Erratum Notes on algebraic log stack

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This note corrects some mistakes in the previous paper ‘Notes on algebraic log stack’.

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1. Notations

For the convenience of the readers, we explain the notations we use in this erratum. Let S be a fine log scheme and $\underline{S}$ be its underlining scheme. Denote by $\mathbf{Flog}_S$ the category of fine log schemes over S , endowed with the big fppf topology. Denote by $\mathbf{Sch}_{\underline{S}}$ the category of $\underline{S}$ -schemes, endowed with the big fppf topology. $\mathbf{Flog}_S^*$ stands for the subcategory of $\mathbf{Flog}_S$ consisting of all strict log morphisms.

- $\mathbf{LogCFG}/\underline{S}$ (respectively $\mathbf{FLogCFG}/\underline{S}$) the 2-category of categories fibered in groupoids over $\mathbf{Sch}_{\underline{S}}$ with a *coherent* (respectively *fine*) log structure $([2, 2.1])$.
- $\mathbf{LAlg}_S$ (respectively $\mathbf{FAlg}_S$) the category of pairs $(X, \mathcal{M}_X)$, where X is an algebraic space and $\mathcal{M}_X$ is a *coherent* (respectively *fine*) log structure on X_{fppf} .

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- **AlgL_S** the category of algebraic log spaces over S . A set-valued sheaf X over **Flog_S** is called an algebraic log space over S if
 - The diagonal $\Delta_X : X \rightarrow X \times_{\mathbf{Flog}_S} X$ is represented by fine log schemes,
 - X admits a morphism $i : U \rightarrow X$, where $U \in \mathbf{Flog}_S$ and i is representable by fine log schemes, strict, surjective and log étale.
- **LAS_S** (respectively **FLAS_S**) the 2-category of pairs $(\mathcal{X}, \mathcal{M}_{\mathcal{X}})$, where $\mathcal{X}$ is an algebraic stack and $\mathcal{M}_{\mathcal{X}}$ is a *coherent* (respectively *fine*) log structure on $\mathcal{X}_{fppf}$.
- **ALS_S** the 2-category of algebraic log stacks (Definition 2.2 in this erratum) over S . A stack over **Flog_S** is called an algebraic log stack over S if
 - The diagonal $\Delta_X : X \rightarrow X \times_{\mathbf{Flog}_S} X$ is representable (by fine log algebraic spaces),
 - X admits a morphism $i : U \rightarrow X$, where $U \in \mathbf{Flog}_S$ and i is representable (by fine log algebraic spaces), strict, surjective and log smooth.

2. Erratum

The aim of this note is to correct some mistakes in [2]. We devote our hearty appreciation to Mattia Talpo who reminds us a mistake in Proposition 3.2 in (loc.cit.) as the following.

Proposition 2.1. *Let $\mathcal{X} \in \mathbf{LogCFG}/\underline{S}$. If $\mathcal{X} \in \mathbf{LAlg}_S$, then $\Phi\mathcal{X}$ is an algebraic log space. If $\mathcal{X} \in \mathbf{FLogCFG}/\underline{S}$, then $\mathcal{X}$ is an algebraic log space if and only if $\Phi\mathcal{X} \in \mathbf{AlgL}_S$.*

In fact, even if $\Phi\mathcal{X}$ is equivalent to a set-valued sheaf on **Flog_S**, $\mathcal{X}$ is not necessarily equivalent to a set-valued sheaf on **Sch_S**. This leads to various interesting phenomena. The corrected statement is.

Proposition 2.2. *Let $\mathcal{X} \in \mathbf{LogCFG}/\underline{S}$. If $\mathcal{X} \in \mathbf{LAlg}_S$, then $\Phi\mathcal{X}$ is an algebraic log space. If $\mathcal{X} \in \mathbf{FLogCFG}/\underline{S}$ and $\Phi\mathcal{X} \in \mathbf{AlgL}_S$, then $\mathcal{X}$ is a fine log algebraic stack.*

The proof is the same as (loc.cit., Proposition 3.2) except the claim that $\mathcal{X}$ is a set-valued sheaf on **Sch_S** if and only if $\Phi\mathcal{X}$ is a set-valued sheaf on **Flog_S** is false.

