GRAVITATION: SL (2,C) GAUGE THEORY AND CONSERVATION LAWS

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GRAVITATION: SL (2,C) GAUGE THEORY AND CONSERVATION LAWS

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World Scientific

Singapore • New Jersey • London • Hong Kong

Published by

World Scientific Publishing Co. Pte. Ltd.

P O Box 128, Farrer Road, Singapore 9128

USA office: 687 Hartwell Street, Teaneck, NJ 07666

UK office: 73 Lynton Mead, Totteridge, London N20 8DH

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ISBN 981-02-0160-5

Printed in Singapore by Utopia Press.

PREFACE

The study of gravitation is one of the most important topics in theoretical physics, because of its implications both on particle and field theories as well as on astrophysics. Of special importance is the formulation of Einstein's general relativity theory as a gauge theory, whereby gravitation is cast as other fields all of which, as is well-known, are described nowadays as gauge fields. The conservation laws in general relativity, from the viewpoint of both Einstein's original classical theory and the $SL(2,C)$ gauge theory of gravitation, is also of great importance since in addition of playing a central role in these theories they bear on the rest of physics.

This monograph deals with the two topics mentioned above; it gives a comprehensive presentation of the $SL(2,C)$ gauge theory of gravitation along with the problem of conservation laws in general relativity. Emphasize is put on quadratic Lagrangians which yield the Einstein field equations, as compared with Hilbert's original linear Lagrangian, thus gravitation follows the other gauge fields all of which are derived from nonlinear Lagrangians. The implications of the quadratic Lagrangians on the conservation laws are then studied. Conserved gravitational currents, which emerge from the $SL(2,C)$ gauge theory of

gravitation, are discussed in detail. Noether's theorem is applied to Carmeli and Kaye's spinorial version of Hilbert's Lagrangian with respect to the $SL(2, C)$ group. Nissani and Leibowitz's work on the existence of a preferred class of coordinate systems in which the energy-momentum tensor satisfies a global continuity equation is outlined, and the physical implication of this class of frames is analyzed.

Beer-Sheva, Israel
October, 1989

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ACKNOWLEDGEMENTS

We wish to thank our colleagues for the encouragement, helpful remarks and discussions we had during the writing of this monograph. In particular we want to thank Professors N. Rosen, S. Malin, A. Gersten, A. Davidson, J. Bekenstein, Y. Avishai and J. Daboul, and Drs. A. Lonke, A. Guendelman, A. Feinstein and M. Schiffer. We are indebted to the late Mrs. Sara Corrogosky for her assistance, and are particularly grateful to Ms. Y. M. Chong, Editor of World Scientific Publishing Company, for the excellent cooperation we had during the course of printing the book. Last, but not least, we wish to express our thanks to our spouses and families for the unlimited personal help, encouragement, patience and support we received during the writing and publishing of this monograph and without which it would never have been published.

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LIST OF SYMBOLS

<i>Symbol</i>	<i>Description</i>
S	Element of the group $SL(2,C)$
ϕ^A	2-component spinor
ϵ_{AB}	Levi-Civita metric
ξ_a^A	Spin frame
$\Gamma_{A\mu}^B$	Spinor affine connection
$\sigma_{AB'}^\mu$	Hermitian matrices
$\epsilon^{\alpha\beta\gamma\delta}$	Levi-Civita skew-symmetric tensor density
$\alpha, \dots, \gamma$	Spin coefficients
Ψ_{ABCD}	Weyl spinor
$\phi_{ABC'D'}$	Trace-free Ricci spinor
$\Psi_0 \dots \Psi_4$	Components of Weyl spinor
$\phi_{00} \dots \phi_{22}$	Components of trace-free Ricci spinor
Λ	Ricci's scalar/24
$T^{\alpha\beta}$	Energy-momentum tensor
$\Gamma_{\alpha\beta}^\gamma$	Christoffel symbols

