Addendum

On deformations of C^* -algebras by actions of Kählerian Lie groups

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We show that two approaches to equivariant deformation of C^* -algebras by actions of negatively curved Kählerian Lie groups, one based on oscillatory integrals and the other on quantizations maps defined by dual 2-cocycles, are still equivalent despite the nonunitarity of our 2-cocycles.

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0. Introduction

The paper [2] relies in a crucial way on unitarity of the cocycle Ω_θ , but it turns out that Ω_θ is only a coisometry. This can be seen already from [2, Theorem A.1]: the map Φ is injective but not surjective, contrary to the claim there that it is a diffeomorphism (see Lemma 2.3). In fact, it seems doubtful that already the simplest Kählerian Lie group G , the connected component of the $ax+b$ group over the reals, supports any nontrivial dual unitary 2-cocycle. At the very least, such a cocycle must be such that the corresponding deformation of the function algebra $L^\infty(G)$ is not a type I factor, see [3, Remark 2.12]. The entire nonconnected $ax+b$ group does support a nontrivial dual cocycle though [3].

The unitarity of Ω_θ was stated for the first time in [4] as a consequence of the identity from [1, Proposition 8.47], which therefore is also wrong. The origin of the mistake is similar: a change of variables closely related to Φ is not bijective, see erratum to [1].

Although the dual cocycle Ω_θ is not unitary, it still has some nice properties that allow one to develop a deformation theory similar to [4]. Within the framework of this extended theory the main results of [2] remain true as stated, with essentially identical proofs. This is what will be explained in this erratum.

1. Deformation by Nonunitary Cocycles

We follow the conventions of [4]. Let G be a locally compact quantum group and $\hat{W}$ be the multiplicative unitary of the dual quantum group. Assume we are given an element $\Omega \in L^\infty(\hat{G}) \bar{\otimes} L^\infty(\hat{G})$ satisfying the cocycle identity

$$(\Omega \otimes 1)(\hat{\Delta} \otimes \iota)(\Omega) = (1 \otimes \Omega)(\iota \otimes \hat{\Delta})(\Omega).$$

We assume in addition that:

- (1) the element Ω is coisometric: $\Omega\Omega^* = 1$;
- (2) there exists a unitary $X \in L^\infty(\hat{G})$ such that $(XJ)^2 = 1$ and $\Omega = (X \otimes X)(\hat{R} \otimes \hat{R})(\Omega_{21}^*) \hat{\Delta}(X)^*$;
- (3) we have $(\hat{\Delta} \otimes \iota)(\Omega)(\iota \otimes \hat{\Delta})(\Omega^*) = (\Omega^* \otimes 1)(1 \otimes \Omega)$.

Recall that $\hat{R}$ denotes the unitary antipode on $L^\infty(\hat{G})$, so that $\hat{R}(x) = Jx^*J$.

We then let

$$\tilde{J} := XJ, \quad L := (\tilde{J} \otimes \hat{J})\Omega\hat{W}^*(J \otimes \hat{J}),$$

and define, for $\nu \in \mathcal{K}^* = \mathcal{B}(L^2(G))_*$, a *quantization map*

$$T_\nu: L^\infty(G) \rightarrow \mathcal{B}(L^2(G)) \quad \text{by } T_\nu(x) := (\iota \otimes \nu)(L(x \otimes 1)L^*).$$

Given a left action $\alpha: A \rightarrow M(C_0(G) \otimes A)$ on a C^* -algebra A , we define the Ω -*deformation* of A as the C^* -subalgebra $A_\Omega \subset M(\mathcal{K} \otimes A)$ generated by the elements $(T_\nu \otimes \iota)\alpha(a)$ for $a \in A$ and $\nu \in \mathcal{K}^*$. Note that although this definition depends on the unitary X , a different choice would give an isomorphic C^* -algebra.