Here is an example of Olsson ([1, Proposition 5.17]) (which Mattia reminds us about).

Example 2.1. Let P be a fine monoid and S be a scheme, then there is a natural action of $\mathrm{Spec}(\mathcal{O}_S[P^{gp}])$ on $\mathrm{Spec}(\mathcal{O}_S[P])$. The log structure on $\mathrm{Spec}(\mathcal{O}_S[P])$ is the one associated to the canonical homomorphism $P \rightarrow \mathcal{O}_S[P]$. The action of

$\mathrm{Spec}(\mathcal{O}_S[P^{gp}])$ lifts to the log scheme $\mathrm{Spec}(\mathcal{O}_S[P])$ by

$$\begin{array}{ccc} P & \longrightarrow & P \oplus P^{gp} \\ \downarrow & & \downarrow \\ \mathcal{O}_S[P] & \longrightarrow & \mathcal{O}_S[P] \otimes_{\mathcal{O}_S} \mathcal{O}_S[P^{gp}]. \end{array}$$

Hence, the log structure of $\mathrm{Spec}(\mathcal{O}_S[P])$ descends to a log structure on the algebraic stack $[\mathrm{Spec}(\mathcal{O}_S[P])/\mathrm{Spec}(\mathcal{O}_S[P^{gp}])]$. We denote this log algebraic stack S_P . In [1], it is proved that $\Phi(S_P)$ is a set-valued sheaf over $\mathbf{Flog}_S$.

It is interesting to know when a log algebraic stack represents an algebraic log space. We provide a criterion (Theorem 3.1) in the appendix on whether a log algebraic stack gives rise to a set-valued sheaf over $\mathbf{Flog}_S$. In the original version of the erratum this criterion is presented in the case of global quotient. According to the suggestion of an anonymous referee, the criterion is stated in full generality in this version. The statement of Theorem 3.1 is suggested by the referee.

Due to the above correction, there are several modifications that should be made throughout the paper.

- (1) Theorem 3.2 is modified into

Theorem 2.1. Φ sends $\mathbf{LAlg}_S$ into $\mathbf{AlgL}_S$ and restricts on $\mathbf{FLAlg}_S$ to be a fully faithful functor

$$\mathbf{FLAlg}_S \rightarrow \mathbf{AlgL}_S.$$

Moreover, $\Phi X \simeq \Phi Y$ if and only if $X^{\mathrm{int}} \simeq Y^{\mathrm{int}}$.

We call $X \in \mathbf{AlgL}_S$ a fine log algebraic space (denoted by $X \in \mathbf{FLAlg}_S$) if there is $X' \in \mathbf{FLAlg}_S$ (unique up to isomorphisms) such that $X \simeq \Phi X'$.

- (2) The definition of representable functor ([2, Definition 4.1]) should be modified into

Definition 2.1. Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism in $\mathbf{CFG}/\mathbf{Flog}_S$. f is called **representable** if for any morphism $U \rightarrow \mathcal{Y}$ with $U \in \mathbf{Flog}_S$, $\mathcal{X} \times_{\mathcal{Y}} U \in \mathbf{FLAlg}_S$.

This definition is more restrictive than the original one (loc.cit., Definition 4.1, where $\mathcal{X} \times_{\mathcal{Y}} U$ is only required to be an algebraic log space).

The definition of algebraic log stack ([2, Definition 4.4]) should be modified into

Definition 2.2. A stack over $\mathbf{Flog}_S$ is called an algebraic log stack over S if

- The diagonal $\Delta_X : X \rightarrow X \times_{\mathbf{Flog}_S} X$ is representable,

- X admits a morphism $i : U \rightarrow X$, where $U \in \mathbf{Flog}_S$ and i is representable, strict, surjective and log smooth.

It turns out that under this definition, the category of algebraic log stacks is equivalent to the category of fine log algebraic stacks. In other words, the main result below (loc.cit., Theorem 4.2) remains true under the above modified definition of algebraic log stack.

