frac{1}{\rho} = \left| \frac{eB_0}{\gamma m v} \right|. \quad (12)$$

Here, we proceed to study a particle in the bunch whose position $\mathbf{r}$ is given by

$$\mathbf{r} = (\rho + y)\mathbf{e}_2 + z\mathbf{e}_3 \quad (13)$$

with respect to the reference orbit.

So let us consider how Eq. (9) is expressed in these profiles.⁸ The velocity of the profile of (13) is

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt} \frac{d\mathbf{r}}{dx} = \frac{dx}{dt} \left[\left(1 + \frac{y}{\rho} \right) \mathbf{e}_1 + y' \mathbf{e}_2 + z' \mathbf{e}_3 \right], \quad (14)$$

where ' indicates the derivative with respect to x . Therefore, the absolute value v of $\mathbf{v}$ is given by

$$v = \frac{dx}{dt} \sqrt{\left(1 + \frac{y}{\rho}\right)^2 + y'^2 + z'^2} \equiv \frac{dx}{dt} \mathcal{N}. \quad (15)$$

In the previous paper, we proceeded the arguments under the condition of $\mathbf{E} = 0$. In this paper, we relax this condition as $\mathbf{E}$ is weak but $\neq 0$. One of the typical examples of focusing fields of $\mathbf{B}$ and $\mathbf{E}$ is

$$\mathbf{B} = \left(0, \frac{\partial B_z}{\partial y} z, B_0 + \frac{\partial B_z}{\partial y} y\right), \quad \mathbf{E} = (0, Ky, -Kz). \quad (16)$$

They satisfy

$$\nabla \times \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = 0. \quad (17)$$

We assume y/ρ , z/ρ , y' , z' are small quantities of same order ϵ and we take up to $O(\epsilon^2)$. Hereafter, we give the approximation formulae for the general $\mathbf{B}$ and $\mathbf{E}$, not assuming the special form of (16) until we will argue at (25) and thereafter.

Here, we list the useful relations and approximations,

$$\begin{aligned} \frac{1}{\gamma v^2} &= \frac{\gamma}{\gamma^2 - 1}, \\ \frac{1}{\mathcal{N}} &= 1 - \frac{y}{\rho} + \frac{y^2}{\rho^2} - \frac{y'^2 + z'^2}{2} + O(\epsilon^3), \\ \frac{1}{\mathcal{N}^2} &= 1 - \frac{2y}{\rho} + \frac{3y^2}{\rho^2} - y'^2 - z'^2 + O(\epsilon^3) \end{aligned} \quad (18)$$

and the divergence and rotation of a general vector $\mathbf{B}$ (and $\mathbf{E}$) in this approximation are

$$\begin{aligned} \mathbf{v} \cdot \mathbf{B} &= \frac{v}{\mathcal{N}} \left(\left(1 + \frac{y}{\rho}\right) B_x + y' B_y + z' B_z \right) \\ &\approx v \left(\left(1 - \frac{y'^2 + z'^2}{2}\right) B_x + \left(1 - \frac{y}{\rho}\right) (y' B_y + z' B_z) \right) + O(\epsilon^3), \end{aligned} \quad (19)$$

$$\begin{aligned} (\mathbf{v} \cdot \mathbf{B}) \mathbf{v} &\approx v^2 \left[\left((1 - (y'^2 + z'^2)) B_x + \left(1 - \frac{y}{\rho}\right) (y' B_y + z' B_z) \right) \mathbf{e}_1 \right. \\ &\quad + y' \left(\left(1 - \frac{y}{\rho}\right) B_x + y' B_y + z' B_z \right) \mathbf{e}_2 \\ &\quad \left. + z' \left(\left(1 - \frac{y}{\rho}\right) B_x + y' B_y + z' B_z \right) \mathbf{e}_3 \right], \end{aligned} \quad (20)$$

$$\begin{aligned}
 \mathbf{v} \times \mathbf{B} &= \frac{v}{\mathcal{N}} \left[(y'B_z - z'B_y) \mathbf{e}_1 + \left(z'B_x - \left(1 + \frac{y}{\rho} \right) B_z \right) \mathbf{e}_2 \right. \\
 &\quad \left. + \left(\left(1 + \frac{y}{\rho} \right) B_y - y'B_x \right) \mathbf{e}_3 \right] \\
 &\approx v \left[\left(1 - \frac{y}{\rho} \right) (y'B_z - z'B_y) \mathbf{e}_1 + \left(\left(1 - \frac{y}{\rho} \right) z'B_x \right. \right. \\
 &\quad \left. \left. - \left(1 - \frac{y'^2 + z'^2}{2} \right) B_z \right) \mathbf{e}_2 \right. \\
 &\quad \left. + \left(\left(1 - \frac{y'^2 + z'^2}{2} \right) B_y - y' \left(1 - \frac{y}{\rho} \right) B_x \right) \mathbf{e}_3 \right]. \quad (21)
 \end{aligned}$$