The definition of A_Ω as such does not require conditions (1)–(3): we could take any unitary in $L^\infty(\hat{G})$ for X in order to define L . However, without additional assumptions it is not clear how reasonable this definition is, since already for the function algebra there is another more transparent definition. We can define a product $\star_\Omega$ on the Fourier algebra $A(G) := L^\infty(\hat{G})_*$ of G by

$$\omega \star_\Omega \nu = (\omega \otimes \nu)(\hat{\Delta}(\cdot)\Omega^*).$$

It is straightforward to check that the cocycle identity for Ω can be written as

$$(\hat{\Delta} \otimes \iota)(\hat{W}\Omega^*)\Omega_{12}^* = (\hat{W}\Omega^*)_{13}(\hat{W}\Omega^*)_{23},$$

see [4, Sec. 2.1]. This implies that we have a representation

$$\pi_\Omega: (A(G), \star_\Omega) \rightarrow \mathcal{B}(L^2(G)) \text{ defined by } \pi_\Omega(\omega) := (\omega \otimes \iota)(\hat{W}\Omega^*).$$

How is the algebra $\pi_\Omega(A(G))$ related to the deformation $C_0(G)_\Omega$ of $C_0(G)$ with respect to the left action of G on itself by translations? This can be answered under conditions (1)–(3), thanks to the following result, cf. [4, Identity (3.1)].

Proposition 1.1. *If an element $\Omega \in L^\infty(\hat{G}) \bar{\otimes} L^\infty(\hat{G})$ satisfies conditions (2) and (3), then*

$$L_{23}\hat{W}_{12}L_{23}^* = (\hat{W}\Omega^*)_{12}L_{13}. \quad (1.1)$$

Proof. By the definition of L we have to prove that

$$\begin{aligned} (J \otimes \tilde{J} \otimes \hat{J})\Omega_{23}\hat{W}_{23}^*(J \otimes J \otimes \hat{J})\hat{W}_{12}(J \otimes J \otimes \hat{J})\hat{W}_{23}\Omega_{23}^*(J \otimes \tilde{J} \otimes \hat{J}) \\ = \hat{W}_{12}\Omega_{12}^*(\tilde{J} \otimes J \otimes \hat{J})\Omega_{13}\hat{W}_{13}^*(J \otimes J \otimes \hat{J}). \end{aligned}$$

As $(J \otimes \hat{J})\hat{W} = \hat{W}^*(J \otimes \hat{J})$ and $\hat{W}_{23}\hat{W}_{12}\hat{W}_{23}^* = \hat{W}_{12}\hat{W}_{13}$, this is equivalent to

$$\begin{aligned} (J \otimes \tilde{J} \otimes \hat{J})\Omega_{23}(J \otimes J \otimes \hat{J})\hat{W}_{12}\hat{W}_{13}(J \otimes J \otimes \hat{J})\Omega_{23}^*(1 \otimes X \otimes 1) \\ = \hat{W}_{12}\Omega_{12}^*(\tilde{J} \otimes J \otimes \hat{J})\Omega_{13}\hat{W}_{13}^*, \end{aligned}$$

hence,

$$\begin{aligned} (J \otimes \tilde{J} \otimes \hat{J})\Omega_{23}(J \otimes J \otimes \hat{J})\hat{W}_{12}(J \otimes J \otimes \hat{J})\hat{W}_{13}^*\Omega_{23}^*(1 \otimes X \otimes 1) \\ = \hat{W}_{12}\Omega_{12}^*(\tilde{J} \otimes J \otimes \hat{J})\Omega_{13}\hat{W}_{13}^*. \end{aligned}$$

Multiplying by $\hat{W}_{12}^*(1 \otimes 1 \otimes J\hat{J})$ on the left and by $\hat{W}_{13}(J \otimes X^*J \otimes J)$ on the right and using that $XJ = JX^*$, we get

$$\begin{aligned} \hat{W}_{12}^*(J \otimes \tilde{J} \otimes J)\Omega_{23}(J \otimes J \otimes \hat{J})\hat{W}_{12}(J \otimes J \otimes \hat{J})\hat{W}_{13}^*\Omega_{23}^*\hat{W}_{13}(J \otimes J \otimes J) \\ = \Omega_{12}^*(\tilde{J} \otimes XJ \otimes J)\Omega_{13}(J \otimes J \otimes J). \end{aligned}$$

Since, we can replace $\hat{J}$ by J in this expression, we get

$$\begin{aligned} \hat{W}_{12}^*(1 \otimes X \otimes 1)(\hat{R} \otimes \hat{R})(\Omega^*)_{23}\hat{W}_{12}(\hat{R} \otimes \hat{R} \otimes \hat{R})(\hat{W}_{13}^*\Omega_{23}\hat{W}_{13}) \\ = \Omega_{12}^*(X \otimes X \otimes 1)(\hat{R} \otimes \hat{R})(\Omega^*)_{13}. \end{aligned}$$