Theorem 2.2. Φ sends $\mathbf{LAS}_S$ to $\mathbf{ALS}_S$, and restricts on $\mathbf{FLAS}_S$ as a strict 2-equivalence:

$$\mathbf{FLAS}_S \rightarrow \mathbf{ALS}_S.$$

$\Phi \mathcal{X} \simeq \Phi \mathcal{Y}$ if and only if $\mathcal{X}^{\text{int}} \simeq \mathcal{Y}^{\text{int}}$.

Other results concerning algebraic log stacks in Sec. 4 in loc.cit. still hold under this new situation. The proofs are almost the same when one replace *algebraic log space* by *fine log algebraic space*. The more detailed changes of Sec. 4 in loc. cit will be listed later. For the convenience of the readers, we summarize the main relations between fine log algebraic stacks and algebraic log stacks as follows:

- The functor of T -points Φ gives an equivalence

$$\Phi : \mathbf{FLAS}_S \rightarrow \mathbf{ALS}_S$$

between the 2-category of fine log algebraic stacks and the 2-category of algebraic log stacks.

- Φ sends a fine log algebraic space to an algebraic log space.
- (Corollary 3.1) $\Phi(\mathcal{X}, \mathcal{M})$ is an algebraic log space if and only if $(\mathcal{X}, \mathcal{M})$ is a fine log DM-stack such that
 - The diagonal $\Delta : \mathcal{X} \rightarrow \mathcal{X} \times_S \mathcal{X}$ is schematic,
 - The automorphism group of every geometric point x faithfully acts on the monoid $\mathcal{M}_x$.

We make other remarks of Sec. 4 in loc.cit..

- Lemma 4.1 and Corollary 4.1 are not necessary any more for the main results. But they still have their own interests.
- In the statement of Lemma 4.3 and Theorem 4.1, the meaning of ‘representable’ is as Definition 2.1 in this erratum. The proof is the same as in loc.cit.

(3) Applications in (loc.cit., Sec. 5):

(loc.cit., Corollary 5.1) is still true with a refined argument.

Corollary 2.1. *Let S be a fine log scheme and $F : \mathcal{X} \rightarrow \mathcal{Y}$ be a 1-morphism of the stacks $\mathcal{X}$ and $\mathcal{Y}$ over $\mathbf{Flog}_{S, fppf}$. If*

- (a) $\mathcal{X}$ is an algebraic log space, and

(b) F is representable, strict, surjective, flat and locally of finite presentation, then $\mathcal{Y}$ is an algebraic log stack.

Proof. Since $\mathcal{X}$ is an algebraic log space, there is a representable, strict, surjective, log étale morphism $U \rightarrow \mathcal{X}$ from a fine log scheme U to $\mathcal{X}$. This gives a morphism $U \rightarrow \mathcal{X}$ which is representable, strict, surjective, flat and locally of finite presentation. Then by the same argument as that in (loc.cit., Corollary 5.1), we see that $\mathcal{Y}$ is an algebraic log stack. $\square$

Section 5.2 in loc.cit is still true if one replaces “algebraic log space” by “fine log algebraic space”.

Corollary 5.4 is still true since its argument only rely on that $\Phi : \mathbf{FLAS}_S \rightarrow \mathbf{ALS}_S$ is an equivalence.

Corollaries 5.5 and 5.6 are incorrect. In fact, there exists a setoid algebraic log stack which is not an algebraic log space. This is a new phenomenon in log geometry. Recall that a setoid algebraic stack is an algebraic space. Here is an example.

Example 2.2. For simplicity, we consider $S_{\mathbb{N}}$ (Example 2.1). Then $\Phi(S_{\mathbb{N}})$ is an algebraic log stack whose underlining stack is $[\mathbb{A}_{\underline{S}}^1/\mathbb{G}_{m,\underline{S}}]$. The diagonal $\Delta : \Phi(S_{\mathbb{N}}) \rightarrow \Phi(S_{\mathbb{N}}) \times_S \Phi(S_{\mathbb{N}})$ is fully faithful. However, $\Phi(S_{\mathbb{N}})$ is not an algebraic log space.