Hereafter, we assume (3) explicitly. Substituting (19)–(21) into (9), we obtain

$$\begin{aligned}
 \boldsymbol{\Omega} &= -\frac{e}{m} \left[\left\{ \frac{1}{\gamma} (a+1) B_x \mathbf{e}_1 + \left(\left(a + \frac{1}{\gamma} \right) B_y - \frac{\eta}{2} v B_z \right) \mathbf{e}_2 + \left(a B_z + \frac{\eta}{2} v B_y \right) \mathbf{e}_3 \right\} \right. \\
 &\quad + \left\{ \left(a \left(1 - \frac{1}{\gamma} \right) \left((y'^2 + z'^2) B_x - \left(1 - \frac{y}{\rho} \right) (y' B_y + z' B_z) \right) \right. \right. \\
 &\quad \left. \left. - \left(a + \frac{1}{\gamma+1} \right) v \left(1 - \frac{y}{\rho} \right) (y' E_z - z' E_y) \right. \right. \\
 &\quad \left. \left. + \frac{\eta}{2} \left(E_x + v \left(1 - \frac{y}{\rho} \right) (y' B_z - z' B_y) \right) \right) \mathbf{e}_1 \right. \\
 &\quad + \left(-a \left(1 - \frac{1}{\gamma} \right) \left(\left(1 - \frac{y}{\rho} \right) B_x + y' B_y + z' B_z \right) y \right. \\
 &\quad \left. \left. - \left(a + \frac{1}{\gamma+1} \right) v \left(\left(1 - \frac{y}{\rho} \right) z' E_x - \left(1 - \frac{y'^2 + z'^2}{2} \right) E_z \right) \right. \right. \\
 &\quad \left. \left. + \frac{\eta}{2} \left(E_y + v \left(\left(1 - \frac{y}{\rho} \right) z' B_x + \frac{y'^2 + z'^2}{2} B_z \right) \right) \right) \mathbf{e}_2 \right. \\
 &\quad + \left(-a \left(1 - \frac{1}{\gamma} \right) \left(\left(1 - \frac{y}{\rho} \right) B_x + y' B_y + z' B_z \right) z' \right. \\
 &\quad \left. \left. + \left(\frac{1}{\gamma^2 - 1} - a \right) v \left(\left(1 - \frac{y'^2 + z'^2}{2} \right) E_y - y' \left(1 - \frac{y}{\rho} \right) E_x \right) \right. \right. \\
 &\quad \left. \left. + \frac{\eta}{2} \left(E_z - v \left(\frac{y'^2 + z'^2}{2} B_y + y' \left(1 - \frac{y}{\rho} \right) B_x \right) \right) \right) \mathbf{e}_3 \right\} \right]. \quad (22)
 \end{aligned}$$

Here, the division of the first $\{\dots\}$ term and the second $\{\dots\}$ term is simply convenient for the reference to the previous paper.¹

Next, we must study the equation of motions of y and z to see the time variation of (22). Constant magnitude of velocity indicates that

$$\frac{dv}{dt} = \frac{d^2x}{dt^2} \mathcal{N} + \mathcal{N}' \left(\frac{dx}{dt} \right)^2 = 0 \quad (23)$$

and, therefore,

$$\begin{aligned} \frac{d\mathbf{v}}{dt} = v^2 & \left[\left\{ \left(1 - \frac{2y}{\rho} \right) \frac{y'}{\rho} - y'y'' - z'z'' \right\} \mathbf{e}_1 \right. \\ & + \left\{ \left(1 - \frac{2y}{\rho} \right) y'' - \frac{1}{\rho} \left(1 - \frac{y}{\rho} + \frac{y^2}{\rho^2} - z'^2 \right) \right\} \mathbf{e}_2 \\ & \left. - \left\{ \frac{y'z'}{\rho} - z'' \left(1 - \frac{2y}{\rho} \right) \right\} \mathbf{e}_3 \right]. \end{aligned} \quad (24)$$

Hereafter, we consider the case of (16). This form together with (3) and (14) gives

$$y^2 - z^2 = \text{const.} \quad (25)$$

Consequently, in the $O(\epsilon^2)$ approximation, it follows from (16), (24) and from the Lorentz equation (4) with $\dot{\gamma} = 0$ that

$$y' \left(y'' + \frac{1-n}{\rho^2} y \right) + z' \left(z'' + \frac{n}{\rho^2} z \right) = 0, \quad (26)$$

$$y'' + \frac{1-n'}{\rho^2} y = \frac{1}{\rho} \left(2yy' + \frac{y^2}{\rho^2} + \frac{y'^2 - z'^2}{2} \right), \quad (27)$$

$$z'' + \frac{n'}{\rho^2} z = \frac{1}{\rho} (2yz'' + y'z'). \quad (28)$$

Here

$$n \equiv -\frac{\rho}{B_0} \frac{\partial B_z}{\partial y}, \quad n' \equiv n \left(1 - \frac{K}{v \partial B_z / \partial y} \right). \quad (29)$$

In (26), we can replace n by n' by virtue of (25). Thus, the well-known Hill's equations,

$$y'' + \frac{1-n'}{\rho^2} y = 0, \quad (30)$$

$$z'' + \frac{n'}{\rho^2} z = 0, \quad (31)$$

are consistent in $O(\epsilon)$ in the presence of $\mathbf{E}$.

In conclusion, we have calculated the spin precession to $O(\epsilon^2)$. This is essential because Farley's pitch correction appears in this order.⁹ It corresponds to $-a(1 - \frac{1}{\gamma})z'^2 B_z$ of $\mathbf{e}_3$ component in (22). Our formula is the generalization of his formula extended to y, y', z, z' and to the presence of the EDM. These extensions

are crucial. This is because in the ongoing and near future experiments at FNAL and J-PARC, they try to measure the $g-2/\text{EDM}$ up to 1 ppm or less than that, whereas ϵ is roughly of order 10^{-3} . Its fully consistent analysis in the presence of $\mathbf{E}$ is left to future work.

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