Recalling that the coproduct on $L^\infty(\hat{G})$ is given by $\hat{\Delta}(x) = \hat{W}^*(1 \otimes x)\hat{W}$, we obtain

$$\begin{aligned} \hat{\Delta}(X)_{12}(\hat{\Delta} \otimes \iota)(\hat{R} \otimes \hat{R})(\Omega^*)(\hat{R} \otimes \hat{R} \otimes \hat{R})(\iota \otimes \hat{\Delta})(\Omega)_{213} \\ = \Omega_{12}^*(X \otimes X \otimes 1)(\hat{R} \otimes \hat{R})(\Omega^*)_{13}. \end{aligned}$$

Applying $\hat{R} \otimes \hat{R} \otimes \hat{R}$ and flipping the first two factors, we then obtain

$$(\iota \otimes \hat{\Delta})(\Omega)(\hat{\Delta} \otimes \iota)(\Omega^*) \hat{\Delta}(X)_{12} = \Omega_{23}^*(X \otimes X \otimes 1)(\hat{R} \otimes \hat{R})(\Omega^*)_{21},$$

where we used that $(\hat{R} \otimes \hat{R})\hat{\Delta} = \hat{\Delta}^{\text{op}}\hat{R}$ and $\hat{R}(X) = X$. By virtue of assumption (2) this is equivalent to

$$(\iota \otimes \hat{\Delta})(\Omega)(\hat{\Delta} \otimes \iota)(\Omega^*) = \Omega_{23}^*\Omega_{12},$$

which is condition (3). $\square$

Corollary 1.2 (cf. [4, Proposition 3.1]). *If Ω is a dual cocycle satisfying conditions (1)–(3), then*

$$\overline{\pi_\Omega(A(G))} = [T_\nu(x) : x \in C_0(G), \nu \in \mathcal{K}^*]. \quad (1.2)$$

It follows that $\overline{\pi_\Omega(A(G))}$ is a nondegenerate C^ -algebra of operators on $L^2(G)$ and*

$$C_0(G)_\Omega = V(\overline{\pi_\Omega(A(G))} \otimes 1)V^*,$$

where $V = (\hat{J} \otimes \hat{J})\hat{W}(\hat{J} \otimes \hat{J})$.

We remind that the square brackets denote the norm closure of the linear span.

Proof. Slicing the first and the third legs of (1.1) and remembering that L_{13} is coisometric we immediately get the first statement. Next, the right-hand side of (1.2) is a self-adjoint space of operators, since the maps T_ν are completely positive for positive ν . On the other hand, the left-hand side is an algebra. Hence, both sides give a C^* -algebra. This C^* -algebra acts nondegenerately on $L^2(G)$, because $C_0(G)$ acts nondegenerately and $T_\nu(1) = \nu(1)1$.

Since, the first leg of L is in $L^\infty(\hat{G})$ while the first leg of V is in $L^\infty(\hat{G})'$, we have

$$V(T_\nu(x) \otimes 1)V^* = (T_\nu \otimes \iota)(V(x \otimes 1)V^*) = (T_\nu \otimes \iota)\Delta(x)$$

for $x \in C_0(G)$. This proves the last statement of the corollary. $\square$

2. Dual Cocycles on Kählerian Lie Groups

As in [2, Sec. 3], let G be a Kählerian Lie group, $\theta \in \mathbb{R}^*$ and Ω_θ be the dual cocycle defined by the kernel K_θ . We want to check that Ω_θ satisfies conditions

(1)–(3). The operator Ω_θ is coisometric by [2, Theorem A.1] and injectivity of the map Φ there (see Lemma 2.3). Condition (2) is also satisfied for $X = 1$, since $\overline{K_\theta(g, h)} = K_\theta(h, g)$. As for (3), the cocycle identity implies that

$$(\Omega_\theta^* \Omega_\theta \otimes 1)(\hat{\Delta} \otimes \iota)(\Omega_\theta)(\iota \otimes \hat{\Delta})(\Omega_\theta^*) = (\Omega_\theta^* \otimes 1)(1 \otimes \Omega_\theta).$$

It follows that if we let $F_\theta := \Omega_\theta^* \Omega_\theta$ then condition (3) is equivalent to

$$(F_\theta \otimes 1)(\hat{\Delta} \otimes \iota)(\Omega_\theta)(\iota \otimes \hat{\Delta})(F_\theta) = (\hat{\Delta} \otimes \iota)(\Omega_\theta)(\iota \otimes \hat{\Delta})(F_\theta).$$

We will show that the following stronger property holds.