$G^{\alpha\beta}$	Einstein's tensor
$R^{\alpha\beta}$	Ricci's tensor
R	Ricci's scalar
$g_{\alpha\beta}$	The metric tensor
κ	Einstein's gravitational constant
g	Determinant of the metric tensor
$\tau^{\mu\nu}$	Gravitational energy-momentum pseudo-tensor
$\mathcal{L}$	Lagrangian
$\mathcal{L}_E$	Einstein's Lagrangian
$\mathcal{L}_H$	Hilbert's Lagrangian
$\tau_E^{\mu\nu}$	Einstein's gravitational energy-momentum pseudo-tensor
$\tau_E^{\mu\nu}$	Einstein's total energy-momentum complex
$\mathcal{L}_M$	Matter Lagrangian
P^α	Energy-momentum vector
$\tau_L^{\mu\nu}$	Landau-Lifshitz's total energy-momentum complex
$\tau_L^{\mu\nu}$	Landau-Lifshitz's gravitational energy-momentum pseudo-tensor
$M^{\alpha\beta\gamma}$	Angular-momentum tensor
$\Phi_L^{\mu\nu\rho}$	Landau-Lifshitz's super-potential
$\Phi_E^{\mu\nu\rho}$	Freud's super-potential
$\Phi_M^{\mu\nu\rho}$	Moller's super-potential
$\tau_M^{\nu\mu}$	Moller's total energy-momentum pseudo-tensor
$\tau_M^{\nu\mu}$	Moller's gravitational energy-momentum complex
h^Λ_μ	Moller's tetrad
$\theta_{\mu\nu\rho}$	Moller's tensor
ϕ^ρ	Moller's vector
$\bar{\tau}_M^{\nu\mu}$	Moller's gravitational energy-momentum second pseudo-tensor
$\bar{\Phi}_M^{\mu\nu\rho}$	Moller's second super-potential
$F_{\mu\nu}{}^b$	Carmeli's gauge field tensor
$B_{\mu a}{}^b$	Carmeli's gauge potential vector
$R^\rho{}_{\sigma\mu\nu}$	Riemann's curvature tensor
$\sigma_{\rho\alpha\epsilon'}$	The metric tetrad
$L_{\mu\nu}$	Gauge potential rotor tensor
$K_{\mu\nu}$	Gauge potential commutator tensor
$*L^{\alpha\beta}$	Antisymmetric conserved tensor density
$\Phi^{\alpha\beta\mu}$	Super-potential
$\mathcal{L}_0$	Hilbert's Lagrangian in vacuum
$\mathcal{L}_{Li}$	Quadratic scalar densities

$\mathcal{L}_{Li}$	($i = 1, 2, 3$) High-order theories Lagrangians
$\mathcal{L}_{Li}$	($i = 4, 5$) Lanczos' invariants
$\mathcal{L}_C$	Carmeli's Lagrangian
F^W	Weyl's part of Carmeli's gauge field tensor
F^R	Ricci's part of Carmeli's gauge field tensor
$\mathcal{L}_K$	Spinorial family of Lagrangians
$U_{\alpha\beta\gamma\delta}$	Four-index energy-momentum tensor
$C_{\alpha\beta\gamma\delta}$	Four-index gravitational tensor (Weyl)
$\mathcal{L}_i$	Scalar density components of the tensorial quadratic Lagrangian
$\mathcal{L}_N$	Tensorial quadratic Lagrangian
$\mathcal{L}_F$	Tensorial quadratic Lagrangian in vacuum
P^μ	Energy-momentum affine vector
J_C	Carmeli's conserved vector density
J_M	Malin's conserved vector density
F^S	Part of Carmeli's gauge field obtained from Ricci's traceless tensor
F^T	Part of Carmeli's gauge field obtained from Ricci's scalar
J_σ	The "gravitational" conserved current
The metric signature used is (1, -1, -1, -1)	