Introduction to **KAC-MOODY ALGEBRA**

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Wan Zhe-xian

Institute of Systems Science
Academia Sinica
Beijing, China

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PREFACE

This book is an introduction to a rapid growing subject of modern mathematics, the Kac-Moody algebra, which were introduced by V. Kac and R. Moody simultaneously and independently in 1968.

In 1948 C. Chevalley published a short note in *Comptes Rendus*. This short note contains not only a unified algebraic proof of a theorem of E. Cartan which asserts that an irreducible representation of a finite-dimensional complex semi-simple Lie algebra is finite-dimensional if and only if it is a highest weight representation with a dominant integral weight as its highest weight, but also many of the basic concepts of Kac-Moody algebras which appeared twenty years later. In particular, in this note he pointed out that in any finite-dimensional complex semi-simple Lie algebra $\mathfrak{g}$ of rank n there exist $3n$ generators $e_1, \dots, e_n, f_1, \dots, f_n, \alpha_1^\vee, \dots, \alpha_n^\vee$, which satisfies the following relations:

$$\left. \begin{aligned} [\alpha_i^\vee, \alpha_j^\vee] &= 0 \\ [e_i, f_j] &= \delta_{ij} \alpha_i^\vee \\ [x_i^\vee, e_j] &= a_{ij} e_j, \quad [\alpha_i^\vee, f_j] = -a_{ij} f_j \\ (\text{ad } e_i)^{1-a_{ii}} e_j &= 0, \quad (\text{ad } f_j)^{1-a_{jj}} f_i = 0, \quad \text{for } i \neq j \end{aligned} \right\} \quad \text{for } i, j = 1, 2, \dots, n, \quad (1)$$

where a_{ij} 's are the Cartan integers of $\mathfrak{g}$. All the a_{ij} 's form the so-called Cartan matrix $A = (a_{ij})_{i,j=1}^n$ which is an $n \times n$ matrix with the following properties:

- (C1) $a_{ii} = 2$ for $i = 1, 2, \dots, n$,
- (C2) a_{ij} are non-positive integers for $i \neq j$,
- (C3) $a_{ij} = 0$ implies $a_{ji} = 0$,
- (C4) all principal minors of A are positive.

Much later J.P. Serre proved in 1966 that the relations (1) are defining relations of the finite-dimensional complex semi-simple Lie algebra.

Kac and Moody's idea was: dropping the condition (C4) and starting with an $n \times n$ matrix A satisfying (C1), (C2) and (C3) only, which is called a *generalized Cartan matrix*, one can define a Lie algebra $\mathfrak{g}'(A)$ with $3n$ generators $e_i, f_i, \alpha_i^\vee$ ($i = 1, 2, \dots, n$) and relations (1), which is in general infinite-dimensional, and obtain a nice algebraic theory of the algebra $\mathfrak{g}'(A)$. Since a matrix A satisfying (C1), (C2) and (C3) only may be singular and then the linear functions $\alpha_i, i = 1, 2, \dots, n$, on the linear span $\mathfrak{h}$ of $\alpha_1^\vee, \alpha_2^\vee, \dots, \alpha_n^\vee$ defined by

$$[h, e_i] = \alpha_i(h) e_i \quad \text{for } h \in \mathfrak{h}$$

may be linearly dependent, they suggested to enlarge the abelian Lie algebra $\mathfrak{h}$ so that the linear functions $\alpha_1, \alpha_2, \dots, \alpha_n$ will be linearly independent. Then the Lie algebra $\mathfrak{g}(A)$ is obtained, which is called *the Kac-Moody algebra associated with the*

generalized Cartan matrix A . Chapter 1 is devoted to the construction of the Kac-Moody algebra $\mathfrak{g}(A)$ and the study of its elementary properties.

To classify the Kac-Moody algebras it is sufficient to classify the generalized Cartan matrices. Chapter 2 is devoted to this problem. Generalized Cartan matrices are classified into three types: finite, affine and indefinite. The generalized Cartan matrices of finite type are simply the Cartan matrices, and their classification is well-known. The classification of the generalized Cartan matrices of affine type were achieved by Kac and Moody independently. However, the indefinite type is far from complete.

In the theory of the finite-dimensional complex semi-simple Lie algebras there are four main tools: the root string (or more generally, the weight-string), the Weyl group, the Killing form and the Casimir operator. All of them were generalized to Kac-Moody algebras by Kac and Moody. The first two of them were generalized to arbitrary Kac-Moody algebras and the last two were generalized to symmetrizable Kac-Moody algebras. A Kac-Moody algebra $\mathfrak{g}(A)$ is called *symmetrizable*, if A can be written as $A = DB$, where D is a nonsingular diagonal matrix and B is a symmetric matrix. These main tools will be studied in later chapters.

Concrete realizations of the Kac-Moody algebras of affine type, their root systems and Weyl groups are also obtained by Kac and Moody and will be studied in more detail in this book.

Finally, an introduction to the representation theory of the Kac-Moody algebras is also given.

In recent years, the Kac-Moody algebras attracted the attention of many mathematicians and mathematical physicists, mainly because they have various connections with and applications in different branches of mathematics and mathematical physics. Since we present only the fundamental theory of Kac-Moody algebras in this book, which is genuinely elementary and requires a minimum knowledge of algebra, these connections and applications will not be included. Interested readers may consult the recent literature.

In preparing this book, Ronghua Xiong, Minan Yu and Hechun Zhang helped me a lot, in particular, Xiong and Yu prepared the camera-ready copy and Zhang corrected several errors and did the proof-reading. Without their help, the book could not be published; to all of them the author pay his sincere thanks.

Z. Wan

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