Proposition 2.1. *We have $(F_\theta \otimes 1)(\hat{\Delta} \otimes \iota)(\Omega_\theta) = (\hat{\Delta} \otimes \iota)(\Omega_\theta)(\iota \otimes \hat{\Delta})(F_\theta)$.*

We don't have any conceptual explanation of this property. It is possible that this is true for any dual cocycle defined using a unitary quantization map as in [3, Sec. 2.2]. In this case the identity is not very difficult to verify by a direct computation as follows.

First of all, it is clearly enough to consider elementary Kählerian Lie groups.

Lemma 2.2. *For any $x \in \mathbb{R}$ and $\theta \in \mathbb{R}^*$, we have, with $g := (x, 0, 0) \in G$:*

$$(\lambda_g \otimes \lambda_g)\Omega_\theta = \Omega_{e^{-2x\theta}}(\lambda_g \otimes \lambda_g).$$

The proof is straightforward.

It follows that it suffices to prove the proposition for $\theta = \pm 2$. In fact, by passing from G to $G \rtimes_\alpha \mathbb{Z}/2\mathbb{Z}$, where α is the involutive automorphism defined at the Lie algebra level by $H \mapsto H$, $E \mapsto -E$, $X_i \mapsto X_{i+d}$ and $X_{i+d} \mapsto X_i$ for $i = 1, \dots, d$, we can similarly relate Ω_2 and Ω_{-2} . Thus, it is enough to consider $\theta = -2$.

From now on we write Ω and F instead of Ω_{-2} and F_{-2} . The operator $\mathcal{F}_{P,-2}: L^2(G) \rightarrow L^2(G)$ from [2] is up to a phase factor the partial Fourier transform $\mathcal{F}_P$ in P coordinate:

$$(\mathcal{F}_P f)(q, p) = (2\pi)^{-(d+1)/2} \int_P e^{-ip \cdot p'} f(q, p') dp'.$$

Therefore by [2, Theorem A.1] we have

$$\Omega = (\mathcal{F}_P^* \otimes \mathcal{F}_P^*)U_\Phi(\mathcal{F}_P \otimes \mathcal{F}_P),$$

where

$$(U_\Phi \varphi)(g_1, g_2) = \text{Jac}_\Phi(g_1, g_2)^{1/2} \varphi(\Phi(g_1, g_2)),$$

and the map $\Phi: G \times G \rightarrow G \times G$ is defined by

$$\begin{aligned} \Phi(g_1, g_2) := & \left(a_1 - \frac{1}{2} \operatorname{arcsinh}(e^{-2a_2} t_2), \frac{\alpha^2}{\alpha'} n_1 + \frac{t_2}{2\alpha'} m_1 + \frac{\alpha t_2}{2\alpha'} n_2 - \frac{\alpha}{\alpha'} m_2, \right. \\ & \frac{t_1^2 t_2}{8\alpha'} n_1 + \frac{\alpha^2}{\alpha'} m_1 - \frac{\alpha t_1 t_2}{4\alpha'} n_2 + \frac{\alpha t_1}{2\alpha'} m_2, t_1; \\ & a_2 + \frac{1}{2} \operatorname{arcsinh}(e^{-2a_1} t_1), -\frac{\alpha t_1}{2\alpha'} n_1 + \frac{\alpha}{\alpha'} m_1 + \frac{\alpha^2}{\alpha'} n_2 + \frac{t_1}{2\alpha'} m_2, \\ & \left. \frac{\alpha t_1 t_2}{4\alpha'} n_1 - \frac{\alpha t_2}{2\alpha'} m_1 + \frac{t_1 t_2^2}{8\alpha'} n_2 + \frac{\alpha^2}{\alpha'} m_2, t_2 \right), \end{aligned}$$

with

$$\alpha = \frac{1}{2} (e^{2a_1} + (e^{4a_1} + t_1^2)^{1/2})^{1/2} (e^{2a_2} + (e^{4a_2} + t_2^2)^{1/2})^{1/2}, \quad \alpha' = \alpha^2 + \frac{1}{4} t_1 t_2.$$

(Note that these formulas slightly differ from those in [2], correcting typos and the erroneous claim that m_1 and m_2 remain unchanged).

