Erratum

Eigenfunction expansions and spacetime estimates for generators in divergence-form

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Matania Ben-Artzi

*Institute of Mathematics, Hebrew University,
 Jerusalem 91904, Israel
 mbartzi@math.huji.ac.il*

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Let $H = -\sum_{j,k=1}^n \partial_j a_{j,k}(x) \partial_k$, where $a_{j,k}(x) = a_{k,j}(x)$, be a formally self-adjoint operator in $L^2(\mathbb{R}^n)$, $n \geq 2$. The notations $\partial_j = \frac{\partial}{\partial x_j}$ and $\partial_t = \frac{\partial}{\partial t}$ are used throughout the paper.

We assume that the real measurable matrix function $a(x) = \{a_{j,k}(x)\}_{1 \leq j,k \leq n}$ satisfies, with some positive constants $a_1 > a_0 > 0, \Lambda_0 > 0$,

$$a_0 I \leq a(x) \leq a_1 I, \quad x \in \mathbb{R}^n, \quad (1)$$

$$a(x) = I \quad \text{for } |x| > \Lambda_0. \quad (2)$$

The first theorem proved in the paper (see Sec. 3) was the following.

Theorem A. *Suppose that $a(x)$ satisfies (1), (2). Then the operator H satisfies the Limiting Absorption Principle (LAP) in $\mathbb{R}$. More precisely, let $s > 1$ and consider the resolvent $R(z) = (H - z)^{-1}$, $\text{Im } z \neq 0$, as a bounded operator from $L^{2,s}(\mathbb{R}^n)$ to $H^{1,-s}(\mathbb{R}^n)$.*

Then:

- $R(z)$ is bounded with respect to the $H^{-1,s}(\mathbb{R}^n)$ norm. Using the density of $L^{2,s}$ in $H^{-1,s}$, we can therefore view $R(z)$ as a bounded operator from $H^{-1,s}(\mathbb{R}^n)$ to $H^{1,-s}(\mathbb{R}^n)$.
- The operator-valued functions, defined respectively in the lower and upper half-planes,

$$z \rightarrow R(z) \in B(H^{-1,s}(\mathbb{R}^n), H^{1,-s}(\mathbb{R}^n)), \quad s > 1, \quad \pm \text{Im } z > 0, \quad (3)$$

can be extended continuously from $\mathcal{C}^\pm = \{z/\pm \operatorname{Im} z > 0\}$ to $\overline{\mathcal{C}^\pm} = \mathcal{C}^\pm \cup \mathbb{R}$ (with respect to the operator-norm topology of $B(H^{-1,s}(\mathbb{R}^n), H^{1,-s}(\mathbb{R}^n))$).

In the case $n = 2$, replace $H^{-1,s}$ by $H_0^{-1,s}$.

In this generality (with respect to the coefficients $a_{j,k}(x)$), however, it is still possible to have a *discrete sequence of eigenvalues* embedded in the continuous spectrum. In order to exclude such eigenvalues (as in the statement of the theorem), one needs the coefficients to be *Lipschitz continuous*. Hence the correct theorem is the following.

Theorem A-Modified. Suppose that $a(x)$ satisfies (1), (2). Then the operator H satisfies the Limiting Absorption Principle (LAP) in $\mathbb{R}$. More precisely, let $s > 1$ and consider the resolvent $R(z) = (H - z)^{-1}$, $\operatorname{Im} z \neq 0$, as a bounded operator from $L^{2,s}(\mathbb{R}^n)$ to $H^{1,-s}(\mathbb{R}^n)$.

Then:

- (a) $R(z)$ is bounded with respect to the $H^{-1,s}(\mathbb{R}^n)$ norm. Using the density of $L^{2,s}$ in $H^{-1,s}$, we can therefore view $R(z)$ as a bounded operator from $H^{-1,s}(\mathbb{R}^n)$ to $H^{1,-s}(\mathbb{R}^n)$.
- (b) There exists (at most) a discrete sequence $\Lambda \subseteq (0, \infty)$, having no finite accumulation points (and, in particular, does not accumulate at zero), of eigenvalues of H .

The operator-valued functions, defined respectively in the lower and upper half-planes,

$$z \rightarrow R(z) \in B(H^{-1,s}(\mathbb{R}^n), H^{1,-s}(\mathbb{R}^n)), \quad s > 1, \quad \pm \operatorname{Im} z > 0, \quad (4)$$

can be extended continuously from $\mathcal{C}^\pm = \{z/\pm \operatorname{Im} z > 0\}$ to $\overline{\mathcal{C}^\pm} \setminus \Lambda = \mathcal{C}^\pm \cup (\mathbb{R} \setminus \Lambda)$ (with respect to the operator-norm topology of $B(H^{-1,s}(\mathbb{R}^n), H^{1,-s}(\mathbb{R}^n))$).

In the case $n = 2$ replace $H^{-1,s}$ by $H_0^{-1,s}$.

- (c) If in addition to (1), (2) the coefficients $a_{j,k}(x)$ are Lipschitz continuous, then there are no positive embedded eigenvalues, namely, $\Lambda = \emptyset$.

Outline of modified proof. We need to modify the argument in Sec. 5, in the paragraph following Eq. (5.11).

Thus, suppose first that $z_0 > 0$. Note that u vanishes for $|x| > \Lambda_0 + 2$. If the coefficients are Lipschitz continuous, then the unique continuation property holds [4] and we get $u \equiv 0$, which leads to a contradiction as in the paper. Thus part (c) is proved.

However, if the coefficients only satisfy (1), (2), then an example by Filonov [3] shows that the unique continuation property does not hold and the function u can still be a nontrivial eigenfunction of H , with the eigenvalue z_0 . Note that it is compactly supported.

Let $K \subseteq (0, \infty)$ be a compact interval with $z_0 \in K$.

From the equation $Hu = z_0u$, we obtain (no weights are needed due to the compact support),

$$\|u\|_1 \leq C\|u\|_0.$$

The constant $C > 0$ depends only on K . By a classical argument of Agmon [1] the last estimate implies that any set of eigenfunctions of H , with eigenvalues in K , is compact in $L^2(\mathbb{R}^n)$. It must therefore be finite, and that establishes part (b).

Also, $z_0 = 0$ cannot be an eigenvalue, since $Hu = 0$ implies $\nabla u = 0$, which in turn entails $u = 0$.

Finally, we show that there are no eigenvalues in $(0, \delta)$, for $\delta > 0$ sufficiently small. Indeed, by the Poincaré inequality there exists $c > 0$ such that, for all H^1 functions supported in $|x| < \Lambda_0 + 2$,

$$\|\nabla u\|_0^2 \geq c\|u\|_0^2.$$

Let $0 < \delta < ca_0$. Now if $z_0 \in (0, \delta)$ is an eigenvalue, and u is the associated normalized eigenfunction, then $(Hu, u) = z_0(u, u) < \delta$. On the other hand we have by (1),

$$(Hu, u) \geq a_0\|\nabla u\|_0^2 \geq ca_0 > \delta,$$

which is a contradiction to the preceding inequality.

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