MATRIX SPACES AND SCHUR MULTIPLIERS

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Matriceal Harmonic Analysis

Lars-Erik Persson

Luleå University of Technology, Sweden & Narvik University College, Norway

Nicolae Popa

*"Simion Stoilov" Institute of Mathematics, Romanian Academy, Romania
& Technical University "Petrol si Gaze", Romania*

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To Irina, Andrei, Manuela and Alexandra

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Preface

In the last two centuries the Fourier analysis, known also as harmonic analysis, experienced a strong development. Roughly speaking it consists mainly in the study of properties of periodical functions connected to their Fourier coefficients. As examples of such properties we may consider the beautiful Fejer's theory [31] about the convergence of the Fourier series to a function with respect to its Cesaro means, or the well-known Fejér-Hardy-Littlewood inequality [96].

On the other hand many mathematicians have observed that there is a linear bijective correspondence between the periodical functions f on the torus $\mathbb{T}$ and its corresponding Toeplitz matrix, that is the infinite matrix having the n th Fourier coefficient of f on the n th diagonal submatrix parallel to the main diagonal, numbered by 0. (See for instance [94], [11].) For instance we began to think about this topic after reading the paper of J. Arazy [1], where a short remark about the analogy between Fourier coefficients and diagonal submatrices was made. In fact, it appears that periodical functions on the torus are particular cases of infinite matrices and it is tempting to develop a matrix version of the classical harmonic analysis, where instead of n th Fourier coefficient of a periodical function you have to consider the n th diagonal of a given infinite matrix.

The present volume is dedicated to this goal, namely to formulate and prove some statements in this new matrix version of classical harmonic analysis in terms of "diagonal" submatrices of an infinite matrix, which are analogous of the well-known statements in harmonic analysis. The current knowledge is presented in a unified way and also some new results are included to complete the picture to a fairly nice theory we hereby call *matriceal harmonic analysis*.

Now we briefly describe some of the most motivating results of this

book.

After an introduction and a presentation of principal notions, in Chapter 2 we present some results from the master dissertation of Victor Lie (see [50]). The main idea of this work is to consider an infinite matrix as a sequence of functions $(\mathcal{L}_k)_{k \geq 1}$ and to exploit this interpretation in order to obtain a useful formula for the operator norm of an infinite matrix, namely:

$$\|A\|_{B(\ell_2)} = \sup_{\|h\|_2 \leq 1} \|V_{\mathcal{L}_B, h}\|_{\infty},$$

where

$$V_{\mathcal{L}_B, h}(x) = \left(\sum_{k=1}^{\infty} |(\mathcal{L}_k * h)(x)|^2 \right)^{1/2}, \quad \forall x \in [0, 1] \text{ and } \forall h \in H_0^2([0, 1]).$$

By using this formula we give new proofs of some classical results of infinite matrix theory. For instance, it is possible to prove, with the same method, some scattered facts like:

- For a Toeplitz matrix A , $A \in B(\ell_2)$ if and only if $f_A \in L^\infty([0, 1])$ (see Theorem 3.2),
- (**Bennett's Theorem**) For a Toeplitz matrix A , $A \in M(\ell_2)$ if and only if f_A is a bounded Borel measure on $[0, 1]$ (see Theorem 3.3),
- (**Nehari's Theorem**) For a Hankel matrix A , $A \in B(\ell_2)$ if and only if $g_A \in BMO$ (see Theorem 3.5),
- (**Theorem of Kwapien-Pelczynski**) If P_n is the main triangle projection of order n , then

$$\sup_{\|A\|_{B(\ell_2)} \leq 1} \|P_n(A)\|_{B(\ell_2)} = \mathcal{O}(\log n) \quad n \rightarrow \infty$$

(see Theorem 3.8).

In Chapter 3 we give a matrix version of Fejer's theory, introducing a subclass of infinite matrices representing bounded linear operators on ℓ_2 , namely the class of *continuous matrices* $C(\ell_2)$.

This class consists of those infinite matrices, which are approximable in the operator norm by matrices of finitely band-type. In particular, we present the results in [17], which were extended later in [48] and [46].

Just as one possible application of the matrix version of Fourier analysis we mention the approximation of an infinite matrix in various ways. For

instance the following natural question arise:

Which infinite matrices can be approximated in the operator norm by its finite-type band submatrices?

In particular, an answer to this question, which extends the well-known Jordan's theorem, is given in this chapter.