Let us first consider the case $d = 0$. Then,

$$\Phi(a_1, t_1; a_2, t_2) = \left(a_1 - \frac{1}{2} \operatorname{arcsinh}(e^{-2a_2} t_2), t_1; a_2 + \frac{1}{2} \operatorname{arcsinh}(e^{-2a_1} t_1), t_2 \right).$$

Lemma 2.3. *The map Φ is a diffeomorphism of $G \times G$ onto the proper subset*

$$\mathcal{O} := \{(a_1, t_1; a_2, t_2) \mid (1 + (e^{-2a_1} t_1 + e^{-2a_2} t_2)^2)^{1/2} > e^{-2a_1} t_1 - e^{-2a_2} t_2\} \subset G \times G.$$

Proof. Consider the map $\phi: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $\phi(a_1, a_2) := (\sinh(a_1)e^{-a_2}, \sinh(a_2)e^{-a_1})$. It is easy to check that it is a diffeomorphism of $\mathbb{R}^2$ onto the set

$$\{(a_1, a_2) \mid (1 + (a_1 - a_2)^2)^{1/2} > a_1 + a_2\} \subset \mathbb{R}^2.$$

Now, we have $\Phi(a_1, t_1; a_2, t_2) = (b_1, s_1, b_2, s_2)$ if and only if $t_i = s_i$ and

$$\sinh(2b_1 - 2a_1) = -e^{-2a_2} s_2, \quad \sinh(2b_2 - 2a_2) = e^{-2a_1} s_1,$$

which in terms of the map ϕ is equivalent to

$$\phi(2b_1 - 2a_1, 2b_2 - 2a_2) = (-e^{-2b_2} s_2, e^{-2b_1} s_1).$$

This gives the result. □

It follows that $F = \Omega^* \Omega$ is indeed a proper projection, namely,

$$F = (\mathcal{F}_P^* \otimes \mathcal{F}_P^*) M(\mathbb{1}_{\mathcal{O}}) (\mathcal{F}_P \otimes \mathcal{F}_P), \quad (2.1)$$

where $M(\mathbb{1}_{\mathcal{O}})$ is the operator of multiplication by the characteristic function $\mathbb{1}_{\mathcal{O}}$ of $\mathcal{O}$.

Next, recall that we have $\hat{\Delta}(x) = \hat{W}^*(1 \otimes x)\hat{W}$, where the multiplicative unitary $\hat{W}: L^2(G \times G) \rightarrow L^2(G \times G)$ is defined by

$$(\hat{W}f)(g, h) := f(hg, h).$$

The following lemma is again a straightforward computation.

Lemma 2.4. *We have $\hat{W} = (\mathcal{F}_P^* \otimes \mathcal{F}_P^*)U_\alpha(\mathcal{F}_P \otimes \mathcal{F}_P)$, where $U_\alpha f = f \circ \alpha$ and $\alpha: G \times G \rightarrow G \times G$ is the diffeomorphism defined by*

$$\alpha(a_1, t_1; a_2, t_2) := (a_1 + a_2, t_1; a_2, t_2 - e^{-2a_1}t_1).$$

Note that the inverse of α is given by

$$\alpha^{-1}(a_1, t_1; a_2, t_2) = (a_1 - a_2, t_1; a_2, t_2 + e^{2(a_2 - a_1)}t_1).$$

It follows that

$$(\hat{\Delta} \otimes \iota)(\Omega) = (\mathcal{F}_P^* \otimes \mathcal{F}_P^* \otimes \mathcal{F}_P^*)(U_\alpha)_{12}^*(U_\Phi)_{23}(U_\alpha)_{12}(\mathcal{F}_P \otimes \mathcal{F}_P \otimes \mathcal{F}_P), \quad (2.2)$$

$$(\iota \otimes \hat{\Delta})(F) = (\mathcal{F}_P^* \otimes \mathcal{F}_P^* \otimes \mathcal{F}_P^*)M(\mathbb{1}_{\alpha_{23}(\mathcal{O}_{13})})(\mathcal{F}_P \otimes \mathcal{F}_P \otimes \mathcal{F}_P). \quad (2.3)$$

From (2.1)–(2.3) we conclude that Proposition 2.1 is equivalent (for $d = 0$) to the following lemma.

