Differential Geometry from a Singularity Theory Viewpoint

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Shyuichi Izumiya

Hokkaido University, Japan

Maria del Carmen Romero Fuster

Universitat de València, Spain

Maria Aparecida Soares Ruas

University of São Paulo, Brazil

Farid Tari

University of São Paulo, Brazil

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Preface

The geometry of surfaces is a subject that has fascinated many mathematicians and users of mathematics. This book offers a new look at this classical subject, namely from the point of view of singularity theory. Robust geometric features on a surface in the Euclidean 3-space, some of which are detectable by the naked eye, can be captured by certain types of singularities of some functions and mappings on the surface. In fact, the mappings in question come as members of some natural families of mappings on the surface. The singularities of the individual members of these families of mappings measure the contact of the surface with model objects such as lines, circles, planes and spheres.

This book gives a detailed account of the theory of contact between manifolds and its link with the theory of caustics and wavefronts. It then uses the powerful techniques of these theories to deduce geometric information about surfaces immersed in the Euclidean 3, 4 and 5-spaces as well as spacelike surfaces in the Minkowski space-time.

In Chapter 1 we argue the case for using singularity theory to study the extrinsic geometry of submanifolds of Euclidean spaces (or of other spaces). To make the book self-contained, we devote Chapter 2 to introducing basic facts about the extrinsic geometry of submanifolds of Euclidean spaces. Chapter 3 deals with singularities of smooth mappings. We state the results on finite determinacy and versal unfoldings which are fundamental in the study of the geometric families of mappings on surfaces treated in the book. Chapter 4 is about the theory of contact introduced by Mather and developed by Montaldi. In Chapter 5 we recall some basic concepts in symplectic and contact geometry and establish the link between the theory of contact and that of Lagrangian and Legendrian singularities. We apply in Chapters 6, 7 and 8 the singularity theory framework exposed in

the previous chapters to the study of the extrinsic differential geometry of surfaces in the Euclidean 3, 4 and 5-spaces respectively. The codimension of the surface in the ambient space is 1, 2 or 3 and this book shows how some aspects of the geometry of the surface change with its codimension. In Chapter 9 we chose spacelike surfaces in the Minkowski space-time to illustrate how to approach the study of submanifolds in Minkowski spaces using singularity theory. Most of the results in the previous chapters are local in nature. Chapter 10 gives a flavour of global results on closed surfaces using local invariants obtained from the local study of the surfaces in the previous chapters.

The emphasis in this book is on how to apply singularity theory to the study of the extrinsic geometry of surfaces. The methods apply to any smooth submanifolds of higher dimensional Euclidean space as well as to other settings, such as affine, hyperbolic or Minkowski spaces. However, as it is shown in Chapters 6, 7 and 8, each pair (m, n) with m the dimension of the submanifold and n of the ambient space needs to be considered separately.

This book is unapologetically biased as it focuses on research results and interests of the authors and their collaborators. We tried to remedy this by including, in the Notes of each chapter, other aspect and studies on the topics in question and as many references as we can. Omissions are inevitable, and we apologise to anyone whose work is unintentionally left out.

Currently, there is a growing and justified interest in the study of the differential geometry of singular submanifolds (such as caustics, wavefronts, images of singular mappings etc) of Euclidean or Minkowski spaces, and of submanifolds with induced (pseudo) metrics changing signature on some subsets of the submanifolds. We hope that this book can be used as a guide to anyone embarking on the study of such objects.

This book has been used (twice so far!) by the last-named author as lecture notes for a post-graduate course at the University of São Paulo, in São Carlos. We thank the following students for their thorough reading of the final draft of the book: Alex Paulo Francisco, Leandro Nery de Oliveira, Lito Edinson Bocanegra Rodríguez, Martin Barajas Sichaca, Mostafa Salarinoghabi and Patricia Tempesta. Thanks are also due to Catarina Mendes de Jesus for her help with a couple of the book's figures and to Asahi Tsuchida, Shunichi Honda and Yutaro Kabata for correcting some typos. Most of the results in Chapter 4 are due to James Montaldi. We thank him for allowing us to reproduce some of his proofs in this book.

We are also very grateful to Masatomo Takahashi for reading the final draft of the book and for his invaluable comments and corrections.

S. Izumiya, M. C. Romero Fuster, M. A. S. Ruas and F. Tari
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