ALMOST COMPLEX HOMOGENEOUS SPACES AND THEIR SUBMANIFOLDS

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PREFACE

These notes contain an exposition of the theory of almost complex homogeneous spaces and their complex submanifolds. Certain closely related class of spaces, what we call partial G -flag manifolds, are also studied. Our emphasis is on the compact case so that there is a pervasive influence of the structure theory of compact Lie groups. In this we are strongly guided by the works of Wang [Wan] and Borel-Hirzebruch [B-H].

An almost complex homogeneous space is a homogeneous space equipped with an invariant almost complex structure. In the compact case fundamental examples of such spaces are provided by homogeneous spaces of the form G/H , where G is a connected compact semisimple Lie group and H is a closed subgroup of maximal rank. (Not all such homogeneous spaces are almost complex, however. We call such a space a partial G -flag manifold.) For these spaces the root decomposition of G relative to a maximal torus T (contained in H) plays a crucial role and the understanding of the flag manifold G/T becomes important.

Recently much attention has been paid to the investigation of minimal surfaces (and harmonic maps) in various Riemannian manifolds. Often times in these studies the ambient space has been an almost complex homogeneous space or at least a partial G -flag manifold and when this happened there appeared a strong relationship between (pseudo-) complex curves and minimal surfaces. Just to mention a few examples we have $CP^n = U(n+1)/U(1) \times U(n)$ ([C-W], [E-W]), $HP^n = Sp(n+1)/Sp(1) \times Sp(n)$ ([G]), $Q_n = SO(n+2; \mathbf{R})/SO(2) \times SO(n)$ ([J-R-Y]), $S^4 = HP^1$ ([Ch1], [B2]), $S^6 = G_2/SU(3)$ ([B1]). No doubt

that a systematic investigation into (pseudo-) complex submanifolds of these spaces is worth pursuing.

The method we adopt, justifiably most suited to our situation, is that of moving frames (*repère mobile*) used so successfully by Cartan and more recently by Chern. Accordingly we reformulate the basic notions regarding invariant almost complex structures on G/H in terms of the Maurer-Cartan form of G which plays an essential role in studying submanifolds of homogeneous spaces via moving frames. The basic reference on the technique of moving frames is Cartan [Ca]. One may also consult [Gr] or [J] for a more recent treatment.

Moving frame theoretic analysis of a given class of submanifolds in a homogeneous space is often complicated by algebraic difficulties associated with computing certain orbit structures of Lie group actions. This problem is already formidable at local level in many instances. (Holomorphic curves in CP^n case is a notable exception to this. Here one is able to concentrate on the global study.) However, in studying complex submanifolds of a flag manifold the “local-algebraic” difficulty disappears. Purely algebraic consideration into the root system of a compact Lie group G relative to a maximal torus T enables us to understand local orbit structures of complex maps rather completely. Thus we are free to pursue the global investigation of these objects in an attempt to mimic the well-known situation of holomorphic curves in CP^n (e.g., Plücker formulae, equidistribution theorems, etc).

Understanding of the flag manifold (G/T) case frequently throws light upon other more general cases. For example, if G/H is an almost complex homogeneous space with H , a closed subgroup of maximal rank (G , compact) then H contains a maximal torus of G . Therefore, by composing complex maps into G/T with the projection $G/T \rightarrow G/H$ we recover complex maps into G/H . Of particular importance among complex submanifolds of G/T in this regard are “horizontal” complex submanifolds. An important theorem regarding horizontal complex curves in G/T and minimal surfaces in G/H , under a setting parallel to ours, was proved by Bryant [B3].

Though we have not proved this in its generality, a sort of converse to the above statement about the projections of complex submanifolds

of G/T to G/H is exhibited in Ch. IV §§18–19. This is achieved by interpreting the “Frenet frames” along complex submanifolds of G/H as horizontal complex liftings into G/T . To be sure there is more work to be done along this area.

Each chapter comes with its own introduction explaining the organization.

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