Another type of approximation of infinite matrices by a special type of matriceal polynomials is also given in Chapter 3 and it extends a well-known theorem of A. Haar from 1910 (see Theorem C on page 49):

- Let $A = (a_k^l)_{l \geq 1, k \in \mathbb{Z}}$ be a matrix belonging to $C(\ell_2)$ such that all sequences $\mathbf{a}_k \stackrel{\text{def}}{=} (a_k^l)_{l \geq 1}$, $k \in \mathbb{Z}$, located on the k th diagonal, belong to the class ms .
Then, for any $\epsilon > 0$ there is an $n_\epsilon \in \mathbb{N}^*$ and sequences $\alpha_k \in ms$, $k \in \{0, \dots, n-1\}$, such that

$$\|A - \sum_{k=0}^{n-1} \alpha_k \odot H_k\|_{B(\ell_2)} < \epsilon.$$

Here ms is a special linear subspace of the space of all bounded sequences ℓ_∞ , and $\odot$ means a product between two matrices, which extends the usual product of a scalar and a function.

Next we mention that in 1983 A. Shields [84] stated and proved the following beautiful theorem:

- Let $M \in C_1$ have upper triangular form with respect to the orthonormal basis $\{e_n\}$ ($n = 1, 2, \dots$) of ℓ_2 . Then

$$\sum_{k=1}^{\infty} \sum_{j=1}^k \frac{|M(j, k)|}{1 + k - j} \leq \pi \|M\|_{T_1},$$

with equality only when $M = 0$.

Here T_1 means the subspace of all trace class infinite upper triangular matrices.

The above result is analogous to the well-known Hardy-Littlewood-Fejer inequality.

We present and discuss this theorem and some other related results in Chapter 4.

For instance we present a new characterization of the elements of the space T_1 with respect to the sequence of a special linear combination of their diagonals, namely:

- Let $A \in B(\ell_2)$ be an upper triangular matrix. Then the following assertions are equivalent:

$$a) A \in T_1;$$

$$b) \sup_n \frac{1}{a_n} \sum_{j=0}^n \frac{1}{j+1} \|s_j A\| < \infty;$$

$$c) \sup_n \|P_n A\| < \infty.$$

Here

$$P_n A = \frac{1}{a_n} \sum_{j=0}^n \frac{1}{j+1} s_j A, \text{ where } a_n = \sum_{j=0}^n \frac{1}{j+1} \quad (n = 0, 1, 2, \dots)$$

$$\text{and } s_j A = \sum_{k=0}^j A_k.$$

It is important to mention that in Chapter 4 and also in the sequel, often the obtained matriceal results are only analogous to but do not extend the known results from harmonic analysis. This happens because most of the results from these chapters refer to matrices connected to Schatten classes of matrices and consequently their proofs cannot be applied to Toeplitz matrices.

Another topic which was touched some years ago is the study of matrix version of Hankel operators. Some especially interesting results about these Hankel operators were obtained by S. Power in [77]. For instance he showed there the following matrix version of Nehari's theorem:

- Let Φ be an infinite matrix such that $\Phi A \in C_2$ for all finite band-type matrices A .

Then the following statements are equivalent:

- H_Φ is a bounded linear operator on T_2 .
- There is $\Psi \in B(\ell_2)$ such that $\Psi_k = \Phi_k$ for all $k < 0$.
- $P_- \Phi \in BMO_F(\ell_2)$.

See Chapter 5 for all unexplained notions and notations in this theorem.

In Chapter 5 we give a different and in our opinion more natural proof of this result and investigate this topic further. For example, we derive a sufficient condition in order that a matrix version of a Hankel operator to be nuclear. For a particular class of its symbol this condition is even necessary.

A class of Banach spaces of analytic functions, which has received great attention in the last two decades is the class of Bergman spaces. See e.g. [94], [28] and the references given there. We investigate some properties of the class of Bergman-Schatten spaces in Chapter 6. For instance we present and discuss some inequalities valid in Bergman-Schatten spaces (see [76]) e.g. the following:

- **(Hausdorff-Young Theorem)** For $1 \leq p \leq \infty$, let q be the conjugate index, i.e. $\frac{1}{p} + \frac{1}{q} = 1$ (for $p = 1$ we have $q = \infty$).
 - (i) If $1 \leq p \leq 2$, then $A \in T_p$ implies that $\left(\sum_{n=0}^{\infty} \|A_n\|_{T_p}^q\right)^{1/q} \leq \|A\|_{T_p}$.
 - (ii) If $2 \leq p \leq \infty$, then $\{\|A_n\|_{T_p}\} \in \ell_q$ implies that $\|A\|_{T_p} \leq \left(\sum_{n=0}^{\infty} \|A_n\|_{T_p}^q\right)^{1/q}$.