Proof. Since $S_{\mathbb{N}}$ is a fine log algebraic stack, $\Phi(S_{\mathbb{N}})$ is an algebraic log stack by ([2, 4.2]). The diagonal $\Delta : \Phi(S_{\mathbb{N}}) \rightarrow \Phi(S_{\mathbb{N}}) \times_S \Phi(S_{\mathbb{N}})$ is fully faithful due to Example 2.1. We show in the following that there is no surjective strict log étale morphism $U \rightarrow \Phi(S_{\mathbb{N}})$, where U is a fine log scheme. Thus, $\Phi(S_{\mathbb{N}})$ is not an algebraic log space.

Assume that $U \rightarrow \Phi(S_{\mathbb{N}})$ is a surjective strict log étale morphism, then by ([2, Theorem 4.2]), this morphism descends to a log morphism $U \rightarrow S_{\mathbb{N}}$ which is surjective, strict and log étale. Therefore, the underlining stack $[\mathbb{A}_{\underline{S}}^1/\mathbb{G}_{m,\underline{S}}]$ of $S_{\mathbb{N}}$ is a DM-stack. This contradicts the fact that the diagonal morphism of $[\mathbb{A}_{\underline{S}}^1/\mathbb{G}_{m,\underline{S}}]$ is not unramified. $\square$

By letting $\mathcal{X} = S_{\mathbb{N}}$, $\mathcal{Y} = S$ and $f : \mathcal{X} \rightarrow \mathcal{Y}$ be the structure morphism, the example above gives a counterexample of Corollary 5.6.

3. Appendix

In this appendix, we provide a geometric description of log algebraic stacks whose associated algebraic log stack is equivalent to a set-valued sheaf on $\mathbf{Flog}_S$.

First, let us make some remarks on the action of the stabilizer on the stalk of the log structure of a point. Let $(\mathcal{X}, \mathcal{M}_{\mathcal{X}})$ be a log algebraic stack, where $\mathcal{M}_{\mathcal{X}}$ is a coherent log structure on $\mathcal{X}_{fppf}$. As in ([2, Sec. 2.2]), the log structure can

be characterized by a functor $\mathcal{M} : \mathcal{X} \rightarrow \mathbf{Flog}_S^*$ over $\mathbf{Sch}_S$. For each $T \in \mathbf{Sch}_S$ and $\xi : T \rightarrow \mathcal{X}$, $\mathcal{M}(\xi) = (T, \xi^* \mathcal{M}_{\mathcal{X}})$ in the tautological sense. The induced group homomorphism

$$\mathcal{M}_{\xi} : \mathrm{Aut}_T(\xi) \rightarrow \mathrm{Aut}_T(\mathcal{M}(\xi)) \simeq \mathrm{Aut}(\xi^* \mathcal{M}_{\mathcal{X}})$$

defines an action of $\mathrm{Aut}_T(\xi)$ on the log structure $\xi^* \mathcal{M}_{\mathcal{X}}$ on T .

Theorem 3.1. *Let $(\mathcal{X}, \mathcal{M}_{\mathcal{X}})$ be a log algebraic stack equipped with a coherent log structure $\mathcal{M}_{\mathcal{X}}$ over a fine log scheme S . The associated algebraic log stack $\Phi \mathcal{X}$ is a set-valued sheaf on $\mathbf{Flog}_S$ if and only if for any geometric point $p : \mathrm{Spec}(k) \rightarrow \mathcal{X}$ with an algebraically closed field k , the automorphism group of p (that is, so-called stabiliser) faithfully acts on the monoid $p^* \mathcal{M}_{\mathcal{X}}$.*

