ERRATUM

A UNIFIED APPROACH TO RESOLVENT EXPANSIONS AT THRESHOLDS

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This note contains an erratum to, and a few remarks on, the paper in the title [1]. The erratum is that the statement of Corollary 2.2 in [1] is incomplete. For the conclusion to be true, one needs that $A_0S = 0$, which is not the case when A_0 has a nilpotent part. The correct form is

Proposition 1. *Let $F \subset \mathbf{C}$ have zero as an accumulation point. Let $A(z)$, $z \in F$, be a family of bounded operators of the form*

$$A(z) = A_0 + zA_1(z), \quad (1)$$

with A_1 uniformly bounded as $z \rightarrow 0$. Suppose 0 is an isolated point of the spectrum of A_0 , and let S be the corresponding Riesz projection. If

$$A_0S = 0, \quad (2)$$

then for sufficiently small z , the operator $B(z): S\mathcal{H} \rightarrow S\mathcal{H}$ defined by

$$B(z) = \frac{1}{z}(S - S(A(z) + S)^{-1}S) = \sum_{j=0}^{\infty} (-z)^j S[A_1(z)(A_0 + S)^{-1}]^{j+1} S, \quad (3)$$

is uniformly bounded as $z \rightarrow 0$. The operator $A(z)$ has a bounded inverse in $\mathcal{H}$, if and only if $B(z)$ has a bounded inverse in $S\mathcal{H}$, and in this case

$$A(z)^{-1} = (A(z) + S)^{-1} + \frac{1}{z}(A(z) + S)^{-1}SB(z)^{-1}S(A(z) + S)^{-1}. \quad (4)$$

The rest of the paper is not affected, since everywhere A_0 is self-adjoint, so that (2) holds true.

We remark that (2) might be true, even if A_0 is not self-adjoint. As we shall prove below, this is the case when one considers the asymptotic expansion of the perturbed

resolvent around an embedded non-threshold eigenvalue, λ_0 , of the unperturbed (self-adjoint) Hamiltonian, as in [2].

More precisely (see [2] for details), if the unperturbed resolvent is

$$R_0(z) = \frac{P_0}{\lambda_0 - z} + \tilde{R}_0(z) \quad (5)$$

and $V = |V|^{1/2}U|V|^{1/2}$ is a bounded self-adjoint perturbation (here we choose U to be unitary by defining it to be 1 on $\text{Ker } V$, such that $U^2 = I$), then A_0 turns out to be

$$A_0 = U + \lim_{\eta \searrow 0} |V|^{1/2} \tilde{R}_0(\lambda_0 + i\eta) |V|^{1/2}. \quad (6)$$

It is assumed that the limit exists and is compact, and then either A_0 has a bounded inverse, or 0 is an isolated part of the spectrum of A_0 . In the latter case, one can define the corresponding Riesz projection, S .

The next proposition shows that under these conditions, (2) holds true.

Proposition 2.

$$SA_0 = SA_0S = 0. \quad (7)$$

Proof. The first equality follows from the fact that S is a Riesz projection. The key observation is that from (6)

$$\text{Im } A_0 \geq 0. \quad (8)$$

Consider now, as an operator in $S\mathcal{H}$,

$$A_1 = SA_0S. \quad (9)$$

On one hand $S\mathcal{H}$ is finite dimensional, and on the other hand for all $\Psi \in S\mathcal{H}$

$$\text{Im} \langle \Psi, A_1 \Psi \rangle = \text{Im} \langle \Psi, SA_0S \Psi \rangle = \text{Im} \langle \Psi, A_0S \Psi \rangle = \text{Im} \langle \Psi, A_0 \Psi \rangle \geq 0,$$

i.e.

$$\text{Im } A_1 \geq 0. \quad (10)$$

Since A_1 is nilpotent,

$$\text{Tr } A_1 = \text{Tr Re } A_1 + i \text{Tr Im } A_1 = 0, \quad (11)$$

which together with (10) implies that

$$\text{Im } A_1 = 0. \quad (12)$$

As a consequence, A_1 is self-adjoint, and since $\sigma(A_1) = \{0\}$, it follows that $A_1 = 0$. $\square$

The last remark is that one can generalize Corollary 2.2, and then the whole procedure in [1], to non-self-adjoint A_0 , under the additional assumption that A_0 is a Fredholm operator with index zero [3] (i.e. $\dim \text{Ker } A_0 = \dim(\text{Ran } A_0)^\perp$). Then

if $A_0 = W|A_0|$ is the polar decomposition of A_0 , one can extend (in a non-unique way) W to a unitary operator U (just take $\{f_j\}$ and $\{g_j\}$ orthonormal bases in $\text{Ker } A_0$ and $(\text{Ran } A_0)^\perp$ respectively, and define $Uf_j = g_j$). Then write

$$A(z) = U(|A_0| + zU^{-1}A_1(z)) \quad (13)$$

and apply Corollary 2.2 to $|A_0| + zU^{-1}A_1(z)$.

There is a price to pay in this case: due to the non-unicity of U , the obtained expansions are not “canonical”. Of course, the coefficients of the expansion do not depend on U , but various identities have to be used in order to see that.

References

- [1] A. Jensen and G. Nenciu, A unified approach to resolvent expansions at threshold, *Rev. Math. Phys.* **13** (2001) 717–754.
- [2] A. Jensen, On a unified approach to resolvent expansions for Schrödinger operators, in *Spectral and Scattering Theory and Related Topics* (RIMS, Kyoto, Japan); *Sūrikaiseikikenkyūsho Kōkyūroku* **1208** (2001) 91–103.
- [3] T. Kato, *Perturbation Theory for Linear Operators* (Springer-Verlag, Berlin, 1966).