Another important result, which appears in Chapter 6 is a matricial analogue of a result obtained by Mateljevic and Pavlovic [61] in 1984, (see [62]):

- Let A be an upper triangular matrix. Then $A \in L_a^1(D, \ell_2)$ if and only if

$$\sum_{n=1}^{\infty} \frac{\|\sigma_n(A)\|_{C_1}}{(n+1)^2} < \infty.$$

Moreover, a matrix version of the usual Bloch space is introduced in Chapter 7. See [72]. The interest of this space consists mainly in the fact that it satisfies the equality

$$\mathcal{B}(D, \ell_2) = (H^1(\ell_2), BMOA(\ell_2)),$$

where $\mathcal{B}(D, \ell_2)$ means the matricial Bloch space, $H^1(\ell_2)$, $BMOA(\ell_2)$ stand for the matrix version of the Hardy space H^1 , respectively, for the matrix version of the space $BMOA$, and (X, Y) is the space of all Schur multipliers between spaces of infinite matrices X and Y .

This equation is the matrix analogue of a result valid for Fourier multipliers of periodical functions proved in 1990 by Mateljevic and Pavlovic [60].

Moreover, in 1995 O. Blasco [12] proved that a vector valued version of the equality of Mateljevic and Pavlovic is valid, generally speaking, only for functions with values in Hilbert spaces.

Consequently, this version of the above equality deserves some special attention. The equality is proved in Chapter 8. This chapter is dedicated to a very important tool in the theory, namely Schur multipliers, which represents the matrix version of classical Fourier multipliers.

We also present, prove and discuss some other results concerning Schur multipliers between Banach spaces of infinite upper triangular matrices. We mention just the following matrix version of a well-known result of Paley:

- If $A = \sum A_n \in H^1(\ell_2)$, then $P(A) := \sum_{k=1}^{\infty} A_{2^k} \in H^2(\ell_2)$.

Here $H^1(\ell_2)$ is a matrix version of Hardy space introduced in Chapter 4 and $H^2(\ell_2)$ is the space of all upper triangular Hilbert-Schmidt matrices.

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The cover images of the midnight sun were taken by Elin Persson from the north harbour of Luleå, Sweden, at midnight of 16th June 2013. These images reflect the empowering senses the authors felt when this special book was finalized under the skies of this famous light.

Luleå, June 2013, under the influence of the magic midnight sun atmosphere close to the Polar Circle.

Lars-Erik Persson and Nicolae Popa

Contents

<i>Preface</i>	vii
1. Introduction	1
1.1 Preliminary notions and notations	1
1.1.1 Infinite matrices	1
1.1.2 Analytic functions on disk	4
1.1.3 Miscellaneous	5
1.1.4 The Bergman metric	7
2. Integral operators in infinite matrix theory	9
2.1 Periodical integral operators	9
2.2 Nonperiodical integral operators	17
2.3 Some applications of integral operators in the classical theory of infinite matrices	18
2.3.1 The characterization of Toeplitz matrices	18
2.3.2 The characterization of Hankel matrices	24
2.3.3 The main triangle projection	27
2.3.4 $B(\ell_2)$ is a Banach algebra under the Schur product	30
3. Matrix versions of spaces of periodical functions	33
3.1 Preliminaries	34
3.2 Some properties of the space $C(\ell_2)$	34
3.3 Another characterization of the space $C(\ell_2)$ and related results	36
3.4 A matrix version for functions of bounded variation	41

3.5	Approximation of infinite matrices by matriceal Haar polynomials	44
3.5.1	Introduction	45
3.5.2	About the space ms	50
3.5.3	Extension of Haar's theorem	56
3.6	Lipschitz spaces of matrices; a characterization	61
4.	Matrix versions of Hardy spaces	65
4.1	First properties of matriceal Hardy space	65
4.2	Hardy-Schatten spaces	69
4.3	An analogue of the Hardy inequality in T_1	75
4.4	The Hardy inequality for matrix-valued analytic functions	79
4.4.1	Vector-valued Hardy spaces H_X^p	79
4.4.2	$(H^p - \ell_q)$ -multipliers and induced operators for vector-valued functions	80
4.5	A characterization of the space T_1	97
4.6	An extension of Shields's inequality	101
5.	The matrix version of $BMOA$	109
5.1	First properties of $BMOA(\ell_2)$ space	109
5.2	Another matrix version of BMO and matriceal Hankel operators	111
5.3	Nuclear Hankel operators and the space $\mathcal{M}^{1,2}$	119
6.	Matrix version of Bergman spaces	121
6.1	Schatten class version of Bergman spaces	121
6.2	Some inequalities in Bergman-Schatten classes	132
6.3	A characterization of the Bergman-Schatten space	136
6.4	Usual multipliers in Bergman-Schatten spaces	141
7.	A matrix version of Bloch spaces	149
7.1	Elementary properties of Bloch matrices	149
7.2	Matrix version of little Bloch space	161
8.	Schur multipliers on analytic matrix spaces	175

<i>Bibliography</i>	185
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<i>Index</i>	191
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