Lemma 2.5. *We have*

$$\mathcal{O} \times G = \{(g_1, g_2, g_3) \mid (\alpha_{12} \circ \Phi_{23} \circ \alpha_{12}^{-1})(g_1, g_2, g_3) \in \alpha_{23}(\mathcal{O}_{13})\}.$$

Proof. From Lemma 2.3, we easily get that the points $(b_1, s_1; b_2, s_2; b_3, s_3)$ of $\alpha_{23}(\mathcal{O}_{13})$ are characterized by the inequality

$$(1 + (e^{-2b_1}s_1 + e^{-2b_2}s_2 + e^{-2b_3}s_3)^2)^{1/2} > e^{-2b_1}s_1 - e^{-2b_2}s_2 - e^{-2b_3}s_3.$$

Since

$$\begin{aligned} & (\alpha_{12} \circ \Phi_{23} \circ \alpha_{12}^{-1})(a_1, t_1; a_2, t_2; a_3, t_3) \\ &= \left(a_1 - \frac{1}{2} \operatorname{arcsinh}(e^{-2a_3}t_3), t_1; a_2 - \frac{1}{2} \operatorname{arcsinh}(e^{-2a_3}t_3), t_2; \right. \\ & \quad \left. a_3 + \frac{1}{2} \operatorname{arcsinh}(e^{-2a_1}t_1 + e^{-2a_2}t_2), t_3 \right), \end{aligned}$$

letting $x_i = e^{-2a_i}t_i$ we see that we have to show that the inequality

$$(1 + (x_1 + x_2)^2)^{1/2} > x_1 - x_2 \quad (2.4)$$

is equivalent to

$$\begin{aligned} & (1 + (e^{\operatorname{arcsinh} x_3}x_1 + e^{\operatorname{arcsinh} x_3}x_2 + e^{-\operatorname{arcsinh}(x_1+x_2)}x_3)^2)^{1/2} \\ & > e^{\operatorname{arcsinh} x_3}x_1 - e^{\operatorname{arcsinh} x_3}x_2 - e^{-\operatorname{arcsinh}(x_1+x_2)}x_3. \end{aligned} \quad (2.5)$$

Put $a := e^{\operatorname{arcsinh}(x_1+x_2)} > 0$ and $b := x_1 - x_2$. Then $x_1 + x_2 = (a - a^{-1})/2$, so (2.4) becomes

$$\frac{a + a^{-1}}{2} > b. \quad (2.6)$$

Similarly, letting $c := e^{\operatorname{arcsinh} x_3} > 0$, (2.5) becomes

$$\frac{ac + a^{-1}c^{-1}}{2} > bc - \frac{a^{-1}c - a^{-1}c^{-1}}{2},$$

which is obviously equivalent to (2.6) for all $a > 0$, $b \in \mathbb{R}$ and $c > 0$. $\square$

This completes the proof of Proposition 2.1 for $d = 0$. The case $d > 0$ can be easily deduced from this. Indeed, from the definition of Φ we see that $\tilde{\mathcal{O}} := \Phi(G \times G)$ is the preimage of the set $\mathcal{O}$ under the map π defined by $\pi(a_1, v_1, t_1; a_2, v_2, t_2) := (a_1, t_1; a_2, t_2)$. It is also easy to check that the transformation $\tilde{\alpha}: G \times G \rightarrow G \times G$ defining the unitary $(\mathcal{F}_P \otimes \mathcal{F}_P)\hat{W}(\mathcal{F}_P^* \otimes \mathcal{F}_P^*)$ satisfies $\pi \circ \tilde{\alpha} = \alpha \circ \pi$. From this, we see that the coordinates v do not play any role, so the result follows from the case $d = 0$.

Therefore, Ω_θ satisfies assumptions (1)–(3) from Sec. 1, so we get a well-behaved notion of Ω_θ -deformation. The main result of [2] — Theorem 3.2 (Theorem 3.4 in the arXiv version) — then holds as stated, and essentially the only change in the proof is that the operator $\hat{W}_{\Omega_\theta}\Omega_\theta$ gets replaced by the operator $(J \otimes \hat{J})\Omega_\theta\hat{W}^*(J \otimes \hat{J})$ everywhere.

In more detail, [2, Corollary 2.1] (Corollary 2.4 in the arXiv version) and [2, Proposition 2.3] (Proposition 2.8 in the arXiv version) and the discussions preceding them are no longer meaningful and should be removed. The remainder of the paper remains unchanged, with [2, Identity (2.12)] understood as a definition of the operator $\hat{W}_{\Omega_\theta}\Omega_\theta$. Note that the place where we need special properties of Ω_θ is the first identity in the proof of [2, Lemma 2.5] (Lemma 2.10 in the arXiv version), which now holds by Proposition 1.1.

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