Proof. According to the construction of Φ ([2, 2.2]), $\Phi(\mathcal{X}, \mathcal{M}_{\mathcal{X}})$ is a set-valued sheaf over $\mathbf{Flog}_S$ if and only if for any $T \in \mathbf{Sch}_S$ and $\xi : T \rightarrow \mathcal{X}$, the induced group homomorphism

$$\mathcal{M}_{\xi} : \mathrm{Aut}_T(\xi) \rightarrow \mathrm{Aut}_T(\mathcal{M}_{\mathcal{X}}(\xi)) \simeq \mathrm{Aut}(\xi^* \mathcal{M}_{\mathcal{X}})$$

is injective. This is equivalent to that for every geometric point $p : \mathrm{Spec} k \rightarrow \mathcal{X}$ with an algebraically closed field k , the induced morphism

$$\mathcal{M}_p : \mathrm{Aut}(p) \rightarrow \mathrm{Aut}(p^* \mathcal{M}_{\mathcal{X}})$$

is injective. In the other word, the automorphism group of p faithfully acts on the monoid $p^* \mathcal{M}_{\mathcal{X}}$. $\square$

As a simple example, we take $P = \mathbb{N}$ and $S = \mathrm{Spec}(k)$ with the trivial log structure in Example 2.1. Then we get the canonical action of $\mathbb{G}_{m,k}$ on $(\mathbb{A}_k^1, \mathcal{M}_{\mathbb{A}_k^1})$. Here $\mathcal{M}_{\mathbb{A}_k^1}$ is induced by $\mathbb{N} \rightarrow k[x]$, $1 \mapsto x$. 0 is the only point that $\mathbb{G}_{m,k}$ acts on with nontrivial stabilizer. Since $\mathbb{G}_{m,k}$ acts on $\mathcal{M}_{\mathbb{A}_k^1,0}$ by

$$\begin{aligned} \mathbb{G}_{m,k} \times \mathcal{M}_{\mathbb{A}_k^1,0} &\rightarrow \mathcal{M}_{\mathbb{A}_k^1,0}, \\ (g, x^n) &\mapsto gx^n, \end{aligned}$$

and it is a faithful action, $\Phi([\mathbb{A}_k^1/\mathbb{G}_{m,k}])$ is a set-valued sheaf on $\mathbf{Flog}_k$.

Corollary 3.1. *Under the equivalence*

$$\Phi : \mathbf{FLAS}_S \rightarrow \mathbf{ALS}_S$$

$\Phi(\mathcal{X}, \mathcal{M})$ is an algebraic log space if and only if $(\mathcal{X}, \mathcal{M})$ is a fine log DM-stack such that

- The diagonal $\Delta : \mathcal{X} \rightarrow \mathcal{X} \times_S \mathcal{X}$ is schematic,
- The automorphism group of every geometric point x faithfully acts on the monoid $\mathcal{M}_x$.

Proof. If $(\mathcal{X}, \mathcal{M})$ is a fine log DM-stack such that the automorphism group of every geometric point x faithfully acts on the monoid $\mathcal{M}_x$, then by Theorem 3.1 $\Phi(\mathcal{X}, \mathcal{M})$ is a set-valued sheaf on $\mathbf{Flog}_S$. By Proposition 4.2 in [2], $\Phi(\mathcal{X}, \mathcal{M})$ is also an algebraic log stack over S that admits a schematic strict covering. Since $\Delta : \mathcal{X} \rightarrow \mathcal{X} \times_{\underline{S}} \mathcal{X}$ is schematic and

$$((\mathcal{X}, \mathcal{M}) \times_S (\mathcal{X}, \mathcal{M}))^{\text{int}} \hookrightarrow (\mathcal{X}, \mathcal{M}) \times_S (\mathcal{X}, \mathcal{M})$$

is a closed immersion ([2, Lemma 4.2]), $\Phi(\mathcal{X}, \mathcal{M})$ is an algebraic space.

Conversely, assume that $\Phi(\mathcal{X}, \mathcal{M})$ is an algebraic log space. By [2, Proposition 4.2], $(\mathcal{X}, \mathcal{M})$ is a fine log algebraic stack. Moreover, $(\mathcal{X}, \mathcal{M})$ admits a strict étale covering $U \rightarrow (\mathcal{X}, \mathcal{M})$, where U is a fine log scheme and the diagonal of $(\mathcal{X}, \mathcal{M})$ is schematic. As a consequence, $\mathcal{X}$ is a DM-stack whose diagonal is schematic. By Theorem 3.1, the automorphism group of every geometric point x faithfully acts on the monoid $\mathcal{M}_x$. $\square$

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